4.1 Inverse Functions

Inverse Operations
One-to-One Functions
Inverse Functions
Equations of Inverses

Inverse Operations

Addition and subtraction are *inverse operations:* starting with a number *x,* adding 5, and subtracting 5 gives *x* back as the result. Similarly, some functions are *inverses* of each other. For example, the functions defined by

$$f(x) = 8x \quad \text{and} \quad g(x) = \frac{1}{8}x$$

are inverses of each other with respect to function composition.

Inverse Operations

This means that if a value of *x* such as $x = 12$ is chosen, then

$$f(12) = 8 \cdot 12 = 96.$$

Calculating $g(96)$ gives

$$g(96) = \frac{1}{8} \cdot 96 = 12.$$

Inverse Operations

Thus $g(f(12)) = 12$. Also, $f(g(12)) = 12$. For these functions f *and* g, it can be shown that

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x$$

for any value of *x.*

One-to-One Functions

Suppose we define the function

$$F = \{(-2,2), (-1,1), (0,0), (1,3), (2,5)\}.$$

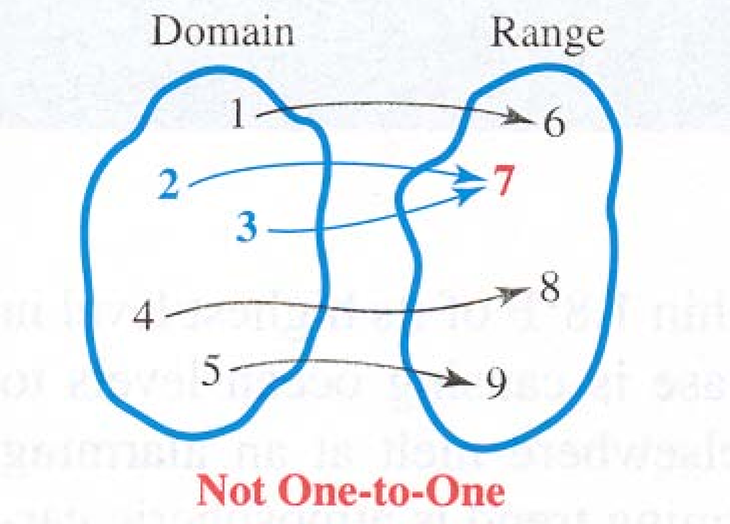
We can form another set of ordered pairs from *F* by interchanging the *x*- and *y*-values of each pair in *F*. We call this set *G*, so

$$G = \{(2,-2), (1,-1), (0,0), (3,1), (5,2)\}.$$

One-to-One Functions

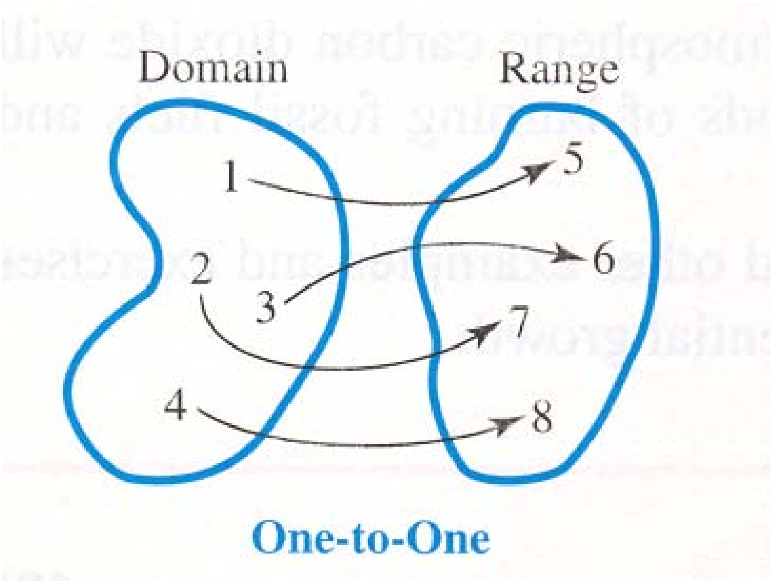
To show that these two sets are related, *G* is called the *inverse* of *F*. For a function f to have an inverse, f must be a *one-to-one function*. ***In a one-to-one function, each x-value corresponds to only one y-value, and each y-value corresponds to only one x-value.***

One-to-One Functions



This function is not one-to-one because the *y*-value 7 corresponds to two *x*-values, 2 and 3. That is, the ordered pairs (2, 7) and (3, 7) both belong to the function.

One-to-One Functions



This function one-to-one.

