Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466
Vol. 10, Issue 2 (December 2015), pp. 835 – 851

Applications and Applied Mathematics:
An International Journal
(AAM)

Accurate spectral solutions of first- and second-order initial value problems by the ultraspherical wavelets-Gauss collocation method

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Received: June 17, 2014; Accepted: February 2, 2015

Abstract

In this paper, we present an ultraspherical wavelets-Gauss collocation method for obtaining direct solutions of first- and second-order nonlinear differential equations subject to homogenous and nonhomogeneous initial conditions. The properties of ultraspherical wavelets are used to reduce the differential equations with their initial conditions to systems of algebraic equations, which then must be solved by using suitable numerical solvers. The function approximations are spectral and have been chosen in such a way that make them easy to calculate the expansion coefficients of the thought-for solutions. Uniqueness and convergence of the proposed function approximation is discussed. Four illustrative numerical examples are considered and these results are comparing favorably with the analytic solutions and proving more accurate than those discussed by some other existing techniques in the literature.

Keywords: Ultraspherical polynomials, wavelets, first- and second-order initial value problems, Gaussian quadrature, Bratu Equation

MSC 2010 No.: 65M70, 65N35, 35C10, 42C10

1. Introduction

In the fields of applied mathematics and physics, first- and second-order ordinary initial value problems (IVPs) are often encountered. An effective method is required to analyze the mathematical model which provides solutions conforming to physical reality, i.e., the real world of physics. Therefore, we should have the ability for solving nonlinear first- and second-order ordinary differential equations (ODEs).

Many practical problems arising in numerous branches of science and engineering require solving high even-order differential equations and, in particular, second-order differential equations.

For example, after spacial discretization, a large class of nonlinear wave equations, including Klein-Gordon and Sine-Gordon equations, are reduced to certain systems of second-order ODEs. Moreover, second-order ODEs appear in celestial mechanics, circuit theory, control theory, chemical kinetics, and biology. In the sequence of papers (Abd-Elhameed (2009); Abd-Elhameed et al. (2014); Abd-Elhameed et al. (2013); Doha and Abd-Elhameed (2002); Doha et al. (2013a)), the authors dealt with the high odd- and high even-order differential equations by the Galerkin and Pertov-Galerkin methods. They constructed suitable basis functions which satisfy the underlying boundary conditions of the given differential equation. The suggested algorithms in these articles are suitable for handling the high-order linear differential equations, while the authors in Doha et al. (2013b) used another technique based on employing operational matrix of derivatives for handling the so-called Lane-Emden equation.

Among the most important second-order differential equations is the one-dimensional Bratu problem, which has a long history. Due to its great importance in various applications, a substantial amount of research work has been done for studying this problem (see, for example, Boyd (1986); Boyd (2003); Buckmire (2004)). In Boyd (1968) and Boyd (2003) Boyd employed Chebyshev polynomial expansions and the Gegenbauer polynomials as base functions. Syam and Hamdan (2006) presented the Laplace Adomian decomposition method for solving such problem. Also, Aksoy and Pakdemirli (2010) suggested new perturbation iteration solutions for Bratu-type equations, while Wazwaz (2005) presented the Adomian Decomposition method for solving this problem. Also, Venkatesh et al. (2012) used Legendre wavelets for solving IVPs of Bratu-type and discussed the analysis concerned with the convergence and error analysis.

The subject of wavelets has recently drawn a great deal of attention from mathematical scientists in various disciplines. It is creating a common link between mathematicians, physicists, and electrical engineers. Wavelet theory is a relatively new and an emerging area in mathematical research. It has been applied to a wide range of engineering disciplines; in particular, wavelets are very successfully used in signal analysis for wave form representation and segmentations, time frequency analysis and fast algorithms for easy implementation. Wavelets permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms (see Constantinides (1987); Newland (1993)).

Recently, wavelets represent a significant approach in different fields of science and engineering. In literature, many authors have used various wavelets (Beylkin et al. (1991); Chen and Hsiao

(1997); Razzaghi and Yousefi (2000)) for analyzing problems of greater computational complexity, and they have pointed out that wavelets represent powerful tools to explore a new direction in solving differential equations. For example, Legendre wavelets are used to obtain an approximate solution of Lane-Emden type equations in Yousefi (2006).

The problem of approximating definite integrals is of central importance in many applications of mathematics. In practice, a mere approximation of an integral very often will not be satisfactory unless it is accompanied by an estimate of the error. For most quadrature formulas of practical interest, error bounds are available in the literature which use, for example, norms of higher-order derivatives or bounds for the integrand in the complex plane. However, in many practical situations such information about the integrand is not available. In particular, automatic quadrature routines are designed such that the user only has to insert the limits of integration, a routine for computing the integrand, a tolerance for the error, and an upper bound for the number of function evaluations (see Krommer and Ueberhuber (1994); Krommer and Ueberhuber (1998); Davis and Rabinowitz (1984)). Most quadrature methods used in modern numerical software packages like those of NAG (1998) and IMSL (1998) are based on Gaussian quadrature formulas. Furthermore, both numerical experience and theoretical results show the superiority of Gaussian formulas over many other quadrature formulas in many function classes (see in particular Brass et al. (1997) and the literature cited therein).

