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Analysis of repairable $M^{[X]}/(G_1,G_2)/1$ - feedback retrial G-queue with balking and starting failures under at most J vacations

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Abstract:

In this paper, we discuss the steady state analysis of a batch arrival feedback retrial queue with two types of services and negative customers. Any arriving batch of positive customers finds the server is free, one of the customers from the batch enters into the service area and the rest of them get into the orbit. The negative customer, is arriving during the service time of a positive customer, will remove the positive customer in-service and the interrupted positive customer either enters the orbit or leaves the system. If the orbit is empty at the service completion of each type of service, the server takes at most J vacations until at least one customer is received in the orbit when the server returns from a vacation. While the busy server may breakdown at any instant and the service channel may fail for a short interval of time. The steady state probability generating function for the system size is obtained by using the supplementary variable method. Numerical illustrations are discussed to see the effect of the system parameters.

Keywords: Bulking; Feedback; Balking; G –queue; Breakdown; Starting failures; Steady-state solution

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1. Introduction

Queueing system is a powerful tool for modeling communication networks, transportation networks, production lines and operating systems. In recent years, computer networks and data communication systems are the fastest growing technologies, which have led to significant development in applications such as swift advance in the internet, audio data traffic, video data traffic, etc. In the retrial literature many of the researchers discussed a retrial queueing models from various viewpoints. Recently, retrial queues have been investigated extensively due to their applications in various fields, such as telephone switching systems, call centers and telecommunication networks with retransmission and computers. The characteristic of retrial queueing systems is that customers who find the server busy upon arrival is obliged to leave the service area and repeat his demand after some time (retrial time). Between trials, a blocked customer who remains in a retrial group is said to be in orbit. For detailed overviews of the related literatures on retrial queues, readers are referred to the books of Falin and Templeton (1997), Artalejo and Gomez-Corral (2008) and the survey papers of Artalejo (2010).

Many queueing situations have the feature that the customers may be served repeatedly for a certain reason. When the service of a customer is unsatisfied, it may be retried again and again until a successful service completion. These queueing models arise in the stochastic modeling of many real-life situations. For example, in data transmission, a packet transmitted from the source to the destination may be returned and it may go on like that until the packet is finally transmitted. Krishnakumar et al. (2013) studied a model with the concept of M/G/1 feedback retrial queueing system with negative customers. Ke and Chang (2009a) have discussed Modified vacation policy for $M/G/1$ retrial queue with balking and feedback.

The concept of balking (customers decide not to join the line at all if he finds the server is unavailable upon arrival) was first studied by Haight in 1957. There are many situations where the customers may be impatient, such as impatient telephone switchboard customers and the hospital emergency rooms handling critical patients, web access, including call centers and computer systems, etc. Ke (2007) studied the $M^{[X]}/G/1$ queue with variant vacations and balking. Some of the authors like Wang and Li (2009) and Gao and Wang (2014) discussed about the concept balking.

Recently, many researchers have studied queueing networks with concept of positive and negative customers. Queues with negative customers (also called *G*–queues) have attracted considerable interests due to their extensive applications, such as computer, communication networks and manufacturing system. The positive customers arrive at the system and get their service in normal manner. The negative customers arrive into the system only at the service time of a positive customer. These customers do not join in the queue and do not get any service. The negative customers will vanish and reduce one positive customer in service, then the positive customer may join into the queue for another regular service or may leave the system. Negative customers have been regarded as virus, inhibitor signals, operation mistakes or system and server disaster in neural and computer communication networks. Some of the authors like, Wang and Zhang (2009), Wang et al. (2011), Yang et al. (2013), Wu and Lian (2013), Krishnakumar et al. (2013), Gao and Wang (2014) discussed different types of queueing models operating with the simultaneous presence of *G*-queues.

In a vacation queueing system, the server may not be available for a period of time due to many reasons such as, being checked for maintenance, working at other queues, scanning for new work (a typical aspect of many communication systems) or simply taking a break. This period of time, when the server is unavailable for primary customers is referred to as a vacation. Krishnakumar and Arivudainambi (2002) have investigated a single server retrial queue with Bernoulli schedule, where the random decision whether to take a vacation or not allowed only at instances when the system is not empty (and a service or vacation has just been completed). If the system is empty, the server must take a vacation that is the assumption for their model. Chang and Ke (2009) examined a batch retrial model with *J* vacations in which, if the orbit becomes empty, the server takes at most *J* vacations repeatedly until at least one customer appears in the orbit upon returning from a vacation. By applying the supplementary variable technique, system characteristics are derived. Later, Ke and Chang (2009a) and Chen et al. (2010), Rajadurai et al. (2014) discussed the different types of queueing model with *J* vacation queueing models.