One-to-One Function

A function f is a **one-to-one function** if, for elements *a* and *b* in the domain of f,

$$a \neq b \quad \text{implies} \quad f(a) \neq f(b).$$

One-to-One Functions

Using the concept of the *contrapositive* from the study of logic, the last line in the preceding box is equivalent to

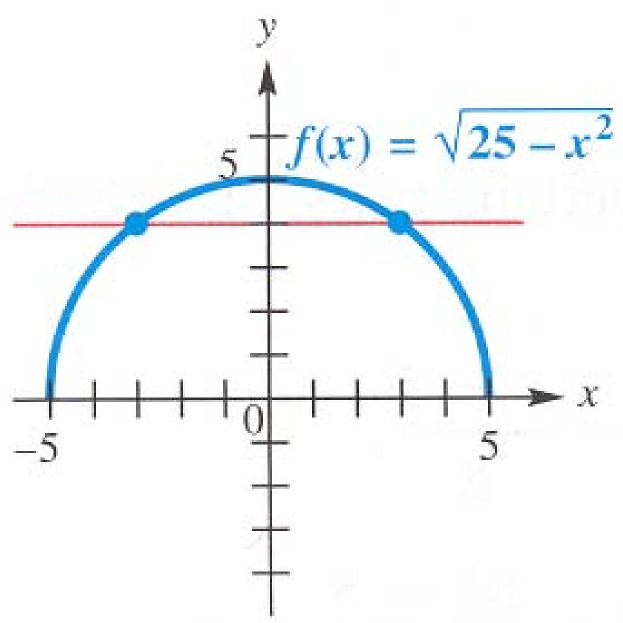
$$f(a) = f(b) \quad \text{implies} \quad a = b.$$

We use this statement to decide whether a function f is one-to-one in the next example.

Horizontal Line Test

As shown in Example 1(b), a way to show that a function is *not* one-to-one is to produce a pair of different numbers that lead to the same function value.

There is also a useful graphical test, the **horizontal line test,** that tells whether or not a function is one-to-one.



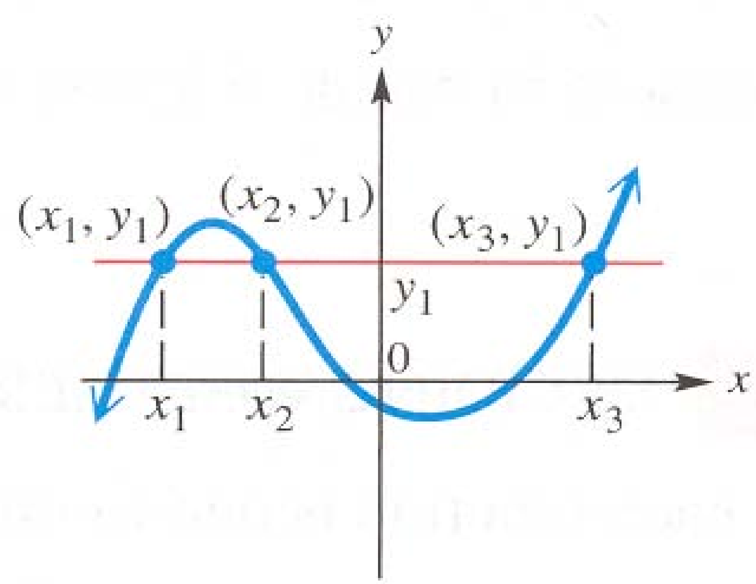
Horizontal Line Test

If any horizontal line intersects the graph of a function in no more than one point, then the function is one-to-one.

Example 2 USING THE HORIZONTAL LINE TEST

Determine whether each graph is the graph of a one-to-one function.

a.



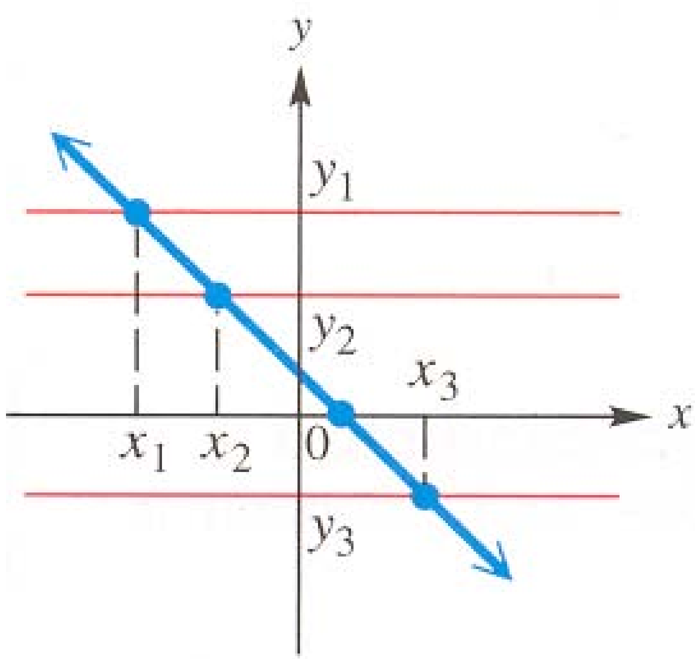
Solution

Each point where the horizontal line intersects the graph has the same value of *y* but a different value of *x*. Since more than one (here three) different values of *x* lead to the same value of *y*, the function is not one-to-one.

Example 2 **USING THE HORIZONTAL LINE TEST**

Determine whether each graph is the graph of a one-to-one function.

b.

