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On the stability of a Pexiderized functional equation in intuitionistic fuzzy Banach spaces

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Abstract

During the last few decades several researchers have been devoted to establishing stability of different kinds of functional equations, differential equations, functional differential equations, fractional differential equations, etc. under different sufficient conditions in different spaces like Banach spaces, Banach modules, fuzzy Banach spaces etc. In this paper, we remain confined in the discussion of stability of functional equations in intuitionistic fuzzy Banach spaces. Ulam was the first person who introduced an open question concerning the stability of a group homomorphism in an international conference. Thereafter several researchers have replied and are still replying to this open question in different contexts. The objective of the present paper is to determine the Hyers-Ulam-Rassias type stability concerning the Pexiderized functional equation in intuitionistic fuzzy Banach spaces. Under a few sufficient conditions, Hyers-Ulam-Rassias type stability of a Pexiderized functional equation has been established in intuitionistic fuzzy Banach spaces.

Keywords: t-norm, t-conorm, Intuitionistic fuzzy Banach space, Pexiderized functional, equation, Hyers-Ulam-Rassias stability

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1. Introduction

The stability problem of a functional equation was posed by Ulam (1960) in 1940 concerning the stability of group homomorphisms and answered in the next year by Hyers (1941) for Cauchy functional equation in Banach spaces and then generalized by T. Aoki (1950) and Th. M. Rassias (1978) for additive mappings and linear mappings by considering an unbounded Cauchy difference, respectively. In the spirit of Rassias's approach, Gavruta (1994) replaced the unbounded Cauchy difference by a general control function to generalize Rassias's theorem. The Hyers-Ulam stability theorem was generalized by F. Skof (1983) for the function $f : X \to Y$, where X is a normed space and Y is a Banach space, and then the result of Skof was extended by P. W. Cholewa (1984) and S. Czerwik (1992). In this way several stability problems for various functional equations have been investigated. Recently, the fuzzy version of different functional equations was discussed by A. K. Mirmostafaee and M. S. Moslehian (2008) and C. Park (2009).

The concept of intuitionistic fuzzy sets was introduced by Atanassov (1986) as a generalization of fuzzy sets. One of the most important problems in intuitionistic fuzzy topology is to obtain an appropriate concept of intuitionistic fuzzy metric spaces and intuitionistic fuzzy normed spaces. J. H. Park (2004), Saadati and Park (2006), and T. K. Samanta and Iqbal (2009) introduced and studied a few notions of intuitionistic fuzzy metric spaces and intuitionistic fuzzy normed spaces.

Several results for the Hyers-Ulam-Rassias stability of many functional equations have been proved by several researchers like A. K. Mirmostafaee and M. S. Moslehian (2008), C. Park (2009), Nabin et al. (2014), Nabin et al. (2014), Shakeri (2009), Samanta et al. (2012) and Samanta et al. (2013) in fuzzy Banach spaces and intuitionistic fuzzy Banach spaces. Our goal is to determine some stability results concerning the Pexiderized functional equation $f(x + y) = g(x) + h(y)$ in intuitionistic fuzzy Banach spaces.

2. Preliminaries

In this section we recall some lemmas, definitions, and examples used in this paper.

Lemma 1. (Deschrijver and Kerre (2003))

Consider the set L^* and the order relation $\leq_{L^*}$ defined by

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1, x_2 \geq y_2, \forall (x_1, x_2), (y_1, y_2) \in L^*.$$

Then $(L^*, \leq_{L^*})$ is a complete lattice. We denote its units by $0_{L^*} = (0, 1)$ and $1_{L^*} = (1, 0)$.

Definition. (Atanassov (1986)) An intuitionistic fuzzy set $A_{\zeta,\eta}$ in a universal set U is an object $A_{\zeta,\eta} = \{(\zeta_A(u), \eta_A(u)) : u \in U\}$, where $\zeta_A(u) \in [0, 1]$ and $\eta_A(u) \in [0, 1]$ for all $u \in U$ are called the membership degree and the non-membership degree, respectively, of u in $A_{\zeta,\eta}$ and furthermore satisfy $\zeta_A(u) + \eta_A(u) \leq 1$.

Definition. (Deschrijver et al. (2004)) A triangular norm (t-norm) on L^* is a mapping $\tau : (L^*)^2 \to L^*$ satisfying the following conditions:
(a) $(\forall x \in L^*)(\tau(x, 1_{L^*}) = x)$ (boundary condition);
(b) $(\forall (x, y) \in (L^*)^2)(\tau(x, y) = \tau(y, x))$ (commutativity);
(c) $(\forall (x, y, z) \in (L^*)^3)(\tau(x, \tau(y, z)) = \tau(\tau(x, y), z))$ (associativity);
(d) $(\forall (x, x', y, y') \in (L^*)^4)(x \leq_{L^*} x'$ and $y \leq_{L^*} y' \Rightarrow \tau(x, y) \leq_{L^*} \tau(x', y'))$ (monotonicity).