The service interruptions are an unavoidable phenomenon in many real life situations. In most of the studies, it is assumed that the server is available in the service station on a permanent basis and the service station never fails. However, these assumptions are practically unrealistic. In practice we often meet the case where service stations may fail and can be repaired. Applications of these models found in the area of computer communication networks and flexible manufacturing system etc. Ke and Chang (2009b) have studied a batch arrival retrial queueing system with two phases of service under the concept of Bernoulli vacation schedules, where the server may meet an unpredictable breakdown subject to starting a failure when a customer requires his service initially. Ke and Choudhury (2012) discussed a batch arrival retrial queueing system with two phases of service under the concept of breakdown and delaying repair. While the busy server may breakdown at any instant and the service channel will fail for a short period of time. The repair process does not start immediately after a breakdown and there is a delay time for repair to start. Choudhury and Deka (2012) have discussed a single server queueing model with two phases of service, where the server is subject to breakdown. Some of the authors like Wang and Li (2009), Chen et al. (2010), Choudhury et al. (2010) and Rajadurai et al. (2015) are discussed about the retrial queueing systems with the concept of breakdown and repair. Recently, Haghighi and Mishev (2013) discussed three possible stages for the handling of job applications in a hiring process as a network queuing model.

However, no work has been published in the queueing literature with the combination batch arrival retrial queue, two types of service, *G*-queues, balking, feedback, modified vacation (at most *J* vacations) and breakdowns. The mathematical results and theory of queues of this model seems to provide a specific and convincing application in the transfer model of an email system. In a Simple Mail Transfer Protocol (SMTP) the mail system is used to deliver the messages between mail servers.

The rest of this work is organized as follows. In Section 2, the detailed description and practical justification of this model are given. In Section 3, we consider the governing equations of the model and also obtain the steady state solutions. Some performance measures are derived in Section 4. In Section 5, some special cases are discussed. In Section 6 the effects of various parameters on the system performance are analyzed numerically. Summary of the work is presented in Section 7.

2. Model Description

In this paper, we consider a batch arrival feedback retrial queueing system with two types of service, negative customers under modified vacation policy where the server is subject to starting failure, breakdown and repair. The detailed description of the model is given as follows:

Arrival Process: Positive customers arrive in batches according to a compound Poisson process with rate λ. Let X_k denote the number of customers belonging to the k^{th} arrival batch, where X_k, k = 1, 2, 3…are with a common distribution $Pr[X_k = n] = \chi_n$, n = 1, 2, 3… and $X(z)$ denotes the probability generating function of X. We denote $X^{[k]}$ as the k^{th} factorial moment of $X(z)$ for (k =1,2).

Retrial process: We assume that there is no waiting space and therefore if an arriving batch finds the server free, one of the customers from the batch begins his service and the rest of them get into the orbit. If an arriving batch of customers find the server busy, vacation or breakdown, the arrivals either leave the service area with probability 1 - b or join the pool of blocked customers called an orbit with probability b. Inter retrial times have an arbitrary distribution $R(t)$ with corresponding Laplace-Stieltjes Transform (LST) $R^*(\vartheta)$.

Service process: A single server provides the two types of service. If any batch of arriving positive customers finds the server free, then one of the customers from the batch is allowed to start the First Type Service (FTS) with probability p_1 or Second Type Service (STS) with probability p_2 ($p_1+p_2=1$) while the others join the orbit. It is assumed that the i^{th} (i = 1, 2) type service times follows a general random variable S_i with distribution function $S_i(t)$ and LST $S_i^*(\vartheta)$.

Starting failure repair process: If any batch of arriving positive customers find a server free, only the customer at the head of batch arriving is allowed to start (turn on) the server and the others leave the service area and join the orbit. On the other hand, the service discipline for the customers in the orbit is first retry success first service (FRSFS). If a returning customer finds a server free (retry successfully), the customer must start (turn on) the server. The startup time of server could be negligible. Moreover, the server may be a starting failure with a probability $\bar{\alpha} = 1-\alpha$. If the server is started successfully, the customer gets service immediately. Otherwise, the server is repaired immediately and the customer must leave the service area and make a retrial at later time. That is, the probability of successful commencement of service is α for a new and returning customer. Note that the repair time of the failure server is of random length H with distribution function $H(t)$, LST $H^*(\vartheta)$ and finite k^{th} moment $h^{(k)}$ (k = 1, 2).