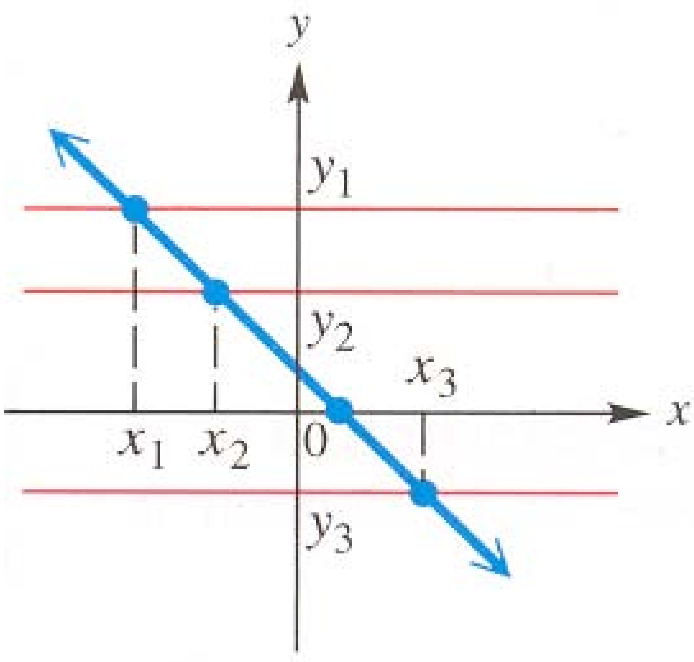


Solution

Since every horizontal line will intersect the graph at exactly one point, this function is one-to-one.

One-to-One Functions

Notice that the function graphed in Example 2(b) decreases on its entire domain. ***In general, a function that is either increasing or decreasing on its entire domain, such as $f(x) = -x$, $f(x) = x^3$, and $g(x) = \sqrt{x}$, must be one-to-one.***



Inverse Functions

Certain pairs of one-to-one functions "undo" one another. For example, if

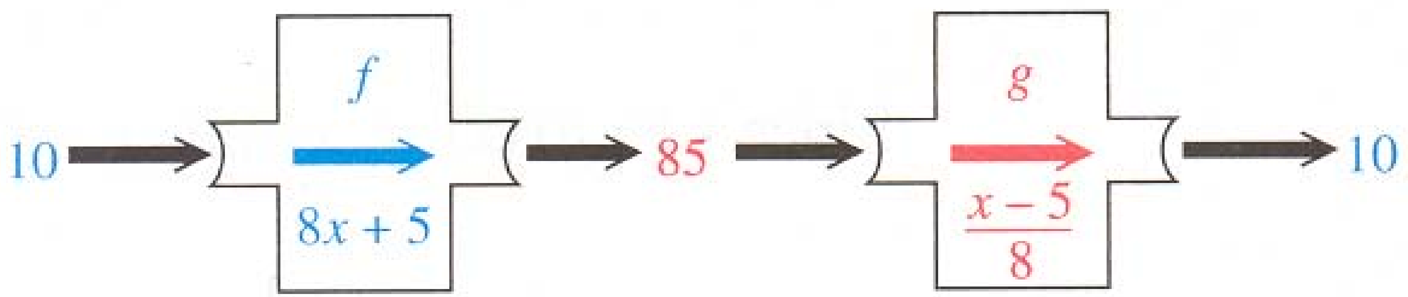
$$f(x) = 8x + 5 \quad \text{and} \quad g(x) = \frac{x-5}{8},$$

then

$$f(10) = 8 \cdot 10 + 5 = 85 \quad \text{and} \quad g(85) = \frac{85-5}{8} = 10.$$

Inverse Functions

Starting with 10, we "applied" function f and then "applied" function g to the result, which returned the number 10.



Inverse Functions

As further examples, check that

$$f(3) = 29 \quad \text{and} \quad g(29) = 3,$$

$$f(-5) = -35 \quad \text{and} \quad g(-35) = -5,$$

$$g(2) = -\frac{3}{8} \quad \text{and} \quad g\left(-\frac{3}{8}\right) = 2,$$

Inverse Functions

In particular, for this pair of functions,

$$f(g(2)) = 2 \quad \text{and} \quad g(f(2)) = 2.$$

In fact, for any value of *x*,

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x,$$

or $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$.

Because of this property, g is called the *inverse* of f.

Inverse Function

Let f be a one-to-one function. Then g is the **inverse function** of f if

$(f \circ g)(x) = x$ for every x in the domain of g,

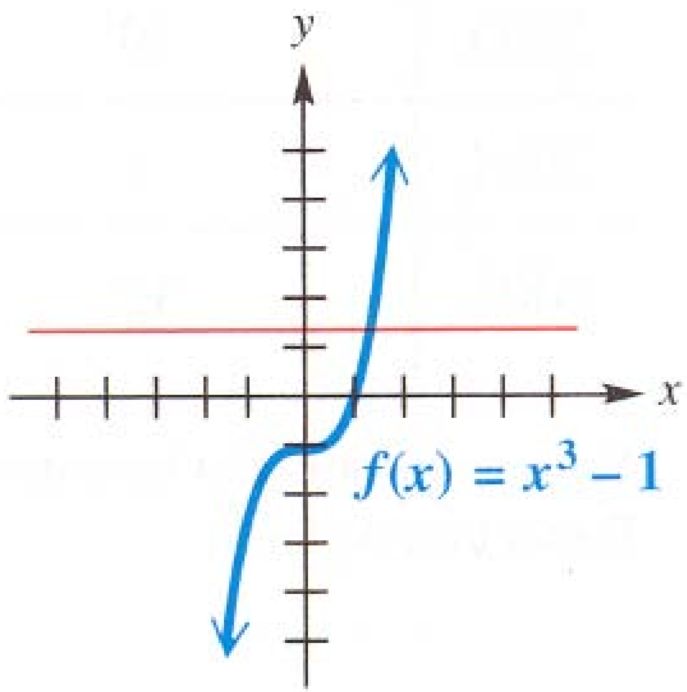
and

$(g \circ f)(x) = x$ for every x in the domain of f.