A t-norm τ on L^* is said to be continuous if for any $x, y \in L^*$ and any sequences $\{x_n\}$ and $\{y_n\}$ which converge to x and y respectively,

$$\lim_{n \to \infty} \tau(x_n, y_n) = \tau(x, y).$$

For example, let $a = (a_1, a_2), b = (b_1, b_2) \in L^*$, consider $\tau(a, b) = (a_1 b_1, \min\{a_2 + b_2, 1\})$ and $M(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$. Then $\tau(a, b)$ and $M(a, b)$ are continuous t-norm.

Now, we define a sequence τ^n recursively by $\tau^1 = \tau$ and

$$\tau^n\left(x^{(1)}, \cdots, x^{(n+1)}\right) = \tau\left(\tau^{n-1}\left(x^{(1)}, \cdots, x^{(n)}\right), x^{(n+1)}\right),$$

for all $n \geq 2$ and $x^{(i)} \in L^*$.

Definition. (Deschrijver et al. (2004)) A continuous t-norm τ on L^* is said to be continuous t-representable if there exists a continuous t-norm $*$ and a continuous t-conorm $\diamond$ on $[0, 1]$ such that, for all $x = (x_1, x_2), y = (y_1, y_2) \in L^*$,

$$\tau(x, y) = (x_1 * y_1, x_2 \diamond y_2).$$

Definition. (Deschrijver et al. (2004)) A negator on L^* is any decreasing mapping $N : L^* \to L^*$ satisfying $N(0_{L^*}) = 1_{L^*}$ and $N(1_{L^*}) = 0_{L^*}$. If $N(N(x)) = x$ for all $x \in L^*$, then N is called an involutive negator. A negator on $[0, 1]$ is a decreasing mapping $N : [0, 1] \to [0, 1]$ satisfying $N(0) = 1$ and $N(1) = 0$. N_s denotes the standard negator on $[0, 1]$ defined by $N_s(x) = 1 - x$ for all $x \in [0, 1]$.

Definition. (Shakeri (2009)) (1) Let $L = (L^*, \leq_{L^*})$. The triple (X, P, τ) is said to be an L-fuzzy normed space if X is a vector space, τ is a continuous t-norm on L^* and P is an L-fuzzy set on $X \times (0, +\infty)$ satisfying the following conditions for all $x, y \in X$ and $t, s > 0$,
(a) $P(x, t) > 0_{L^*}$;
(b) $P(x, t) = 1_{L^*}$ if and only if $x = 0$;
(c) $P(\alpha x, t) = P\left(x, \frac{t}{|\alpha|}\right)$ for all $\alpha \neq 0$;
(d) $P(x + y, t + s) \geq_{L^*} \tau(P(x, t), P(y, s))$;
(e) $P(x, .) : (0, \infty) \to L^*$ is continuous;
(f) $\lim_{t \to 0} P(x, t) = 0_{L^*}$ and $\lim_{t \to \infty} P(x, t) = 1_{L^*}$.

In this case P is called an L-fuzzy norm (briefly, L^*-fuzzy norm).

(2) If $P = P_{\mu,\nu}$ is an intuitionistic fuzzy set, then the triple $(X, P_{\mu,\nu}, \tau)$ is said to be an intuitionistic fuzzy normed space (briefly, IFN-space). In this case $P = P_{\mu,\nu}$ is called an intuitionistic fuzzy norm on X.

Note that, if P is an L^*-fuzzy norm on X, then the following are satisfied:
(i) $P(x, t)$ is nondecreasing with respect to t for all $x \in X$.
(ii) $P(x - y, t) = P(y - x, t)$ for all $x, y \in X$ and $t > 0$.

Example 1.

Let $(X, \|.\|)$ be a normed space.

Let $\tau(a, b) = (a_1 b_1, \min\{a_2 + b_2, 1\})$ for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and μ, ν be membership and non-membership degree of an intuitionistic fuzzy set defined by

$$P_{\mu,\nu}(x, t) = (\mu_x(t), \nu_x(t)) = \left(\frac{t}{t + m\|x\|}, \frac{\|x\|}{t + \|x\|}\right),$$

for all $t \in \mathbb{R}^+$ in which $m > 1$. Then, $(X, P_{\mu,\nu}, \tau)$ is an IFN-space. Here, $\mu(x, t) + \nu(x, t) = 1$ for $x = 0$ and $\mu(x, t) + \nu(x, t) < 1$ for $x \neq 0$.

