4.1 Synthetic Division

Synthetic Division
Evaluating Polynomial Functions Using the Remainder Theorem
Testing Potential Zeros
Division Algorithm

Let \( f(x) \) and \( g(x) \) be polynomials with \( g(x) \) of lower degree than \( f(x) \) and \( g(x) \) of degree one or more. There exists unique polynomials \( q(x) \) and \( r(x) \) such that

\[
f(x) = g(x)q(x) + r(x),
\]

where either \( r(x) = 0 \) or the degree of \( r(x) \) is less than the degree of \( g(x) \).
Synthetic division is a shortcut method of performing long division with polynomials. It is used only when a polynomial is divided by a first-degree binomial of the form $x - k$, where the coefficient of $x$ is 1.
With synthetic division it is helpful to change the sign of the divisor, so the $-4$ at the left is changed to $4$, which also changes the sign of the numbers in the second row. To compensate for this change, subtraction is changed to addition.
Caution  To avoid errors, use 0 as the coefficient for any missing terms, including a missing constant, when setting up the division.
Example 1

USING SYNTHETIC DIVISION

Use synthetic division to divide

\[
\frac{5x^3 - 6x^2 - 28x - 2}{x + 2}
\]

**Solution** Express \(x + 2\) in the form \(x - k\) by writing it as \(x - (-2)\). Use this and the coefficients of the polynomial to obtain

\[-2 \begin{array}{ccccc}
5 & -6 & -28 & -2 \\
\end{array}
\]

\(x + 2\) leads to \(-2\)
Example 1

USING SYNTHETIC DIVISION

Use synthetic division to divide

\[ 5x^3 - 6x^2 - 28x - 2 \]
\[ x + 2 \]

Solution

Bring down the 5, and multiply:

\[-2(5) = -10\]

\[ \begin{array}{r}
-2 \downarrow \\
5 \quad -6 \quad -28 \quad -2 \\
\hline \\
\end{array} \]

\[ \begin{array}{r}
\downarrow \\
-10 \\
\hline \\
5 \\
\end{array} \]
Example 1

USING SYNTHETIC DIVISION

Use synthetic division to divide

\[ \frac{5x^3 - 6x^2 - 28x - 2}{x + 2} \].

Solution Add \(-6\) and \(-10\) to obtain \(-16\). Multiply \(-2(-16) = 32\).

\[
\begin{array}{cccc}
-2 & \sqrt{5} & -6 & -28 & -2 \\
-10 & & 32 & & \\
5 & & -16 & & \\
\end{array}
\]
Example 1

USING SYNTHETIC DIVISION

Use synthetic division to divide

\[
5x^3 - 6x^2 - 28x - 2 \div x + 2
\]

**Solution** Add \(-28\) and 32 to obtain 4. Finally, \(-2(4) = -8\).

\[
\begin{array}{cccc}
-2 & \sqrt{5} & -6 & -28 & -2 \\
-10 & & 32 & -8 \\
5 & -16 & 4 \\
\end{array}
\]

Add columns. Watch your signs.
Example 1

USING SYNTHETIC DIVISION

Use synthetic division to divide

\[
\frac{5x^3 - 6x^2 - 28x - 2}{x + 2}.
\]

Solution  Add \(-2\) and \(-8\) to obtain \(-10\).

\[
\begin{array}{cccc}
-2 & \sqrt{5} & -6 & -28 & -2 \\
-10 & 32 & -8 & \\
-10 & 32 & -8 & \\
5 & -16 & 4 & -10 & \textup{Remainder}
\end{array}
\]

Quotient
Example 1

USING SYNTHETIC DIVISION

Use synthetic division to divide

\[ \frac{5x^3 - 6x^2 - 28x - 2}{x + 2}. \]

Since the divisor \( x - k \) has degree 1, the degree of the quotient will always be written one less than the degree of the polynomial to be divided. Thus,

\[ \frac{5x^3 - 6x^2 - 28x - 2}{x + 2} = 5x^2 - 16x + 4 + \frac{-10}{x + 2}. \]