Example 3 **DECIDING WHETHER TWO FUNCTIONS ARE INVERSES**

Let functions f and g be defined by $f(x) = x^3 - 1$ and $g(x) = \sqrt[3]{x+1}$, respectively.
Is g the inverse function of f?

Solution The horizontal line test applied to the graph indicates that f is one-to-one, so the function does have an inverse. Since it is one-to-one, we now find $(f \circ g)(x)$ and $(g \circ f)(x)$.



Example 3

DECIDING WHETHER TWO FUNCTIONS ARE INVERSES

Let functions f and g be defined by $f(x) = x^3 - 1$ and $g(x) = \sqrt[3]{x+1}$, respectively.
Is g the inverse function of f?

Solution

$$(f \circ g)(x) = f(g(x)) = \left(\sqrt[3]{x+1}\right)^3 - 1$$

$$= x + 1 - 1$$

$$= x$$

Example 3 DECIDING WHETHER TWO FUNCTIONS ARE INVERSES

Let functions f and g be defined by $f(x) = x^3 - 1$ and $g(x) = \sqrt[3]{x+1}$, respectively.
Is g the inverse function of f?

Solution

$$f(x) = x^3 - 1; \quad g(x) = \sqrt[3]{x+1}$$

$$(g \circ f)(x) = g(f(x)) = \sqrt[3]{(x^3 - 1) + 1}$$

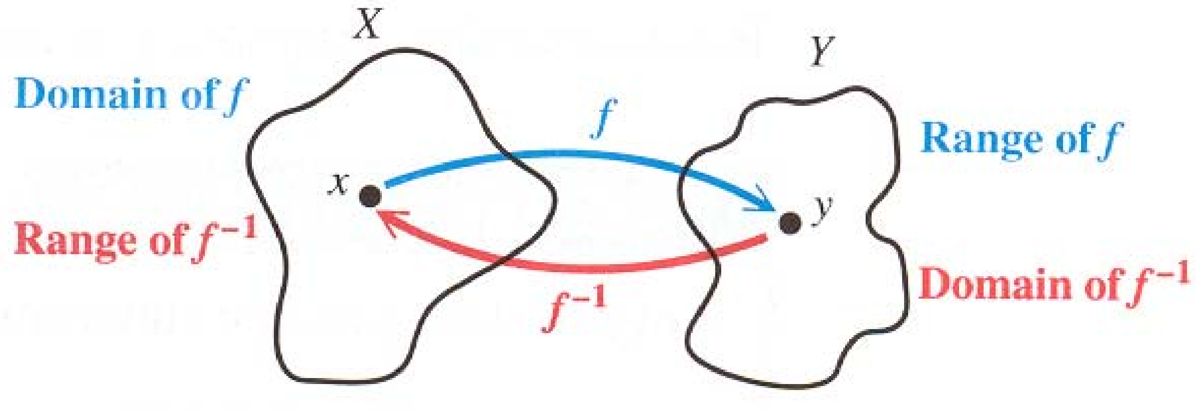
$$= \sqrt[3]{x^3}$$

$$= x$$

Since $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$, function g is the inverse of function f.

Inverse Function

By the definition of inverse function, the domain of f is the range of f^{-1}, and the range of f is the domain of f^{-1} .



Example 4 **FINDING THE INVERSES OF ONE-TO-ONE FUNCTIONS**

Find the inverse of each function that is one-to-one.

a. $F = \{(-2,1), (-1,0), (0,1), (1,2), (2,2)\}$

Solution Each *x*-value in *F* corresponds to just one *y*-value. However, the *y*-value 2 corresponds to two *x*-values, 1 and 2. Also, the *y*-value 1 corresponds to both -2 and 0. Because some *y*-values correspond to more than one *x*-value, *F* is not one-to-one and does not have an inverse.

Example 4

FINDING THE INVERSES OF ONE-TO-ONE FUNCTIONS

Find the inverse of each function that is one-to-one.

b. $G = \{(3,1), (0,2), (2,3), (4,0)\}$

Solution Every *x*-value in *G* corresponds to only one *y*-value, and every *y*-value corresponds to only one *x*-value, so *G* is a one-to-one function. The inverse function is found by interchanging the *x*- and *y*-values in each ordered pair.

$$G^{-1} = \{(1,3), (2,0), (3,2), (0,4)\}$$

Notice how the domain and range of *G* becomes the range and domain, respectively, of G^{-1}.

Equations of Inverses

By definition, the inverse of a one-to-one function is found by interchanging the x- and y-values of each of its ordered pairs. The equation of the inverse of a function defined by $y = f(x)$ is found in the same way.

Finding the Equation of the Inverse of $y = f(x)$

For a one-to-one function f defined by an equation $y = f(x)$, find the defining equation of the inverse as follows. (You may need to replace $f(x)$ with y first.)

Step 1 Interchange x and y.

Step 2 Solve for y.

Step 3 Replace y with $f^{-1}(x)$.