Collocation methods (see, for instance, Guo and Yan (2009); Doha et al. (2013b)) have become increasingly popular for solving ordinary and partial differential equations. In particular, they are very useful in providing highly accurate solutions to nonlinear differential equations.

In this paper, we aim to introduce a collocation wavelets algorithm for handling first- and second-order IVPs based on using ultraspherical wavelets collocation algorithm. This algorithm is based on making use of the ultraspherical wavelets together with the Gaussian integration formula. In this article, we discuss the convergence analysis and uniqueness of the proposed function approximations. Numerical examples and some applications are also considered.

The paper is organized as follows. In Section 2, we enumerate some properties of ultraspherical polynomials and their shifted ones. Section 3 describes the properties of ultraspherical wavelets. Sections 4 and 5 are devoted for solving first- and second-order IVPs based on using ultraspherical wavelets collocation method. Uniqueness and convergence theorems for second-order IVPs are discussed in detail in Section 6. In Section 7, we give some numerical examples to demonstrate the validity and applicability of our proposed algorithm. Some concluding remarks are given in Section 8.

2. Some properties of ultraspherical polynomials and their shifted ones

In this section we give some useful properties of ultraspherical polynomials and their shifted forms.

2.1. Some properties of ultraspherical polynomials

The ultraspherical polynomials (a special type of Jacobi polynomials) associated with the real parameter ($\lambda > -\frac{1}{2}$), are a sequence of orthogonal polynomials defined on $(-1, 1)$, with respect to the weight function $w(x) = (1 - x^2)^{\lambda-\frac{1}{2}}$. The orthogonality relation is given by

$$\int_{-1}^{1} (1 - x^2)^{\lambda-\frac{1}{2}}\ C_m^{(\lambda)}(x) C_n^{(\lambda)}(x)\ dx = \begin{cases} \dfrac{\sqrt{\pi}\ n!\ \Gamma(\lambda + \frac{1}{2})}{(2\lambda)_n\ (n + \lambda)\ \Gamma(\lambda)}, & m = n, \\ 0, & m \neq n, \end{cases} \tag{1}$$

where $(2\lambda)_n$ is the Pochhammer notation, i.e., $(2\lambda)_n = \dfrac{\Gamma(n + 2\lambda)}{\Gamma(2\lambda)}$.

It should be noted here that the ultraspherical polynomials $C_n^{(\lambda)}(x)$ are normalized such that $C_n^{(\lambda)}(1) = 1$. This normalization is characterized by an advantage that the polynomials $C_n^{(0)}(x)$ are identical with the Chebyshev polynomials of the first kind $T_n(x)$, $C_n^{(\frac{1}{2})}(x)$ are the Legendre polynomials $L_n(x)$, and $C_n^{(1)}(x)$ is equal to $(1/(n + 1))U_n(x)$, where $U_n(x)$ are the Chebyshev polynomials of the second kind.

The polynomials $C_n^{(\lambda)}(x)$ may be generated by using the recurrence relation

$$(n + 2\,\lambda)\, C_{n+1}^{(\lambda)}(x) = 2(n + \lambda)\ x\, C_n^{(\lambda)}(x) - n\, C_{n-1}^{(\lambda)}(x), \quad n = 1, 2, 3, \ldots,$$

with the initial values: $C_0^{(\lambda)}(x) = 1$ and $C_1^{(\lambda)}(x) = x$.

For more properties and relations of ultraspherical polynomials, see, for instance, Andrews et al. (1999).

The shifted ultraspherical polynomials $\tilde{C}_n^{(\lambda)}(x) = C_n^{(\lambda)}(2x - 1)$ are a sequence of orthogonal polynomials defined on (0,1), with respect to the weight function $\tilde{w}(x) = (x - x^2)^{\lambda-\frac{1}{2}}$, i.e.,

$$\int_{0}^{1} (x - x^2)^{\lambda-\frac{1}{2}}\ \tilde{C}_m^{(\lambda)}(x)\ \tilde{C}_n^{(\lambda)}(x)\ dx = \begin{cases} \dfrac{\pi\, 2^{1-4\lambda}\, \Gamma(n + 2\lambda)}{n!\, (n + \lambda)\ (\Gamma(\lambda))^2}, & m = n, \\ 0, & m \neq n. \end{cases} \tag{2}$$

They also may generated by using the recurrence relation

$$(n + 2\lambda)\tilde{C}_{n+1}^{(\lambda)}(x) = 2(n + \lambda)\ (2x - 1)\, \tilde{C}_n^{(\lambda)}(x) - n\, \tilde{C}_{n-1}^{(\lambda)}(x), \quad n = 1, 2, 3, \ldots,$$

with the initial values: $\tilde{C}_0^{(\lambda)}(x) = 1$ and $\tilde{C}_1^{(\lambda)}(x) = 2x - 1$.

All relations and properties of ultraspherical polynomials can be easily transformed to give the corresponding relations and properties of the shifted ultraspherical polynomials.