Remember to add \( \text{remainder} \) \( \text{divisor} \).
Special Case of the Division Algorithm

For any polynomial \( f(x) \) and any complex number \( k \), there exists a unique polynomial \( q(x) \) and number \( r \) such that

\[
f(x) = (x - k)q(x) + r.
\]
In the synthetic division in Example 1,

\[ 5x^3 - 6x^2 - 28x - 2 = (x + 2)(5x^2 - 16x + 4) + (-10). \]

\[ f(x) = (x - k) \quad q(x) + r \]

Here \( g(x) \) is the first-degree polynomial \( x - k \).
Remainder Theorem

If the polynomial $f(x)$ is divided by $x - k$, the remainder is equal to $f(k)$. 
Remainder Theorem

A simpler way to find the value of a polynomial is often by using synthetic division. By the remainder theorem, instead of replacing \( x \) by \(-2\) to find \( f(-2) \), divide \( f(x) \) by \( x + 2 \) using synthetic division as in Example 1. Then \( f(-2) \) is the remainder, \(-10\).

\[
\begin{array}{cccc}
\text{\(-2\)} & 5 & -6 & -28 & -2 \\
\hline
& -10 & 32 & -8 \\
\hline
& 5 & -16 & 4 & -10
\end{array}
\]

\( f(-2) \)
Example 2

APPLYING THE REMAINDER THEOREM

Let \( f(x) = -x^4 + 3x^2 - 4x - 5 \). Use the remainder theorem to find \( f(-3) \).

Solution

Use synthetic division with \( k = -3 \).

\[
\begin{array}{c|ccccc}
-3 & -1 & 0 & 3 & -4 & -5 \\
\hline
 & & 3 & -9 & 18 & -42 \\
-1 & 3 & -6 & 14 & -47 \\
\end{array}
\]

By this result, \( f(-3) = -47 \).
A zero of a polynomial function \( f \) is a number \( k \) such that \( f(k) = 0 \). The real number zeros are the \( x \)-intercepts of the graph of the function. The remainder theorem gives a quick way to decide if a number \( k \) is a zero of the polynomial function defined by \( f(x) \). Use synthetic division to find \( f(k) \); if the remainder is 0, then \( f(k) = 0 \) and \( k \) is a zero of \( f(x) \). A zero of \( f(x) \) is called a root or solution of the equation \( f(x) = 0 \).
Example 3

DECIDING WHETHER A NUMBER IS A ZERO

Decide whether the given number $k$ is a zero of $f(x)$.

a. $f(x) = x^3 - 4x^2 + 9x - 6; \quad k = 1$

Solution

Proposed zero

\[
\begin{array}{cccc}
1 & -4 & 9 & -6 \\
1 & -3 & 6 & \\
1 & -3 & 6 & 0 \\
\end{array}
\]

Since the remainder is 0, $f(1) = 0$, and 1 is a zero of the polynomial function defined by $f(x) = x^3 - 4x^2 + 9x - 6$. An x-intercept of the graph $f(x)$ is 1, so the graph includes the point (1, 0).
### Example 3

DECIDING WHETHER A NUMBER IS A ZERO

Decide whether the given number $k$ is a zero of $f(x)$.

b. $f(x) = x^4 + x^2 - 3x + 1$; $k = -4$

#### Solution

Remember to use 0 as coefficient for the missing $x^3$-term in the synthetic division.

<table>
<thead>
<tr>
<th>Proposed zero</th>
<th>$-4$</th>
<th>1</th>
<th>0</th>
<th>1</th>
<th>-3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>-4</td>
<td>16</td>
<td>-68</td>
<td>284</td>
<td></td>
</tr>
<tr>
<td>Remainder</td>
<td></td>
<td>1</td>
<td>-4</td>
<td>17</td>
<td>-71</td>
<td>285</td>
</tr>
</tbody>
</table>

The remainder is not 0, so $-4$ is not a zero of $f(x) = x^4 + x^2 - 3x + 1$. In fact, $f(-4) = 285$, indicating that $(-4, 285)$ is on the graph of $f(x)$. 

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