Example 5 **FINDING EQUATIONS OF INVERSES**

Decide whether each equation defines a one-to-one function. If so, find the equation of the inverse.

a. $f(x) = 2x + 5$

Solution The graph of $y = 2x + 5$ is a non-horizontal line, so by the horizontal line test, f is a one-to-one function. To find the equation of the inverse, follow the steps in the preceding box, first replacing $f(x)$ with y.

Example 5

FINDING EQUATIONS OF INVERSES

Solution

$$y = 2x + 5$$ $y = f(x)$

$$x = 2y + 5$$ Interchange x and y.

$$2y = x - 5$$ Solve for y.

$$y = \frac{x - 5}{2}$$

$$f^{-1}(x) = \frac{1}{2}x - \frac{5}{2}$$ Replace y with $f^{-1}(x)$.

Example 5 **FINDING EQUATIONS OF INVERSES**

Solution

In the function, the value of *y* is found by starting with a value of *x*, multiplying by 2, and adding 5. The first form for the equation of the inverse has us *subtract* 5 and then *divide* by 2. This shows how an inverse is used to "undo" what a function does to the variable *x*.

Example 5 **FINDING EQUATIONS OF INVERSES**

Decide whether each equation defines a one-to-one function. If so, find the equation of the inverse.

b. $y = x^2 + 2$

Solution The equation has a parabola opening up as its graph, so some horizontal lines will intersect the graph at two points. For example, both $x = 3$ and $x = -3$ correspond to $y = 11$. Because of the x^2-term, there are many pairs of x-values that correspond to the same y-value. This means that the function defined by $y = x^2 + 2$ is not one-to-one and does not have an inverse.

Example 5

FINDING EQUATIONS OF INVERSES

Decide whether each equation defines a one-to-one function. If so, find the equation of the inverse.

b. $y = x^2 + 2$

Solution If we did not notice this, then following the steps for finding the equation of an inverse leads to

$$y = x^2 + 2$$

$$x = y^2 + 2$$ Interchange *x* and *y*.

$$x - 2 = y^2$$ Solve for *y*.

$$\pm\sqrt{x-2} = y.$$ Square root property

Remember *both* roots.

Example 5 FINDING EQUATIONS OF INVERSES

Decide whether each equation defines a one-to-one function. If so, find the equation of the inverse.

b. $y = x^2 + 2$

Solution If we did not notice this, then following the steps for finding the equation of an inverse leads to

$$y = x^2 + 2$$
$$x = y^2 + 2$$
$$x - 2 = y^2$$
$$\pm\sqrt{x-2} = y.$$

Remember *both* roots.

The last step shows that there are two *y*-values for each choice of *x* greater than 2, so the given function is not one-to-one and cannot have an inverse.

Example 5 FINDING EQUATIONS OF INVERSES

Decide whether each equation defines a one-to-one function. If so, find the equation of the inverse.

c. $f(x) = (x-2)^3$

Solution Refer to **Sections 2.6 and 2.7** to see that translations of the graph of the cubing function are one-to-one.

$$f(x) = (x-2)^3$$

$$y = (x-2)^3 \quad \text{Replace } f(x) \text{ with } y.$$

$$x = (y-2)^3 \quad \text{Interchange } x \text{ and } y.$$

Example 5 **FINDING EQUATIONS OF INVERSES**

Solution

$$\sqrt[3]{x} = \sqrt[3]{(y-2)^3}$$

Take the cube root on each side.

$$\sqrt[3]{x} = y - 2$$

$$\sqrt[3]{x} + 2 = y$$

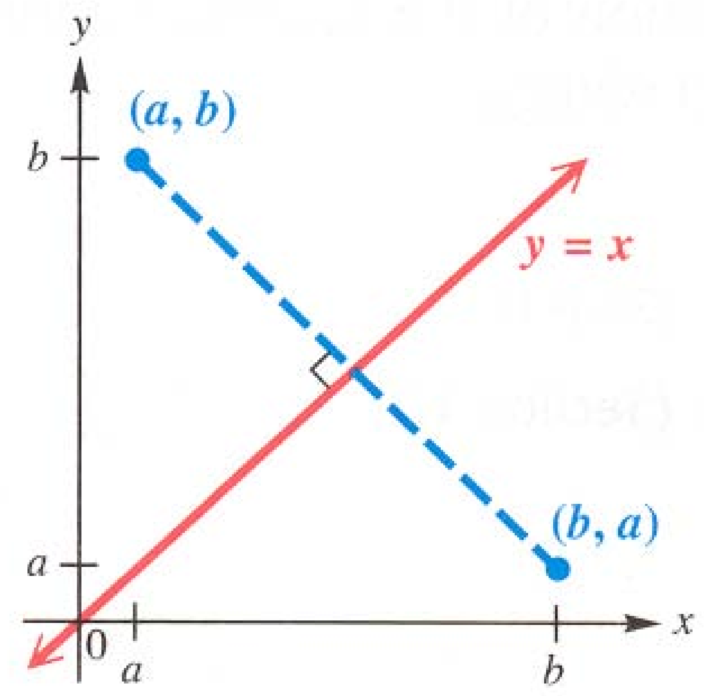
Solve for *y*.

$$f^{-1}(x) = \sqrt[3]{x} + 2$$

Replace *y* with $f^{-1}(x)$.