Let $M(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$ for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and μ, ν be membership and non-membership degree of an intuitionistic fuzzy set defined by

$$P_{\mu,\nu}(x, t) = (\mu_x(t), \nu_x(t)) = \left(e^{-\frac{\|x\|}{t}}, e^{-\frac{\|x\|}{t}}\left(e^{\frac{\|x\|}{t}} - 1\right)\right),$$

for all $t \in \mathbb{R}^+$. Then $(X, P_{\mu,\nu}, M)$ is an IFN-space.

Definition. (1) A sequence $\{x_n\}$ in an IFN-space $(X, P_{\mu,\nu}, \tau)$ is said to be convergent to a point $x \in X$ (denoted by $x_n \to x$) if $P_{\mu,\nu}(x_n - x, t) \to 1_{L^*}$ as $n \to \infty$ for every $t > 0$.

(2) A sequence $\{x_n\}$ in an IFN-space $(X, P_{\mu,\nu}, \tau)$ is said to be a Cauchy sequence if, for any $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$P_{\mu,\nu}(x_n - x_m, t) >_{L^*} (N_s(\epsilon), \epsilon),$$

for all $n, m \geq n_0$, where N_s is the standard negator.

(3) An IFN-space $(X, P_{\mu,\nu}, \tau)$ is said to be complete if every Cauchy sequence in $(X, P_{\mu,\nu}, \tau)$ is convergent in $(X, P_{\mu,\nu}, \tau)$. A complete intuitionistic fuzzy normed space is called an intuitionistic fuzzy Banach space.

3. Stability Of The Functional Equation

Throughout this section X, Y, Z are assumed to be real vector spaces.

Theorem 1.

Let $(Y, P_{\mu,\nu}, \tau)$ be a complete IFN-space and $(Z, P'_{\mu,\nu}, \tau)$ be an IFN-space. Let $\phi : X^2 \to Z$ be a mapping such that

$$P'_{\mu,\nu}(\phi(3x, 3y), t) \geq_{L^*} P'_{\mu,\nu}(\alpha\,\phi(x, y), t), \tag{1}$$

for all $x, y \in X$, $t > 0$ and for some $0 < \alpha < 3$. If $f, g, h : X \to Y$ are mappings such that

$$P_{\mu,\nu}(f(x + y) - g(x) - h(y), t) \geq_{L^*} P'_{\mu,\nu}(\phi(x, y), t), \tag{2}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique additive mapping $A : X \to Y$ such that for all $x \in X$, $t > 0$,

$$P_{\mu,\nu}(f(x) - A(x) - f(0), t) \geq_{L^*} M_1\left(x, (3 - \alpha)\frac{t}{2}\right) \tag{3}$$

and

$$\frac{f(3^n x)}{3^n} \to A(x), \text{ as } n \to \infty, \tag{4}$$

$$\frac{g(3^n x)}{3^n} \to A(x), \ \frac{h(3^n x)}{3^n} \to A(x) \text{ as } n \to \infty, \tag{5}$$

where

$$M_1(x, t) := \tau^6\left(P'_{\mu,\nu}\left(\phi\left(\frac{-x}{2}, \frac{3x}{2}\right), \frac{t}{4}\right), P'_{\mu,\nu}\left(\phi\left(\frac{3x}{2}, \frac{-x}{2}\right), \frac{t}{4}\right),\right.$$
$$P'_{\mu,\nu}\left(\phi\left(\frac{-x}{2}, \frac{-x}{2}\right), \frac{t}{4}\right), P'_{\mu,\nu}\left(\phi\left(\frac{3x}{2}, \frac{3x}{2}\right), \frac{t}{4}\right), P'_{\mu,\nu}\left(\phi\left(\frac{x}{2}, \frac{-x}{2}\right), \frac{t}{4}\right),$$
$$\left.P'_{\mu,\nu}\left(\phi\left(\frac{-x}{2}, \frac{x}{2}\right), \frac{t}{4}\right), P'_{\mu,\nu}\left(\phi\left(\frac{x}{2}, \frac{x}{2}\right), \frac{t}{4}\right)\right).$$

Proof:

Here, for all $x, y \in X$ and $t > 0$, we have

$$P_{\mu,\nu}\left(2f\left(\frac{x + y}{2}\right) - f(x) - f(y), t\right)$$
$$\geq_{L^*} \tau^3\left(P'_{\mu,\nu}\left(\phi\left(\frac{x}{2}, \frac{y}{2}\right), \frac{t}{4}\right), P'_{\mu,\nu}\left(\phi\left(\frac{y}{2}, \frac{x}{2}\right), \frac{t}{4}\right),\right.$$
$$\left.P'_{\mu,\nu}\left(\phi\left(\frac{x}{2}, \frac{x}{2}\right), \frac{t}{4}\right), P'_{\mu,\nu}\left(\phi\left(\frac{y}{2}, \frac{y}{2}\right), \frac{t}{4}\right)\right) \quad [\text{by } (2)]. \tag{6}$$

Let us now define $F(x) = f(x) - f(0)$ for all $x \in X$. Clearly, $F(0) = 0$ and F satisfies (6). Putting $y = -x$ in (6) for the function F, we get for all $x \in X$, $t > 0$,

$$P_{\mu,\nu}(-F(x) - F(-x), t) \geq_{L^*} \tau^3\left(P'_{\mu,\nu}\left(\phi\left(\frac{x}{2}, \frac{-x}{2}\right), \frac{t}{4}\right),\right.$$
$$\left.P'_{\mu,\nu}\left(\phi\left(\frac{-x}{2}, \frac{x}{2}\right), \frac{t}{4}\right), P'_{\mu,\nu}\left(\phi\left(\frac{x}{2}, \frac{x}{2}\right), \frac{t}{4}\right), P'_{\mu,\nu}\left(\phi\left(\frac{-x}{2}, \frac{-x}{2}\right), \frac{t}{4}\right)\right). \tag{7}$$

Replacing x by $-x$ and y by $3x$ in (6), we get for the function F

$$
\begin{aligned}
&P_{\mu,\nu}(2F(x) - F(-x) - F(3x), t) \\
&\geq_{L^*} \tau^3 \left(P'_{\mu,\nu}\left(\phi\left(\frac{-x}{2}, \frac{3x}{2}\right), \frac{t}{4}\right), P'_{\mu,\nu}\left(\phi\left(\frac{3x}{2}, \frac{-x}{2}\right), \frac{t}{4}\right), \right. \\
&\left. P'_{\mu,\nu}\left(\phi\left(\frac{-x}{2}, \frac{-x}{2}\right), \frac{t}{4}\right), P'_{\mu,\nu}\left(\phi\left(\frac{3x}{2}, \frac{3x}{2}\right), \frac{t}{4}\right)\right),
\end{aligned} \tag{8}
$$

for all $x \in X$, $t > 0$. Now, by using (7) and (8), we get for all $x \in X$, $t > 0$,

$$
P_{\mu,\nu}\left(F(x) - \frac{F(3x)}{3}, t\right) \geq_{L^*} M_1\left(x, \frac{3t}{2}\right). \tag{9}
$$

Clearly, $M_1(3x, t) \geq_{L^*} M_1\left(x, \frac{t}{\alpha}\right)$ and $\lim_{t \to \infty} M_1(x, t) = 1_{L^*}$. Replacing x by $3^n x$ in (9), we get

$$
P_{\mu,\nu}\left(\frac{F(3^n x)}{3^n} - \frac{F(3^{n+1} x)}{3^{n+1}}, t\right) \geq_{L^*} M_1\left(x, \frac{3^{n+1} t}{2\,\alpha^n}\right), \tag{10}
$$

for all $x \in X$, $t > 0$, $n \in \mathbb{N}$. Clearly, for all $x \in X$ and $n \in \mathbb{N}$,

$$
F(x) - \frac{F(3^n x)}{3^n} = \sum_{r=0}^{n-1} \left(\frac{F(3^r x)}{3^r} - \frac{F(3^{r+1} x)}{3^{r+1}}\right).
$$

Now, for all $x \in X$, $t > 0$, $n \in \mathbb{N}$,

$$
\begin{aligned}
&P_{\mu,\nu}\left(F(x) - \frac{F(3^n x)}{3^n}, t \sum_{r=0}^{n-1} \frac{\alpha^r}{3^{r+1}}\right) \\
&\geq_{L^*} \tau^{n-1}\left(P_{\mu,\nu}\left(F(x) - \frac{F(3x)}{3}, \frac{t}{3}\right), P_{\mu,\nu}\left(\frac{F(3x)}{3}) - \frac{F(3^2 x)}{3^2}, \frac{t\alpha}{3^2}\right), \right. \\
&\qquad \left. \cdots, P_{\mu,\nu}\left(\frac{F(3^{n-1} x)}{3^{n-1}} - \frac{F(3^n x)}{3^n}, \frac{t\alpha^{n-1}}{3^n}\right)\right) \\
&= M_1(x, \frac{t}{2}) \quad [\text{by (9) and (10)}].
\end{aligned}
$$