2.2 Some properties of shifted ultraspherical polynomials

The shifted ultraspherical polynomials (a special type of shifted Jacobi polynomials) associated with the real parameter ($\lambda > -\frac{1}{2}$) are a sequence of orthogonal polynomials on the interval (0,1),

with respect to the weight function $w(x) = (x - x^2)^{\lambda - \frac{1}{2}}$, i.e.,

$$\int_0^1 (x - x^2)^{\lambda - \frac{1}{2}} \; C_m^{(\lambda)}(x) C_n^{(\lambda)}(x) \; dx = \begin{cases} \dfrac{\pi \, 2^{1-4\lambda} \, \Gamma(n + 2\lambda)}{n! \, (n + \lambda) \; (\Gamma(\lambda))^2}, & m = n, \\ 0, & m \neq n. \end{cases} \tag{3}$$

Here, we present some properties of ultraspherical polynomials (see, for instance, Andrews et al. (1999)). These polynomials are eigenfunctions of the following singular Sturm-Liouville equation

$$(x - x^2) \; D^2 \, \phi_k(x) - \left(\lambda + \frac{1}{2}\right) (2x - 1) \, D \, \phi_k(x) + k(k + 2\lambda) \; \phi_k(x) = 0,$$

and they may be generated with the aid of the recurrence relation

$$(k + 2\lambda) C_{k+1}^{(\lambda)}(x) = 2(k + \lambda) \; (2x - 1) \, C_k^{(\lambda)}(x) - k \, C_{k-1}^{(\lambda)}(x), \quad k = 1, 2, 3, \ldots,$$

starting from $C_0^{(\lambda)}(x) = 1$ and $C_1^{(\lambda)}(x) = 2\lambda \, (2x - 1)$, or obtained from the Rodrigues' formula

$$C_n^{(\lambda)}(x) = \frac{(-4)^n \, \Gamma(n + \lambda) \, \Gamma(n + 2\lambda)}{n! \, \Gamma(\lambda) \, \Gamma(2n + 2\lambda)} (x - x^2)^{\frac{1}{2} - \lambda} \, D^n [(x - x^2)^{n + \lambda - \frac{1}{2}}],$$

where $D \equiv \frac{d}{dx}$.

3. Ultraspherical wavelets

Wavelets constitute a family of functions constructed from dilation and translation of single function called the mother wavelet. If the dilation parameter a and the translation parameter b vary continuously, then the following family of continuous wavelets are obtained:

$$\psi_{a,b}(t) = |a|^{-1/2} \, \psi \left(\frac{t - b}{a} \right) \qquad a, b, \in \mathbb{R}, \quad a \neq 0. \tag{4}$$

We define the ultraspherical wavelets $\psi_{nm}^{(\lambda)}(t) = \psi(k, n, m, \lambda, t)$ as they have five arguments: k, n can be assumed to be any positive integer, m is the order for the ultraspherical polynomial, λ is the known ultraspherical parameter, and t is the normalized time. Explicitly, they are defined on the interval $[0, 1]$ as:

$$\psi_{nm}^{(\lambda)}(t) = \begin{cases} 2^{\frac{k}{2}} \, \xi_{m,\lambda} \, C_m^{(\lambda)} \left(2^k t - 2n + 1\right), & t \in [\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}], \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

where $0 \leq m \leq M - 1, \quad 1 \leq n \leq 2^{k-1}$, and

$$\xi_{m,\lambda} = 2^\lambda \, \Gamma(\lambda) \sqrt{\frac{m! \, (m + \lambda)}{2 \, \pi \Gamma(m + 2\lambda)}}. \tag{6}$$

Remark 1: It is worthy noting here that $\psi_{nm}^{(\frac{1}{2})}(t)$ is identical to Legendre wavelets (Razzaghi and Yousefi (2000); Yousefi (2006)), $\psi_{nm}^{(0)}(t)$ is identical to first kind Chebyshev wavelets (Babolian and Fattahzadeh (2007); Li (2010)) and $\psi_{nm}^{(1)}(t)$ is identical to second kind Chebyshev wavelets (Maleknejad et al. (2007)).

Now, assume a function $f(t)$ defined on $[0,1]$ and suppose that it may be expanded in terms of ultraspherical wavelets as

$$f(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{nm}\psi_{nm}^{(\lambda)}(t), \tag{7}$$

where

$$c_{nm}=\big(f(t),\psi_{nm}^{(\lambda)}(t)\big)_w=\int_0^1 (t-t^2)^{\lambda-\frac{1}{2}}\, f(t)\,\psi_{nm}^{(\lambda)}(t)\,dt.$$

Assume that $f(t)$ is approximated in terms of ultraspherical wavelets as

$$f(t)\approx\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi_{nm}^{(\lambda)}(t)=\mathbf{C}^T\,\Psi^{(\lambda)}(t), \tag{8}$$

where $\mathbf{C}$ and $\Psi^{(\lambda)}(t)$ are $2^{k-1}M\times 1$ matrices given by

$$\mathbf{C}=\left[c_{1,0},c_{1,1},\ldots,c_{1,M-1},\ldots,c_{2^{k-1},M-1},\ldots,c_{2^{k-1},1},\ldots,c_{2^{k-1},M-1}\right]^T, \tag{9}$$

$$\Psi^{(\lambda)}(t)=\left[\psi_{1,0}^{(\lambda)},\psi_{1,1}^{(\lambda)},\ldots,\psi_{1,M-1}^{(\lambda)},\psi_{2,0}^{(\lambda)},\ldots,\psi_{2,M-1}^{(\lambda)},\ldots\psi_{2^{k-1},0}^{(\lambda)},\ldots,\psi_{2^{k-1},M-1}^{(\lambda)}\right]^T. \tag{10}$$