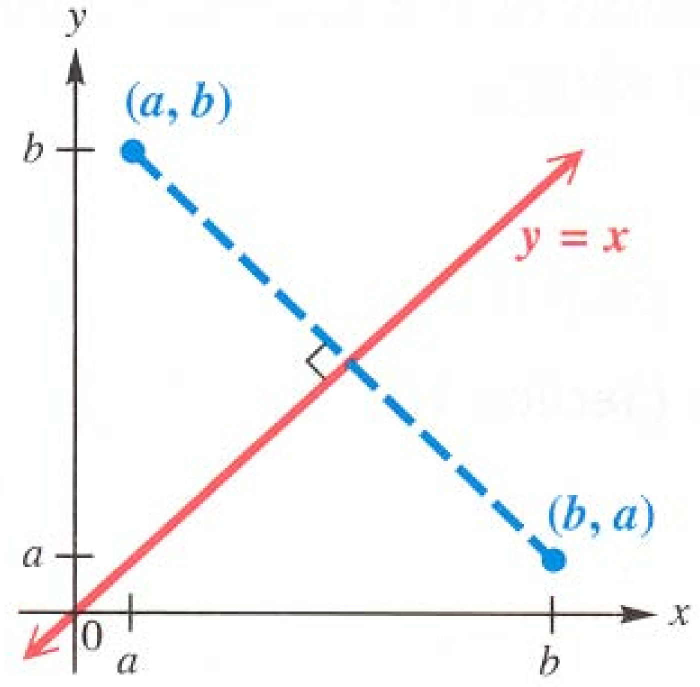
Inverse Function

One way to graph the inverse of a function f whose equation is known is to find some ordered pairs that are on the graph of f, interchange x and y to get ordered pairs that are on the graph of f^{-1}, plot those points, and sketch the graph of f^{-1} through the points.



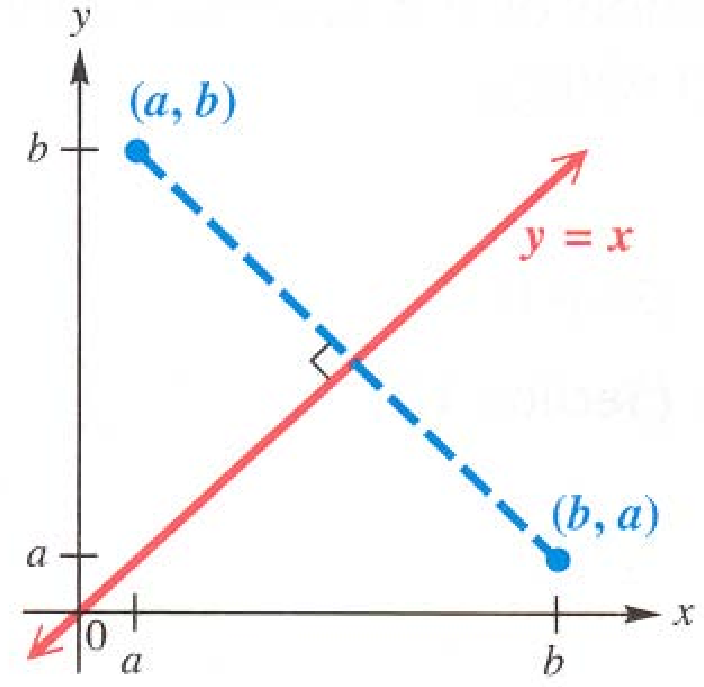
Inverse Function

A simpler way is to select points on the graph of f and use symmetry to find corresponding points on the graph of f^{-1}.



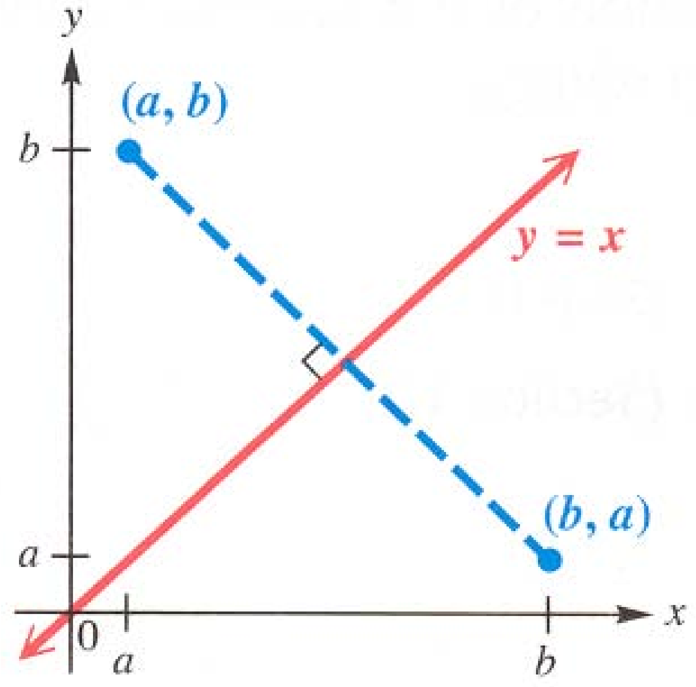
Inverse Function

For example, suppose the point (a, b) shown here is on the graph of a one-to-one function f.