That is,

$$
P_{\mu,\nu}\left(F(x) - \frac{F(3^n x)}{3^n}, t\right) \geq_{L^*} M_1\left(x, \frac{t}{2\sum_{r=0}^{n-1} \frac{\alpha^r}{3^{r+1}}}\right), \tag{11}
$$

for all $x \in X$, $t > 0$, $n \in \mathbb{N}$. Replacing x by $3^m x$ in (11), we get

$$
P_{\mu,\nu}\left(\frac{F(3^m x)}{3^m} - \frac{F(3^{n+m} x)}{3^{n+m}}, t\right) \geq_{L^*} M_1\left(x, \frac{t}{\frac{\alpha^m}{3^m} 2\sum_{r=0}^{n-1} \frac{\alpha^r}{3^{r+1}}}\right), \tag{12}
$$

for all $x \in X$, $t > 0$, $m, n \in \mathbb{N}$. Since $\frac{\alpha^m}{3^m} \sum_{r=0}^{n-1} \frac{\alpha^r}{3^{r+1}} \to 0$ as $m \to \infty$,

$$M_1\left(x, \frac{t}{\frac{\alpha^m}{3^m} 2 \sum_{r=0}^{n-1} \frac{\alpha^r}{3^{r+1}}}\right) \to 1_{L^*} \; as \; m \to \infty.$$

Thus from (12) we see that $\left\{\frac{F(3^n x)}{3^n}\right\}$ is a Cauchy sequence in $(Y, P_{\mu,\nu}, \tau)$. Since $(Y, P_{\mu,\nu}, \tau)$ is a complete IFN-space, there exists a mapping $A : X \to Y$ such that $\frac{F(3^n x)}{3^n} \to A(x)$ as $n \to \infty$. This proves (4). Let $\delta > 0$. Now, for all $x \in X$, $t > 0$ and $n \in \mathbb{N}$,

$$\begin{aligned} &P_{\mu,\nu}(F(x) - A(x), t + \delta) \\ &\quad \geq_{L^*} \tau\left(M_1\left(x, \frac{t}{2 \sum_{r=0}^{n-1} \frac{\alpha^r}{3^{r+1}}}\right), P_{\mu,\nu}\left(\frac{F(3^n x)}{3^n} - A(x), \delta\right)\right) \quad [\text{by } (11)]. \end{aligned}$$

Taking the limit as $n \to \infty$, we get, by using (4),

$$P_{\mu,\nu}(F(x) - A(x), t + \delta) \geq_{L^*} \tau\left(M_1\left(x, \frac{t}{2 \sum_{r=0}^{\infty} \frac{\alpha^r}{3^{r+1}}}\right), 1_{L^*}\right),$$

for all $x \in X$, $t > 0$. Taking the limit as $\delta \to 0$, we get (3). From the definition of A we get for all $x \in X$, $n \in \mathbb{N}$,

$$A(3^n x) = 3^n A(x) \text{ and } A(0) = 0. \tag{13}$$

Now, for all $x \in X$, $t > 0, n \in \mathbb{N}$, we have, by using (13),

$$\begin{aligned} &P_{\mu,\nu}(2A(2x) - 4A(x), t) \\ &\quad \geq_{L^*} \tau^3\left(P_{\mu,\nu}\left(2A(2x) - \frac{2f(3^n 2x)}{3^n}, \frac{t}{4}\right), P_{\mu,\nu}\left(A(3x) - \frac{f(3^{n+1} x)}{3^n}, \frac{t}{4}\right),\right. \\ &\quad\quad \left. P_{\mu,\nu}\left(A(x) - \frac{f(3^n x)}{3^n}, \frac{t}{4}\right), P_{\mu,\nu}\left(2f(3^n 2x) - f(3^{n+1} x) - f(3^n x), \frac{3^n t}{4}\right)\right). \end{aligned}$$

By using (6) and (1), we find that last term tends to 1_{L^*} as $n \to \infty$. This implies that

$$P_{\mu,\nu}(2A(2x) - 4A(x), t) = 1_{L^*} \quad [\text{by } (4)],$$

for all $x \in X$, $t > 0$, i.e., for all $x \in X$,

$$A(2x) = 2A(x). \tag{14}$$

Now, using (14) for all $x, y \in X$, $t > 0$ and $n \in \mathbb{N}$,

$$P_{\mu,\nu}(A(x+y)-A(x)-A(y),\,t)$$
$$\geq_{L^*}\ \tau^3\left(P_{\mu,\nu}\left(2A\left(\frac{x+y}{2}\right)-\frac{2}{3^n}f\left(3^n\left(\frac{x+y}{2}\right)\right),\,\frac{t}{4}\right),P_{\mu,\nu}\left(A(x)-\frac{f(3^nx)}{3^n},\,\frac{t}{4}\right),\right.$$
$$\left.P_{\mu,\nu}\left(A(y)-\frac{f(3^ny)}{3^n},\,\frac{t}{4}\right),P_{\mu,\nu}\left(2f\left(3^n\left(\frac{x+y}{2}\right)\right)-f(3^nx)-f(3^ny),\,\frac{3^nt}{4}\right)\right).$$