4. Solution of first-order initial value problem

In this section, we present an ultraspherical Gaussian quadrature wavelets method (UGQWM) for solving the following first-order IVP:

$$y'(x)=f\big(x,y(x)\big),\quad y(0)=a,\qquad x\in[0,1]. \tag{11}$$

If we define the integral operator $\mathcal{L}(.)=\displaystyle\int_0^x(.)\,dt$, then applying $\mathcal{L}$ to both sides of Equation (11) yields,

$$y(x)=a+\int_0^x f\big(t,y(t)\big)\,dt.$$

Let $y_{k,M}(x)=\displaystyle\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi_{nm}^{(\lambda)}(x)=\mathbf{C}^T\,\Psi^{(\lambda)}(x)$ be an approximation of $y(x)$. Then we have

$$\mathbf{C}^T\,\Psi^{(\lambda)}(x)\approx a+\int_0^x f\big(t,\mathbf{C}^T\,\Psi^{(\lambda)}(t)\big)\,dt. \tag{12}$$

Now, we collocate Equation (12) at the points x_i, to obtain

$$\mathbf{C}^T\,\Psi^{(\lambda)}(x_i)=a+\int_0^{x_i} f\big(t,\mathbf{C}^T\,\Psi^{(\lambda)}(t)\big)\,dt,\qquad i=1,2,3,\ldots,2^{k-1}M, \tag{13}$$

where x_i are the distinct $\big(2^{k-1}M\big)$ roots of $C_{2^{k-1}M}^{(\lambda)}(2x-1)$. For the sake of applying Gaussian integration formula on Equation (13), we make use of the transformation: $t=\frac{x_i}{2}(z+1)$, which

transforms the intervals $[0, x_i]$ into the interval $[-1, 1]$, and therefore Equation (13) may be written in the form

$$\mathbf{C}^T \Psi^{(\lambda)}(x_i) = a + \frac{x_i}{2} \int_{-1}^{1} f\left(\frac{x_i}{2}(z+1), \mathbf{C}^T \Psi^{(\lambda)}\big(\frac{x_i}{2}(z+1)\big)\right) dz, \qquad i = 1, 2, 3, \ldots, 2^{k-1}M. \tag{14}$$

If we apply the Gaussian integration formula on the right hand side of (14), then we get

$$\mathbf{C}^T \Psi^{(\lambda)}(x_i) \approx a + \frac{x_i}{2} \sum_{j=0}^{s} \omega_j \, f\left(\frac{x_i}{2}(z_j+1), \mathbf{C}^T \Psi^{(\lambda)}\big(\frac{x_i}{2}(z_j+1)\big)\right), \qquad i = 1, 2, 3, \ldots, 2^{k-1}M, \tag{15}$$

where z_j's are the $(s+1)$ zeros of the polynomial $C_{s+1}^{(\lambda)}(x)$, and the weights ω_j (see Ralston and Rabinowitz (1978)) can be computed by the formula

$$\omega_j = \int_{-1}^{1} \prod_{\substack{j=0 \\ j \neq i}}^{s} \left(\frac{z - z_j}{z_i - z_j}\right) dz. \tag{16}$$

The idea behind the approximation in Equation (15) is the exactness of the Gaussian integration formula for polynomials of degree not exceeding $(2s+1)$.

5. Solution of second-order initial value problem

In this section, we are interested in solving the following second-order IVP subject to homogenous initial conditions

$$y''(x) + \alpha\, f(x)\, y'(x) + \beta\, g\big(x, y(x)\big) = 0, \quad y(0) = y'(0) = 0, \qquad x \in [0, 1], \tag{17}$$

where α, β are real constants.
Applying $\mathcal{L}$ on both sides of Equation (17) yields

$$y'(x) + \alpha \int_0^x f(t)\, y'(t)\, dt + \beta \int_0^x g\big(t, y(t)\big)\, dt = 0. \tag{18}$$

Equation (18) gives after integration by parts

$$y'(x) = -\alpha\, f(x)\, y(x) + \int_0^x \left[\alpha\, f'(t)\, y(t) - \beta\, g\big(t, y(t)\big)\right] dt. \tag{19}$$

Now, the application of the operator $\mathcal{L}$ again on both sides of Equation (19), and performing some manipulations, yield

$$y(x) = -\int_0^x \alpha\, f(t)\, y(t)\, dt + \int_0^x (x-t) \left[\alpha\, f'(t)\, y(t) - \beta\, g\big(t, y(t)\big)\right] dt. \tag{20}$$