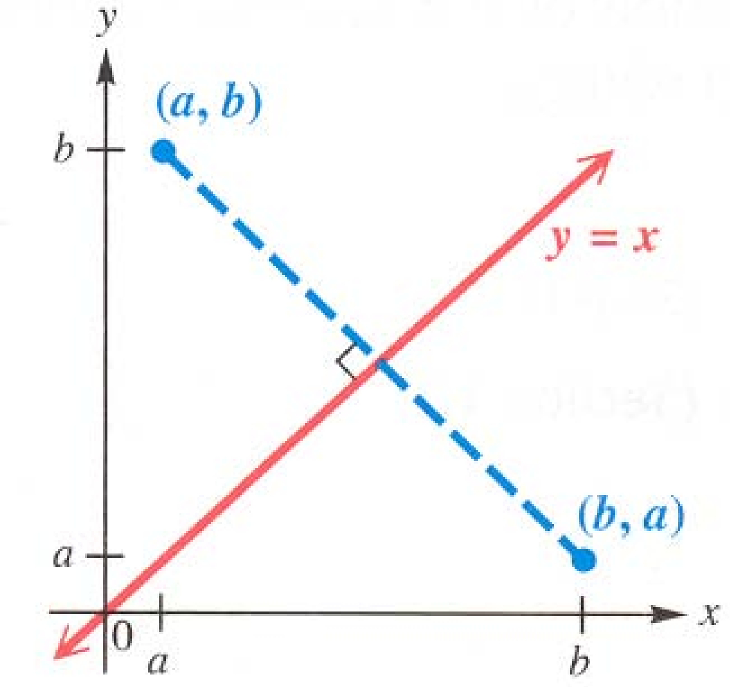
Inverse Function

Then the point (b, a) is on the graph of f^{-1}. The line segment connecting (a, b) and (b, a) is perpendicular to, and cut in half by, the line $y = x$. The points (a, b) and (b, a) are "mirror images" of each other with respect to $y = x$.



Inverse Function

Thus, we can find the graph of f^{-1} from the graph of f by locating the mirror image of each point in f with respect to the line y = x.



Example 6

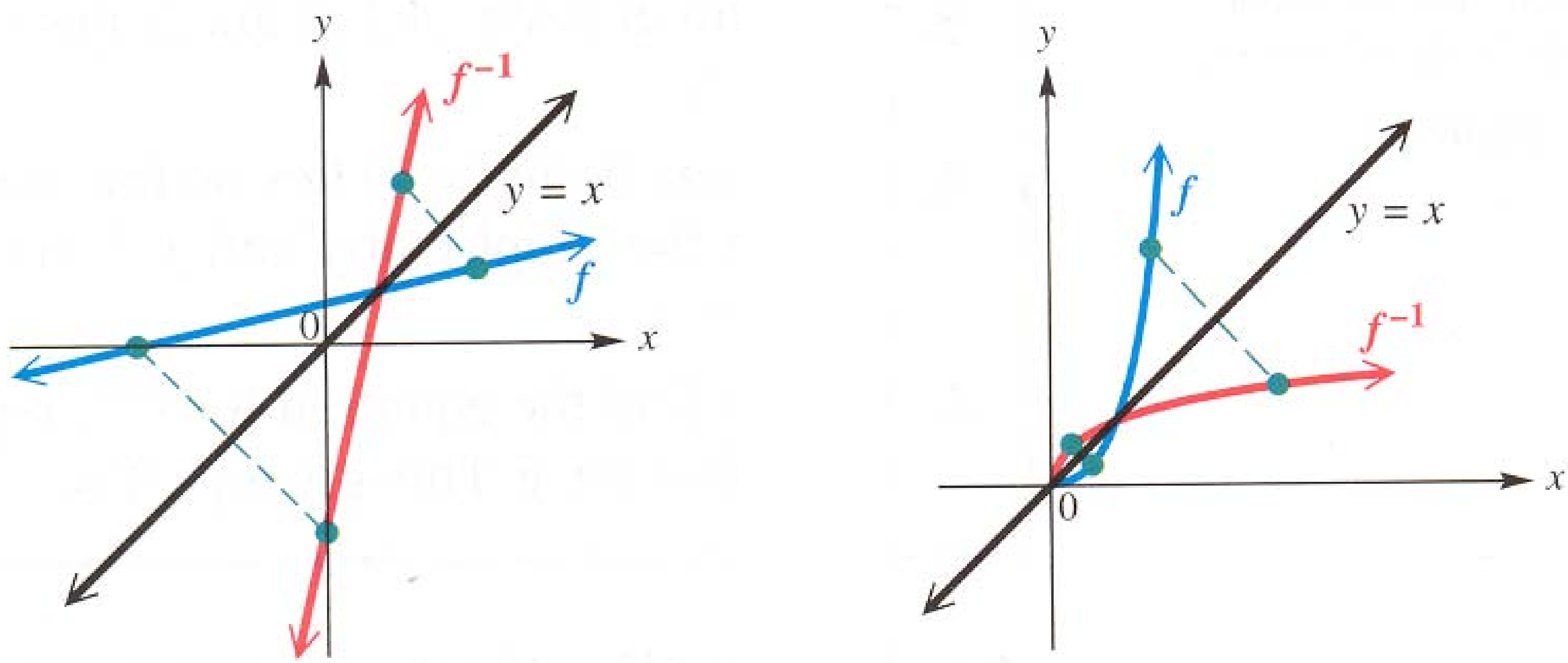
GRAPHING THE INVERSE

In each set of axes, the graph of a one-to-one function f is shown in blue. Graph f^{-1} in red.