By using (4), (6), and (1) and taking the limit as $n\to\infty$, we get for all $x,y\in X$, $t>0$,

$$P_{\mu,\nu}(A(x+y)-A(x)-A(y),\,t)=1_{L^*}.$$

Therefore, $A(x+y)=A(x)+A(y)$ for all $x,y\in X$, i.e., A is additive. To prove the uniqueness, let us assume that $A':X\to Y$ is a mapping satisfying (3) and (13). Now, using (13) and (3), we get for all $x\in X$, $t>0$ and $n\in\mathbb{N}$,

$$P_{\mu,\nu}(A(x)-A'(x),\,t)$$
$$\geq_{L^*}\ \tau\left(P_{\mu,\nu}\left(\frac{A(3^nx)}{3^n}-\frac{f(3^nx)}{3^n}+\frac{f(0)}{3^n},\,\frac{t}{2}\right),P_{\mu,\nu}\left(\frac{f(3^nx)}{3^n}-\frac{A'(3^nx)}{3^n}-\frac{f(0)}{3^n},\,\frac{t}{2}\right)\right)$$
$$\geq_{L^*}\ \tau\left(M_1\left(x,\,(3-\alpha)\frac{3^nt}{4\alpha^n}\right),M_1\left(x,\,(3-\alpha)\frac{3^nt}{4\alpha^n}\right)\right).$$

Taking the limit as $n\to\infty$, we get $P_{\mu,\nu}(A(x)-A'(x),\,t)=1_{L^*}$ for all $x\in X$, $t>0$, i.e., $A'(x)=A(x)$ for all $x\in X$. Thus A is unique. Now using (1) and (2) we get for all $x\in X$, $t>0$, $n\in\mathbb{N}$,

$$P_{\mu,\nu}\left(\frac{g(3^nx)}{3^n}+\frac{h(3^nx)}{3^n}-A(2x),\,t\right)$$
$$\geq_{L^*}\ \tau\left(P'_{\mu,\nu}\left(\phi(3^nx,3^nx),\,\frac{3^nt}{2}\right),P_{\mu,\nu}\left(A(2x)-\frac{f(3^n2x)}{3^n},\,\frac{t}{2}\right)\right)$$
$$\geq_{L^*}\ \tau\left(P'_{\mu,\nu}\left(\phi(x,x),\,\frac{3^nt}{2\alpha^n}\right),P_{\mu,\nu}\left(A(2x)-\frac{f(3^n2x)}{3^n},\,\frac{t}{2}\right)\right).$$

Taking limit as $n\to\infty$, we get by (4) for all $x\in X$, $t>0$,

$$P_{\mu,\nu}\left(\frac{g(3^nx)}{3^n}+\frac{h(3^nx)}{3^n}-A(2x),\,t\right)\to 1_{L^*}\ as\ n\to\infty.$$

Then, by using (14), we get for all $x\in X$,

$$\frac{g(3^nx)}{3^n}+\frac{h(3^nx)}{3^n}\to 2A(x)\ \text{ as }\ n\to\infty. \tag{15}$$

Replacing x and y by $3^{n+1}x$ and 3^nx respectively in (2), we get for all $x\in X$, $t>0$ and $n\in\mathbb{N}$,

$$P_{\mu,\nu}(f(3^{n+1}x+3^nx)-g(3^{n+1}x)-h(3^nx),\,t)\geq_{L^*}P'_{\mu,\nu}(\phi(3^{n+1}x,3^nx),\,t). \tag{16}$$

Again, replacing x and y by $3^n x$ and $3^{n+1} x$ respectively in (2), we get for all $x \in X$, $t > 0$ and $n \in \mathbb{N}$,

$$P_{\mu,\nu}(\, f(3^n x + 3^{n+1} x) - g(3^n x) - h(3^{n+1} x), t\,) \geq_{L^*} P'_{\mu,\nu}(\,\phi(3^n x, 3^{n+1} x), t). \tag{17}$$