Therefore we have

$$\mathbf{C}^T \Psi^{(\lambda)}(x) \approx \int_0^x (x-t) \left[\alpha\, f'(t)\, \mathbf{C}^T \Psi^{(\lambda)}(t) - \beta\, g\big(t, \mathbf{C}^T \Psi^{(\lambda)}(t)\big)\right] dt - \int_0^x \alpha\, f(t)\, \mathbf{C}^T \Psi^{(\lambda)}(t)\, dt. \tag{21}$$

We now collocate Equation (21) (see Canuto et al. (1988)) as

$$\begin{aligned}\mathbf{C}^T\,\Psi^{(\lambda)}(x_i) = &\int_0^{x_i}(x_i-t)\left[\alpha\,f'(t)\,\mathbf{C}^T\,\Psi^{(\lambda)}(t)-\beta\,g\big(t,\mathbf{C}^T\,\Psi^{(\lambda)}(t)\big)\right]\,dt\\ &-\int_0^{x_i}\alpha\,f(t)\,\mathbf{C}^T\,\Psi^{(\lambda)}(t)\,dt, 1\le i\le 2^{k-1}M,\end{aligned}\tag{22}$$

where x_i are the distinct $(2^{k-1}M)$ roots of $C^{(\lambda)}_{2^{k-1}M}(2x-1)$. As in Section 4, and if we make use of the transformation $t=\frac{x_i}{2}(z+1)$, then after applying Gaussian integration formula, Equation (22) may be written in the form

$$\begin{aligned}\mathbf{C}^T\,\Psi^{(\lambda)}(x_i) \;\approx\;& \frac{x_i^2}{4}\sum_{j=0}^{s}\omega_j\,(1-z_j)\left[\alpha\,f'\left(\frac{x_i}{2}(z_j+1)\right)\,\mathbf{C}^T\,\Psi^{(\lambda)}\left(\frac{x_i}{2}(z_j+1)\right)\right.\\ -\;& \left.\beta\,g\left(\frac{x_i}{2}(z_j+1)\mathbf{C}^T\,\Psi^{(\lambda)}\left(\frac{x_i}{2}(z_j+1)\right)\right)\right]\\ -\;& \frac{x_i}{2}\sum_{j=0}^{s}\omega_j\,\alpha\,f\left(\frac{x_i}{2}(z_j+1)\,\mathbf{C}^T\,\Psi^{(\lambda)}\left(\frac{x_i}{2}(z_j+1)\right)\right),\qquad 1\le i\le 2^{k-1}M,\end{aligned}\tag{23}$$

where z_j's are the $(s+1)$ zeros of $C^{(\lambda)}_{s+1}(x)$, and the weights ω_j can be computed from the formula in Equation (16).

Remark 2: For the second-order IVP with nonhomogeneous initial conditions

$$y''(x)+\alpha\,f(x)\,y'(x)+\beta\,g(x,y(x))=0,\quad y(0)=a_0,\ y'(0)=a_1,\qquad x\in[0,1],\tag{24}$$

we use the transformation

$$u(x)=y(x)-a_0-a_1\,x,$$

which turns Equation (24) into the homogeneous second order IVP

$$u''(x)+\alpha\,f(x)\,u'(x)+\beta\,\bar{g}(x,u(x))=0,\quad u(0)=u'(0)=0,\qquad x\in[0,1],\tag{25}$$

where

$$\bar{g}\big(x,u(x)\big)=\alpha\,a_1\,f(x)+\beta\,g\big(x,u(x)+a_0+a_1\,x\big).$$

Now Equation (25) can be treated as Equation (17).

6. Uniqueness and convergence theorems

In this section, the theoretical analysis of uniqueness and convergence of the proposed algorithms is discussed in detail. The uniqueness of the solution of Equation (11) was discussed in Burden and Faires (2011).

Now, we state and prove two theorems. In the first we prove the uniqueness of the solution of the second-order initial value problem (17), while in the second we show that the truncated series given in (8) converges to the exact solutions of the two initial value problems (11) and (17).

Theorem 1. (Uniqueness Theorem)

Assume that $|\alpha\, f^{(i)}(x)| \leqslant L_i$, where L_0, L_1 are two positive constants, $g(x,y)$ is continuous and satisfies Lipschitz condition in the variable y with Lipschitz constant L_2. Then Equation (17) has a unique solution whenever $0 < L < 1$, where $L = x \sum_{i=0}^{2} L_i$.

Proof:

Let y_1 and y_2 be two solutions of Equation (17). Then,

$$y_1(x) = -\int_0^x \alpha\, f(t)\, y_1(t)\, dt + \int_0^x (x-t)\, \alpha\, f'(t)\, y_1(t)\, dt - \int_0^x (x-t)\, \alpha\, f'(t)\, \beta\, g\big(t, y_1(t)\big)\, dt,$$

and

$$y_2(x) = -\int_0^x \alpha\, f(t)\, y_2(t)\, dt + \int_0^x (x-t)\, \alpha\, f'(t)\, y_2(t)\, dt - \int_0^x (x-t)\, \alpha\, f'(t)\, \beta\, g\big(t, y_2(t)\big)\, dt.$$

Now

$$\begin{aligned} |y_1 - y_2| &\leqslant \int_0^x |\alpha\, f(t)|\, |y_1 - y_2|\, dt + \int_0^x (x-t)\, |\alpha\, f'(t)|\, |y_1 - y_2|\, dt \\ &+ \int_0^x (x-t) |\beta\, g(t, y_1) - \beta\ g(t, y_2)|\, dt \\ &\leqslant L|y_1 - y_2|, \end{aligned}$$

and hence

$$|y_1 - y_2|(1 - L) \leqslant 0,$$

but since $0 < L < 1$, one should has $|y_1 - y_2| = 0$. This proves the uniqueness and hence completes the proof. □

Theorem 2. (Convergence Theorem)

The truncated series expansion of Equations (11) and (17) converge to the exact ones.