Solution The graphs of two functions f *are* shown in blue. Their inverses are shown in red. In each case, the graph of f^{-1} is a reflection of the graph of f with respect to the line $y = x$.

GRAPHING THE INVERSE

Solution



Example 7 **FINDING THE INVERSE OF A FUNCTION WITH A RESTRICTED DOMAIN**

Let $f(x)=\sqrt{x+5}$. Find $f^{-1}(x)$.

Solution First, notice that the domain of f is restricted to the interval $[-5, \infty)$. Function f is one-to-one because it is increasing on its entire domain and, thus, has an inverse function. Now we find the equation of the inverse.

Example 7 **FINDING THE INVERSE OF A FUNCTION WITH A RESTRICTED DOMAIN**

Solution

$$f(x) = \sqrt{x+5}, \qquad x \geq -5$$

$$y = \sqrt{x+5}, \qquad x \geq -5 \qquad y = f(x)$$

$$x = \sqrt{y+5}, \qquad y \geq -5 \qquad \text{Interchange } x \text{ and } y.$$

$$x^2 = \left(\sqrt{y+5}\right)^2 \qquad \text{Square both sides.}$$

$$y = x^2 - 5 \qquad \text{Solve for } y.$$

Example 7 **FINDING THE INVERSE OF A FUNCTION WITH A RESTRICTED DOMAIN**

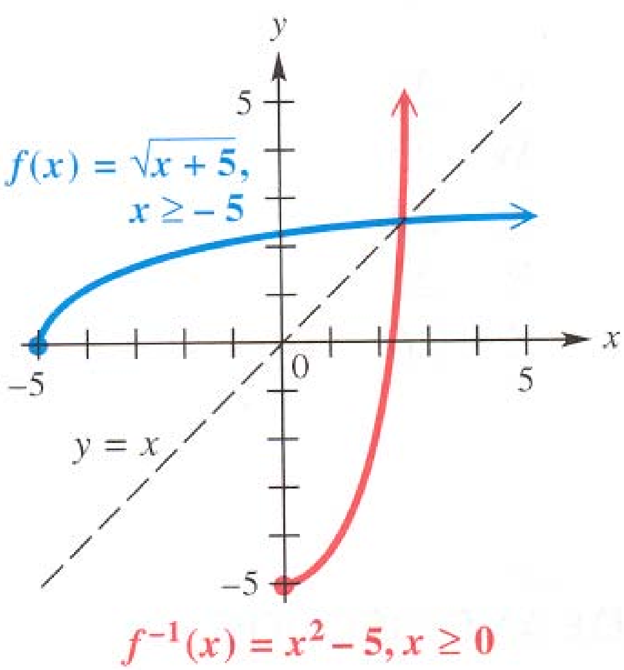
Solution However, we cannot define f^{-1} as $x^2 – 5$. The domain of f is $[–5, \infty)$, and its range is $[0, \infty)$. The range of f is the domain of f^{-1}, so f^{-1} must be defined as

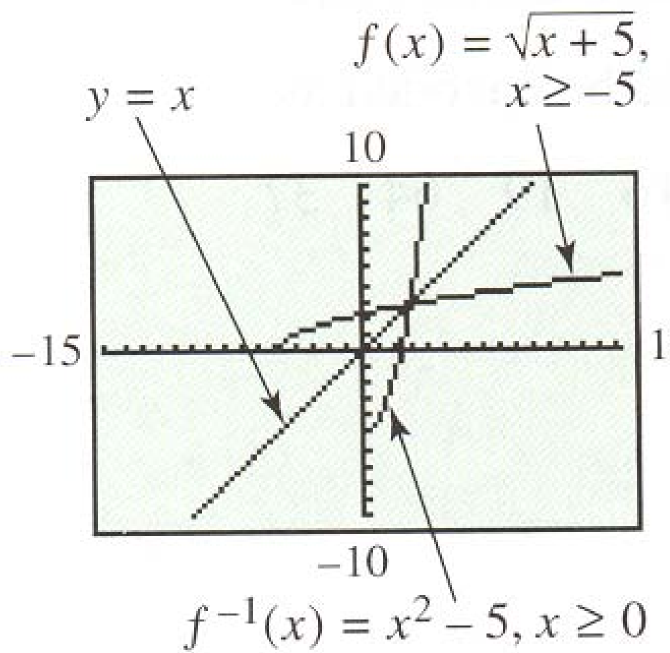
$$f^{-1}(x) = x^2 - 5, \qquad x \geq 0.$$

Example 7

FINDING THE INVERSE OF A FUNCTION WITH A RESTRICTED DOMAIN

As a check, the range of f^{-1}, $[-5, \infty)$, is the domain of f. Graphs of f and f^{-1} are shown. The line $y = x$ is included on the graphs to show that the graphs are mirror images with respect to this line.





Important Facts About Inverses

1. If f is one-to-one, then f^{-1} exists.

2. The domain of f is the range of f^{-1},and the range of f is the domain of f^{-1}.

3. If the point (a, b) lies on the graph of f, then (b, a) lies on the graph of f^{-1}, so the graphs of f and f^{-1} are reflections of each other across the line $y = x$.

4. To find the equation for f^{-1}, replace $f(x)$ with y, interchange x and y, and solve for y.
This gives $f^{-1}(x)$.