

5.1

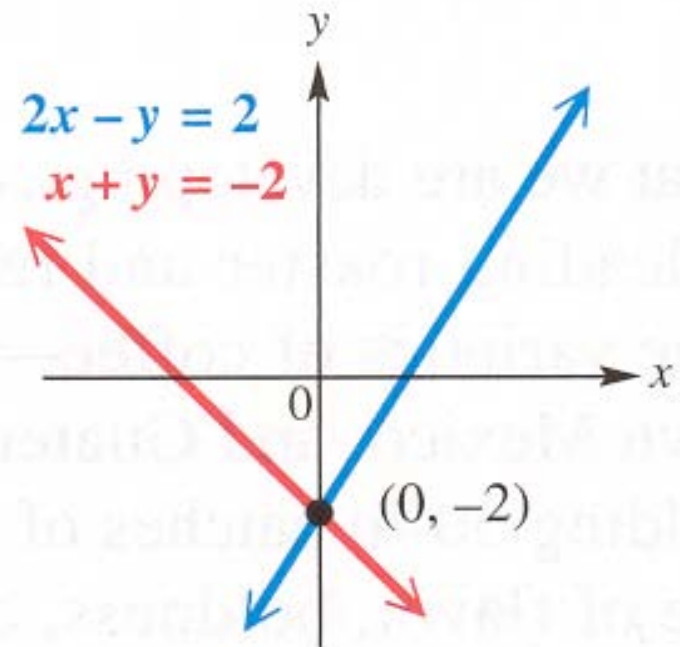
Systems of Linear Equations

Linear Systems
Substitution Method
Elimination Method
Special Systems

Linear Systems

The possible graphs of a linear system in two unknowns are as follows.

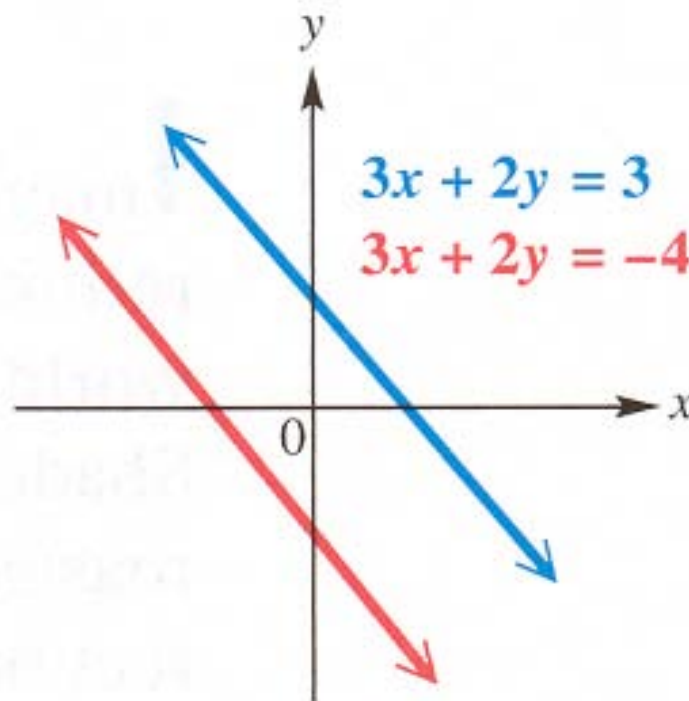
- 1. The graphs intersect at exactly one point**, which gives the (single) ordered pair solution of the system. The **system is consistent** and the **equations are independent**.



One solution

Linear Systems

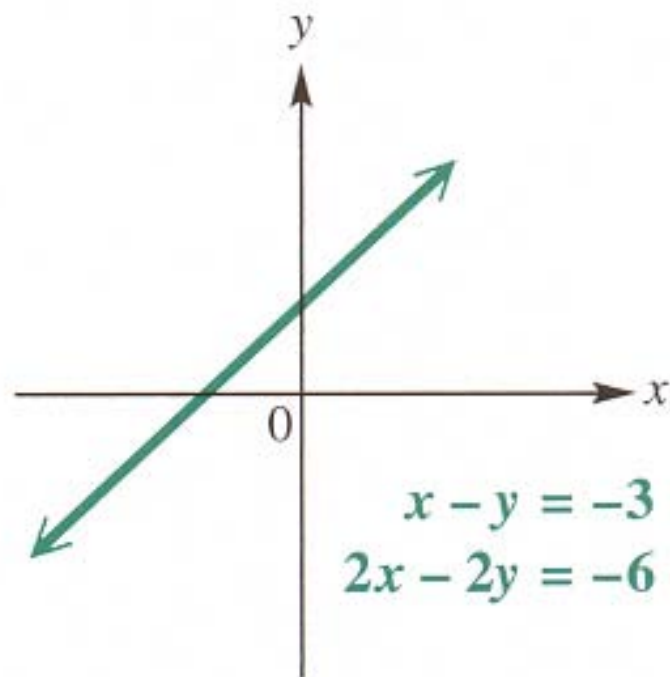
2. The graphs are **parallel lines**, so there is no solution and the solution set is \emptyset . The **system is inconsistent** and the equations are **independent**.