Proof: Let

$$\begin{aligned} y(t) &= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}^{(\lambda)}(t), \\ y_{k,M}(t) &= \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}^{(\lambda)}(t), \\ y_{k,N}(t) &= \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{N-1} c_{nm} \psi_{nm}^{(\lambda)}(t), \end{aligned}$$

be the exact and approximate solutions (partial sums) to Equations (11) and (17) with $N \geqslant M$. Then we have

$$\begin{aligned}\left(y(t), y_{k,N}(t)\right)_w &= \left(y(t), \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{N-1} c_{nm}\psi_{nm}^{(\lambda)}(t)\right)_w \\ &= \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{N-1} \bar{c}_{nm}\left(y(t), \psi_{nm}^{(\lambda)}(t)\right)_w \\ &= \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{N-1} \bar{c}_{nm}\, c_{nm} \\ &= \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{N-1} |c_{nm}|^2.\end{aligned}$$

We show that $y_{k,N}(t)$ is a Cauchy sequence in the complete Hilbert space $L^2(\mathbb{R})$ and hence converges.

Now

$$\begin{aligned}\left\|y_{k,N}(t) - y_{k,M}(t)\right\|_w^2 &= \left\|\sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{N-1} c_{nm}\psi_{nm}^{(\lambda)}(t)\right\|_w^2 \\ &= \sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{N-1} |c_{nm}|^2.\end{aligned}$$

Bessel's inequality implies that $\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \left|c_{nm}\right|^2$ is convergent and this leads to

$$\left\|y_{k,N}(t) - y_{k,M}(t)\right\|_w^2 \to 0 \quad \text{as} \quad M, N \to \infty,$$

and hence $y_{k,N}(t)$ converges to say $s(t)$. We prove that $s(t) = y(t)$,

$$\begin{aligned}\left(s(t) - y(t), \psi_{nm}^{(\lambda)}(t)\right)_w &= \left(s(t), \psi_{nm}^{(\lambda)}(t)\right)_w - \left(y(t), \psi_{nm}^{(\lambda)}(t)\right)_w \\ &= \left(\lim_{N\to\infty} y_{k,N}(t), \psi_{nm}^{(\lambda)}(t)\right)_w - c_{nm} \\ &= \lim_{N\to\infty}\left(y_{k,N}(t), \psi_{nm}^{(\lambda)}(t)\right)_w - c_{nm} \\ &= 0,\end{aligned}$$

and this proves that $\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}^{(\lambda)}(t)$ converges to $y(t)$. □

7. Illustrative examples

In this section, we apply UGQWM to solve first- and second-IVPs as well as Bratu's equation.

Example 1.

Consider the first-order nonlinear IVP (see, Rajabi et al. (2007); Domairry and Nadim (2008); Caruntu and Bota (2012)):

$$y' + y + y^4 = 0; \quad x \in [0,1], \qquad y(0) = 1. \tag{26}$$

We solve Equation (26) using UGQWM for the case corresponds to $\lambda = k = 1, M = 8$ and $s = 10$. Table I illustrates the maximum absolute error of Example 1. The comparison between the error resulting from the numerical solution obtained by UGQWM with the error resulting from the application of the following methods:

- Homotopy perturbation method (HPM) in Rajabi et al. (2007),
- Homotopy analysis method (HAM) in Domairry and Nadim (2008),
- Squared remainder minimization method (SRMM) in Caruntu and Bota (2012),

is presented in Table I. Since no exact solution for Equation (26) is known, the relative error is computed as the difference (in absolute value) between the approximate solution and the numerical solution given by the Mathematica Wolfram software. In addition, in Figure 1, we give a comparison between different approximate solutions of (26). This figure shows that the bigger values of M attains accurate solution.

Table I: Comparison of HPM, HAM, SRMM, and UGQWM for Example 1

x	HPM	HAM	SRMM	UGQWM
0.1	2.24219×10^{-3}	1.18868×10^{-4}	6.08673×10^{-5}	9.20708×10^{-12}
0.2	9.47661×10^{-3}	8.71208×10^{-4}	1.25683×10^{-3}	2.82149×10^{-11}
0.3	1.79917×10^{-2}	1.25548×10^{-3}	3.12286×10^{-4}	4.99824×10^{-11}
0.4	2.51387×10^{-2}	1.54052×10^{-3}	9.67612×10^{-4}	7.33763×10^{-11}
0.5	3.00464×10^{-2}	2.09832×10^{-3}	1.12410×10^{-3}	9.40933×10^{-11}
0.6	3.27857×10^{-2}	3.04594×10^{-3}	1.78443×10^{-4}	1.13213×10^{-10}
0.7	3.37783×10^{-2}	4.35783×10^{-3}	8.14607×10^{-4}	1.33089×10^{-10}
0.8	3.35023×10^{-2}	5.95087×10^{-3}	7.43428×10^{-4}	1.48971×10^{-10}
0.9	3.23752×10^{-2}	7.72966×10^{-3}	3.59748×10^{-4}	1.84797×10^{-10}

Example 2.