Now, by using (1), (16), and (17), we get for all $x \in X$, $t > 0$ and $n \in \mathbb{N}$,

$$\begin{aligned} &P_{\mu,\nu}\left(\frac{g(3^{n+1} x) - h(3^{n+1} x)}{3^n} - \frac{g(3^n x) - h(3^n x)}{3^n}, t\right) \\ &\geq_{L^*} \tau\left(P'_{\mu,\nu}\left(\phi(3^{n+1} x, 3^n x), \frac{3^n t}{2}\right), P'_{\mu,\nu}\left(\phi(3^n x, 3^{n+1} x), \frac{3^n t}{2}\right)\right) \\ &\geq_{L^*} \tau\left(P'_{\mu,\nu}\left(\phi(3x, x), \frac{3^n t}{2\alpha^n}\right), P'_{\mu,\nu}\left(\phi(x, 3x), \frac{3^n t}{2\alpha^n}\right)\right). \end{aligned}$$

Taking limit as $n \to \infty$, we get for all $x \in X$, $t > 0$,

$$P_{\mu,\nu}\left(\frac{g(3^{n+1} x) - h(3^{n+1} x)}{3^n} - \frac{g(3^n x) - h(3^n x)}{3^n}, t\right) \to 1_{L^*} \ as\, n \to \infty.$$

Corresponding to $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that

$$P_{\mu,\nu}\left(\frac{g(3^{n+1} x) - h(3^{n+1} x)}{3^n} - \frac{g(3^n x) - h(3^n x)}{3^n}, t\right) \geq_{L^*} (N_s(\epsilon), \epsilon), \tag{18}$$

for all $x \in X$, $t > 0$, $n \geq m$. For fixed $x \in X$ and $m \in \mathbb{N}$, there exists $m' \in \mathbb{N}$ with $m' \geq m$ such that

$$P_{\mu,\nu}\left(\frac{g(3^m x) - h(3^m x)}{3^{m'}}, t\right) \geq_{L^*} (N_s(\epsilon), \epsilon), \tag{19}$$

for all $t > 0$. Now, for $n \geq m'$:

$$\begin{aligned} &P_{\mu,\nu}\left(\frac{g(3^n x) - h(3^n x)}{3^n}, t\right) \geq_{L^*} P_{\mu,\nu}\left(\frac{g(3^n x) - h(3^n x)}{3^m}, t\right) \quad [\because n \geq m] \\ &\geq_{L^*} \tau^{n-m}\left(P_{\mu,\nu}\left(\frac{g(3^m x) - h(3^m x)}{3^{m'}}, \frac{t}{(n-m+1)3^{m'-m}}\right),\right. \\ &\qquad P_{\mu,\nu}\left(\frac{g(3^{m+1} x) - h(3^{m+1} x)}{3^m} - \frac{g(3^m x) - h(3^m x)}{3^m}, \frac{t}{n-m+1}\right), \ldots, \\ &\qquad \left. P_{\mu,\nu}\left(\frac{g(3^n x) - h(3^n x)}{3^{n-1}} - \frac{g(3^{n-1} x) - h(3^{n-1} x)}{3^{n-1}}, \frac{t}{(n-m+1)3^{n-m-1}}\right)\right) \\ &\qquad \geq_{L^*} (N_s(\epsilon), \epsilon) \quad [\text{by } (18), (19)]. \end{aligned}$$

Thus,

$$\frac{g(3^n x)}{3^n} - \frac{h(3^n x)}{3^n} \to 0 \text{ as } n \to \infty. \tag{20}$$

From (14) and (20) we get (5). This completes the proof of the theorem. □

Corollary 1.

Let $\psi : [a, \infty) \to \mathbb{R}^+$ be a mapping such that

$$(i)\, \psi(ts) \le \psi(t)\psi(s),\ (ii)\, \frac{\psi(3)}{3} < 1,$$

where a is a fixed real number satisfying $0 \le a \le 3$. Let $(Y, P_{\mu,\nu}, M)$ be a complete IFN-space and $(Z, P'_{\mu,\nu}, M)$ be an IFN-space, where M is given in Example 1.Let $z_0 \in Z$. If $f, g, h : X \to Y$ are mappings such that

$$P_{\mu,\nu}(f(x + y) - g(x) - h(y), t) \ge_{L^*} P'_{\mu,\nu}((\psi(\|x\|) + \psi(\|y\|))z_0, t),$$

for all $x, y \in X, t > 0$ with $\|x\|, \|y\| \ge a$, then there exists a unique additive mapping $A : X \to Y$ such that

$$P_{\mu,\nu}(f(x) - A(x) - f(0), t) \\ \ge_{L^*} P'_{\mu,\nu}\left(\psi(\|x\|)z_0, \min\left\{\frac{1}{\psi\left(\frac{1}{2}\right) + \psi\left(\frac{3}{2}\right)}, \frac{1}{2\psi\left(\frac{1}{2}\right)}, \frac{1}{2\psi\left(\frac{3}{2}\right)}\right\}(3 - \psi(3))\frac{t}{8}\right)$$

and

$$\frac{f(3^n x)}{3^n} \to A(x),\ \frac{g(3^n x)}{3^n} \to A(x),\ \frac{h(3^n x)}{3^n} \to A(x) \quad \text{as } n \to \infty,$$

for all $x \in X$ with $\|x\| \ge 2a$ and $t > 0$.