No solution

Linear Systems

3. The graphs are the **same line**, and there is an infinite number of solutions. The **system is consistent** and the **equations are dependent**.



Infinitely many solutions

Substitution Method

In a system of two equations with two variables, the **substitution method** involves using one equation to find an expression for one variable in terms of the other, and then substituting into the other equation of the system.

► Example 1

SOLVING A SYSTEM BY SUBSTITUTION

Solve the system.

$$3x + 2y = 11 \quad (1)$$

$$-x + y = 3 \quad (2)$$

Solution

Begin by solving one of the equations for one of the variables. We solve equation (2) for y .

$$-x + y = 3 \quad (2)$$

$$y = x + 3 \quad \text{Add } x.$$

▶ Example 1

SOLVING A SYSTEM BY SUBSTITUTION

Now replace y with $x + 3$ in equation (1), and solve for x .

Note the careful
use of
parentheses.

$$3x + 2y = 11 \quad (1)$$

$$3x + 2(x + 3) = 11 \quad \text{Let } y = x + 3 \text{ in (1).}$$

$$3x + 2x + 6 = 11 \quad \text{Distributive property}$$

$$5x + 6 = 11 \quad \text{Combine terms.}$$

$$5x = 5 \quad \text{Subtract.}$$

$$x = 1$$

▶ Example 1

SOLVING A SYSTEM BY SUBSTITUTION

Replace x with 1 in equation (3) to obtain $y = 1 + 3 = 4$. The solution of the system is the ordered pair (1, 4). ***Check this solution in both equations (1) and (2).***

Check:

$3x + 2y = 11$	(1)	$-x + y = 3$	(2)
$3(1) + 2(4) = 11$?	$-1 + 4 = 3$?
$11 = 11$	True	$3 = 3$	True

Both check; the solution set is $\{(1, 4)\}$.

Elimination Method

Another way to solve a system of two equations, called the **elimination method**, uses multiplication and addition to eliminate a variable from one equation. To eliminate a variable, the coefficients of that variable in the two equations must be additive inverses. To achieve this, we use properties of algebra to change the system to an **equivalent system**, one with the same solution set. The three transformations that produce an equivalent system are listed here.

Transformations of a Linear System

- 1.** Interchange any two equations of the system.
- 2.** Multiply or divide any equation of the system by a nonzero real number.
- 3.** Replace any equation of the system by the sum of that equation and a multiple of another equation in the system.

▶ Example 2

SOLVING A SYTEM BY ELIMINATION

Solve the system.

$$3x - 4y = 1 \quad (1)$$

$$2x + 3y = 12 \quad (2)$$

Solution

One way to eliminate a variable is to use the second transformation and multiply both sides of equation (2) by -3 , giving the equivalent system

$$3x - 4y = 1 \quad (1)$$

$$-6x - 9y = -36 \quad \text{Multiply (2) by } -3 \quad (3)$$

► Example 2

SOLVING A SYTEM BY ELIMINATION

Now multiply both sides of equation (1) by 2, and use the third transformation to add the result to equation (3), eliminating x . Solve the result for y .

$$6x - 8y = 2$$

Multiply (1) by 2

$$-6x - 9y = -36$$

(3)

$$-17y = -34$$

Add.

$$y = 2$$

Solve for y .

▶ Example 2

SOLVING A SYTEM BY ELIMINATION

Substitute 2 for y in either of the original equations and solve for x .