Consider the first-order linear IVP (see Caruntu and Bota (2012)):

$$y' = 100\,(\sin x - y)\,; \quad x \in [0,1], \qquad y(0) = 0, \tag{27}$$

with the exact solution $y(x) = 0.9999\,(\sin x - 0.01\cos x + 0.01\,e^{-100\,x})$. We solve Equation (27) using UGQWM for the case corresponds to $k = 2, M = 4$ and $s = 7$.

Integrating Equation (27), we get

$$y(x) = 100\left(1 - \cos x - \int_0^x y(t)\,dt\right). \tag{28}$$

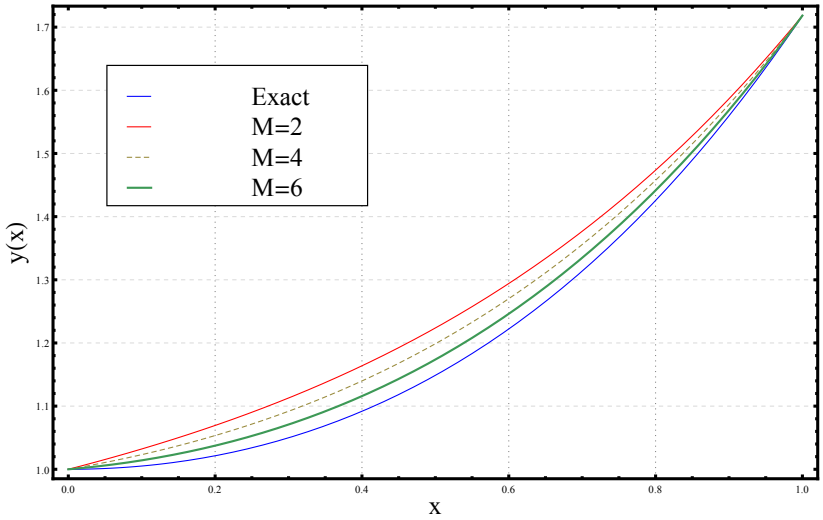

Figure 1: Different solutions of Example 1

The approximate solution for (27) is given by

$$y_{2,4}(x) = c_{1,0}\,\psi^{(\lambda)}_{1,0}(x) + c_{1,1}\,\psi^{(\lambda)}_{1,1}(x) + \cdots + c_{2,3}\,\psi^{(\lambda)}_{2,3}(x),$$

then the application of formula (15) yields

$$\mathbf{C}^T\,\Psi^{(\lambda)}(x_i) \approx 100\,(1-\cos x) + 50\,x_i \sum_{j=0}^{s} \omega_j\,\mathbf{C}^T\,\Psi^{(\lambda)}\left(\frac{x_i}{2}(z_j+1)\right), \qquad i = 1, 2, \ldots 8, \quad (29)$$

and hence a system of equations in the unknown expansion coefficients $c_{n,m}$ is obtained and then solved by a suitable numerical solver. Table II illustrates the maximum absolute error of Example 2 for various values of λ, while in Table III, we give a comparison between the best errors obtained by using the methods in Caruntu and Bota (2012) with UGQWM. This table shows that our method is more accurate if compared with all methods developed in Caruntu and Bota (2012). In addition, a comparison between the exact solution and some approximate solutions is displayed in Figure 2.

Table II: Maximum absolute error of Example 2

λ	1	$\frac{1}{2}$	0	-0.49
E	$7.658\ .10^{-8}$	$6.298\ .10^{-8}$	$4.927\ .10^{-8}$	$3.851\ .10^{-8}$

Example 3.

Consider the second-order nonlinear IVP, (see Al-Khaled and Anwar (2007)):

$$y'' + e^{-2y} = 0; \quad x \in [0,1], \qquad y(0) = 1, \quad y'(0) = \frac{1}{e}, \quad (30)$$

Table III: Different solutions of Example 2

Method	Best error
UGQWM	$3.85\ .10^{-8}$
Cheb I	$2.59\ .10^{-5}$
Cheb II	$2.00\ .10^{-5}$
Eq. pts	$4.45\ .10^{-5}$
Lobatto III A	$7.52\ .10^{-5}$
Runge-Kutta	$8.82\ .10^{-6}$