Proof:

Define $\phi(x, y) = (\psi(\|x\|) + \psi(\|y\|))z_0$ and take $\alpha = \psi(3)$. Clearly, (1) is satisfied and $0 < \alpha < 3$. Then

$$M_1(x, t) \ge_{L^*} P'_{\mu,\nu}\left(\psi(\|x\|)z_0, \min\left\{\frac{1}{\psi\left(\frac{1}{2}\right) + \psi\left(\frac{3}{2}\right)}, \frac{1}{2\psi\left(\frac{1}{2}\right)}, \frac{1}{2\psi\left(\frac{3}{2}\right)}\right\}\frac{t}{4}\right).$$

□

Corollary 2. Let $(Y, P_{\mu,\nu}, M)$ be a complete IFN-space and $(Z, P'_{\mu,\nu}, M)$ be an IFN-space, where M is given in Example 2.7. Let $z_0 \in Z$ and $p < 1$. If $f, g, h : X \to Y$ are mappings such that

$$P_{\mu,\nu}(f(x + y) - g(x) - h(y), t) \ge_{L^*} P'_{\mu,\nu}(((\|x\|)^p + (\|y\|)^p)z_0, t),$$

for all $x, y \in X, t > 0$, then there exists a unique additive mapping $A : X \to Y$ such that

$$P_{\mu,\nu}(f(x) - A(x) - f(0), t) \ge_{L^*} P'_{\mu,\nu}\left(\|x\|^p z_0, \frac{2^p(3 - 3^p)\, t}{3^p\, 16}\right)$$

and

$$\frac{f(3^n x)}{3^n} \to A(x),\ \frac{g(3^n x)}{3^n} \to A(x),\ \frac{h(3^n x)}{3^n} \to A(x) \quad as\, n \to \infty,$$

for all $x \in X, t > 0$.

Proof:

Define $\phi(x, y) = (\|x\|^p + \|y\|^p)z_0$ and take $\alpha = 3^p$. Clearly, (1) is satisfied and $0 < \alpha < 3$. Then

$$M_1(x, t) = P'_{\mu,\nu}\left(\|x\|^p z_0, \frac{2^p t}{3^p 8}\right).$$

□

Theorem 2.

Let $(Y, P_{\mu,\nu}, \tau)$ be a complete IFN-space and $(Z, P'_{\mu_1,\nu_1}, \tau)$ be an IFN-space. Let $\phi : X^2 \to Z$ be a mapping such that

$$P'_{\mu,\nu}\left(\phi\left(\frac{x}{3}, \frac{y}{3}\right), t\right) \geq_{L^*} P'_{\mu,\nu}\left(\frac{1}{\alpha}\phi(x, y), t\right),$$

for all $x, y \in X, t > 0$ and for some $\alpha > 3$. If $f, g, h : X \to Y$ are mappings such that

$$P_{\mu,\nu}(f(x + y) - g(x) - h(y), t) \geq_{L^*} P'_{\mu,\nu}(\phi(x, y), t),$$

for all $x, y \in X$ and $t > 0$, then there exists a unique additive mapping $A : X \to Y$ such that for all $x \in X, t > 0$,

$$P_{\mu,\nu}(f(x) - A(x) - f(0), t) \geq_{L^*} M_1\left(x, (\alpha - 3)\frac{t}{2}\right)$$

and

$$3^n(f(3^{-n}x) - f(0)) \to A(x), 3^n(g(3^{-n}x) - f(0)) \to A(x), 3^n(h(3^{-n}x) - f(0)) \to A(x),$$

as $n \to \infty$, where M_1 is given in Theorem 1.

4. Conclusion

In this paper, the Hyers-Ulam-Rassias stability of $f(x + y) = g(x) + h(y)$, the Pexiderized functional equation, has been discussed in intuitionistic fuzzy Banach spaces. But instead of considering the crisp mappings f, g, h if we consider the fuzzy mappings how the Hyers-Ulam-Rassias stability of the corresponding Pexiderized functional equation can be established in intuitionistic fuzzy Banach spaces. It is very important to investigate the Hyers-Ulam-Rassias stability of the fuzzy functional equations in intuitionistic fuzzy Banach spaces.

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