$$3x - 4y = 1 \quad (1)$$

$$3x - 4(\textcolor{red}{2}) = 1 \quad \text{Let } y = 2 \text{ in (1).}$$

$$3x - 8 = 1 \quad \text{Multiply.}$$

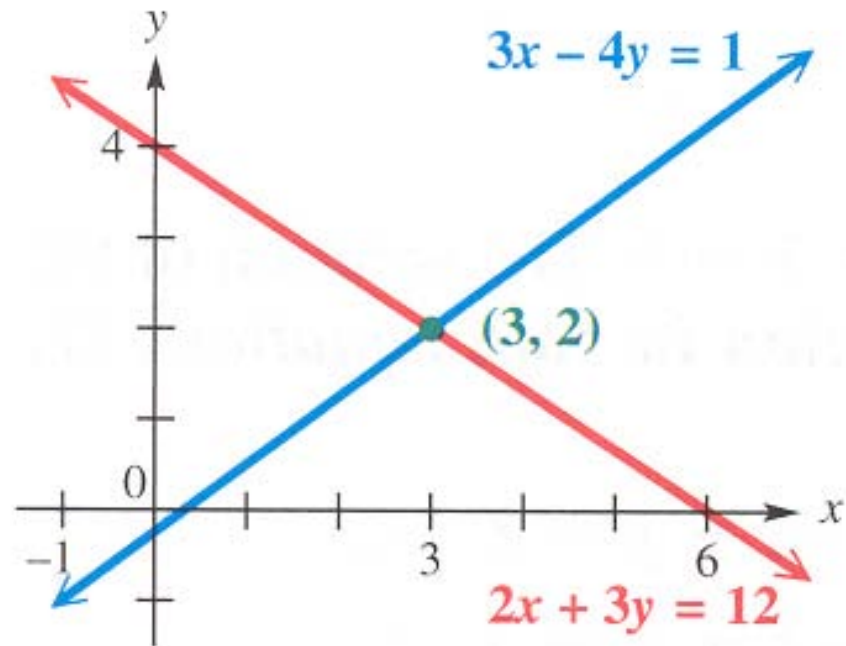
$$3x = 9 \quad \text{Add 8.}$$

$$\textcolor{blue}{x} = \textcolor{blue}{3}$$

► Example 2

SOLVING A SYTEM BY ELIMINATION

A check shows that $(3, 2)$ satisfies both equations (1) and (2); the solution set is $\{(3, 2)\}$.



Consistent system

▶ Example 3

SOLVING AN INCONSISTENT SYSTEM

Solve the system.

$$3x - 2y = 4 \quad (1)$$

$$-6x + 4y = 7 \quad (2)$$

Solution

To eliminate the variable x , multiply both sides of equation (1) by 2.

$$6x - 4y = 8 \quad \text{Multiply (1) by 2}$$

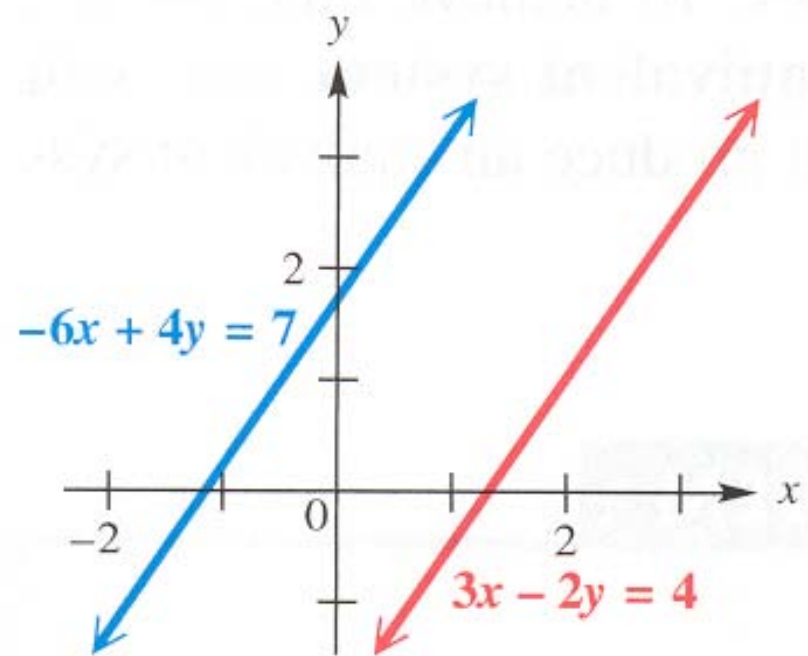
$$\begin{array}{r} 6x - 4y = 8 \\ -6x + 4y = 7 \\ \hline \end{array} \quad (2)$$

$$0 = 15 \quad \text{False.}$$

► Example 3

SOLVING AN INCONSISTENT SYSTEM

Since $0 = 15$ is false, the system is inconsistent and has no solution. As suggested here by the graph, this means that the graphs of the equations of the system never intersect. (The lines are parallel.) The solution set is the empty set.



Inconsistent system

▶ Example 4

SOLVING A SYSTEM WITH INFINITELY MANY SOLUTIONS

Solve the system.

$$8x - 2y = -4 \quad (1)$$

$$-4x + y = 2 \quad (2)$$

Solution

Divide both sides of equation (1) by 2, and add the result to equation (2).

$$4x - y = -2$$

Divide (1) by 2.

$$\begin{array}{r} -4x + y = 2 \\ \hline \end{array}$$

(2)

$$0 = 0$$

True.

► Example 4

SOLVING A SYSTEM WITH INFINITELY MANY SOLUTIONS

The result, is a true statement, which indicates that the equations of the original system are equivalent. Any ordered pair that satisfies either equation will satisfy the system. From equation (2),

$$-4x + y = 2 \quad (2)$$

$$y = 2 + 4x.$$

► Example 4

SOLVING A SYSTEM WITH INFINITELY MANY SOLUTIONS

The solutions of the system can be written in the form of a set of ordered pairs $(x, 2 + 4x)$, for any real number x . Some ordered pairs in the solution set are $(0, 2 + 4 \cdot 0) = (0, 2)$, $(1, 2 + 4 \cdot 1) = (1, 6)$, $(3, 14)$, and $(-2, -6)$. As shown here, the equations of the original system are dependent and lead to the same straight line graph. The solution set is written $\{(x, 2 + 4x)\}$.