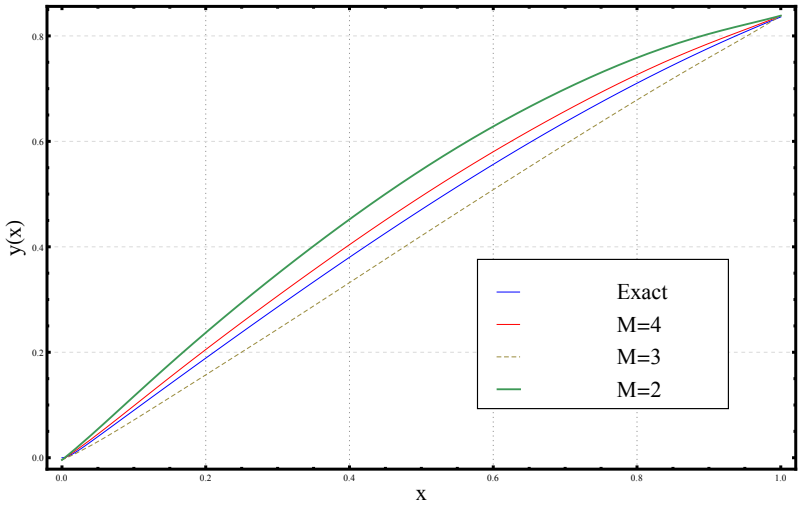

Figure 2: Different solutions of Example 3

with the exact solution $y(x) = \ln(e + x)$. Equation (30) is solved using UGQWM for the case corresponds to $k = 1, M = 9$ and $s = 8$. Table IV illustrates the maximum absolute error of Example 3 for various values of λ, and in Table V we give a comparison between the best errors obtained by using Adomian decomposition method (ADM), quintic spline method (C^2-spline) in Al-Khaled and Anwar (2007) with the best error obtained by UGQWM.

Table IV: Maximum absolute error of Example 3

λ	1	$\frac{1}{2}$	0	-0.49
E	8.322×10^{-13}	5.284×10^{-13}	5.293×10^{-13}	6.004×10^{-14}

Example 4.

Consider the second-order nonlinear Bratu problem (see Venkatesh et al. (2012)):

$$y'' - 2\,e^y = 0; \quad x \in [0,1], \qquad y(0) = y'(0) = 0, \tag{31}$$

Table V: Different solutions of Example 3

Method	Best error
UGQWM	6.004×10^{-14}
ADM	2.20×10^{-6}
C^2-spline	1.68×10^{-13}

with the exact solution $y(x) = -2\log(\cos x)$. Equation (31) is solved using UGQWM for the case corresponds to $k = 1, M = 9$ and $s = 8$. Table VI lists some approximate solutions of (31) for various values of λ if UGQWM is applied, and the corresponding maximum absolute error for each case is displayed also in Table VII. In Table VIII, we give a comparison between the best absolute error obtained in Venkatesh et al. (2012) by using Legendre wavelets method (LWM) with that obtained with UGQWM.

Table VI: The approximate solution and the maximum absolute error of Example 4

λ	Numerical solution by $UGQWM$	E
$\frac{1}{2}$	$0.216\,x^8 - 0.607\,x^7 + 0.82\,x^6 - 0.533\,x^5 + 0.37\,x^4 - 0.045\,x^3 + 1.005\,x^2 - 0.00023\,x$	6.470×10^{-6}
1	$0.212\,x^8 - 0.598\,x^7 + 0.81\,x^6 - 0.533\,x^5 + 0.38\,x^4 - 0.048\,x^3 + 1.005\,x^2 - 0.00031\,x$	9.553×10^{-6}
0	$0.222\,x^8 - 0.623\,x^7 + 0.83\,x^6 - 0.533\,x^5 + 0.39\,x^4 - 0.042\,x^3 + 1.004\,x^2 - 0.00016\,x$	3.162×10^{-6}

Table VII: Maximum pointwise error of Example 4

x	0	0.2	0.4	0.6	0.8	1
Exact	0.	0.0402695	0.164458	0.38393	0.722781	1.23125
$\lambda = \frac{1}{2}$	2.607×10^{-6}	0.0402694	0.164459	0.383929	0.722782	1.23125
$\lambda = 1$	4.731×10^{-6}	0.0402697	0.164459	0.38393	0.722781	1.23124
$\lambda = 0$	1.021×10^{-6}	0.040269	0.164459	0.383929	0.722782	1.23125

Table VIII: Different solutions of Example 4

Method	Best error
UGQWM	1.02×10^{-6}
LWM	1.38×10^{-4}

Remark 3: It is worthy noting here that the obtained numerical results in the previous solved four examples are very accurate, although the number of retained modes in the spectral expansion is few, and the numerical results are comparing favorably with the known analytical solutions. Moreover, the numerical results show that our method is more accurate than those discussed by some other existing techniques in literature.

8. Conclusion

In this paper, a new ultraspherical wavelets algorithm for obtaining numerical spectral solutions for first- and second-order IVPs is presented and analyzed. The derivation of this algorithm is essentially based on ultraspherical wavelets basis functions and the Gaussian quadrature formula. The spectral collocation method is used for handling these equations. One of the main advantages of the presented algorithms is their availability for application on both linear and non linear first- and second-order IVPs including some important equations and also a Bratu-type equation. Another advantage of the developed algorithms is highly accurate approximate solutions are achieved by using a few number of terms of the suggested expansion.

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