Feedback rule: After completion of a type1 service (type 2 service) for each positive customer, the unsatisfied positive customers may rejoin the orbit as a feedback customer for receiving another regular service with probability p ($0 \le p \le 1$) or may leave the system with probability q ($p+q=1$).

Negative arrival process: The negative customers arrive from outside the system according to a Poisson arrival rate δ. These negative customers arrive only at the service time of the positive customers. Once the negative customer comes into the system it will remove the positive customer in-service and the interrupted positive customer either enters into the orbit with probability θ $(0 \leq \theta \leq 1)$ or leaves the system forever with probability $1\text{-}\theta$.

Vacation process: Whenever the orbit is empty, the server leaves for a vacation of random length V. If no customer appears in the orbit when the server returns from a vacation, it leaves again for another vacation with the same length. This pattern continues until it returns from a vacation to find at least one customer in the orbit or it has already taken J vacations. If the orbit is empty at the end of the J^{th} vacation, the server remains idle for new arrivals in the system. At a vacation completion epoch the orbit is nonempty, the server waits for the customers in the orbit or for a new arrival. The vacation time V has a distribution function $V(t)$, LST $V^*(\vartheta)$ and finite k^{th} moment $\nu^{(k)}$ $(k = 1, 2)$.

Breakdown process: While the server is working with any type of service, it may breakdown at any time and the service channel will fail for a short interval of time, i.e., the server is down for a short interval of time. The breakdowns, i.e., the server's life times are generated by exogenous Poisson processes with rates α_1 for FTS and α_2 for STS, which we may call some sort of disaster during FTS and STS periods respectively.

Repair process: As soon as a breakdown occurs the server is sent for repair, during that time it stops providing service to the primary customers till the service channel is repaired. The customer who was just being served before server breakdown waits for the remaining service to be completed. The repair time (denoted by G_1 for FTS and G_2 for STS) distributions of the server for both types of service are assumed to be arbitrarily with distribution function $G_i(t)$, Laplace-Stieltjes Transform $G_i^*(\vartheta)$ and finite k^{th} moment $g_i^{(k)}$ (for $i = 1, 2$ and $k = 1, 2$). The various stochastic processes involved in the system are assumed to be independent of each other.

2.1. Practical justification of the suggested model

The suggested model has potential application in the transfer model of an email system. In a Simple Mail Transfer Protocol (SMTP) the mail system is used to deliver the messages between mail servers. When a mail transfer program contacts a server on a remote machine, it forms a Transmission Control Protocol (TCP) connection over which it communicates. Once the connection is in place, the two programs follow SMTP that allows the sender to identify it, specify a recipient and transfer an e-mail message. For receiving a group of messages, client applications usually use either the Post Office Protocol (POP) or the Internet Message Access Protocol (IMAP) to access their mail box accounts on a mail server. Typically, contacting a group of messages arrive at the mail server following the Poisson stream.

When messages arrive at the mail server, it will be free and one of the messages from the group is selected to access successfully (in POP or IMAP) and the rests will join the buffer. In the buffer, each message waits and requires its service again after some time. If the server is initially a failure, all the arriving group of messages will join into the buffer and try his service after some time. The target server is the same as sender's mail server and the sending message will be possibly retransmitted to the server to request the receiving service once again from the buffer. The mail server may be subject to electronic failure during the service period and receive repair immediately. Meanwhile, the working server may receive additional tasks like the flow of triggers/viruses. Upon arrival at the e-mail servers system, the trigger instantaneously displaces the message being served at the receiver mail server from the receiver mail to the buffer or is forced to leave the system.

To keep the mail server functioning well, virus scanning is an important maintenance activity for the mail server. It can be performed when the mail server is idle. This type of maintenance can be programmed to perform on a regular basis. However, these maintenance activities do not repeat continuously. When these activities are finished, the mail server will enter the idle state again and wait for the contact messages to arrive. Because there is no mechanism to record how many contacting messages from various senders currently, it is appropriate to design a program for collecting information of contacting messages for the reason of efficiency. In this queueing scenario, the buffer in the sender mail server, the receiver mail server, the POP and IMAP, the initial failure, the retransmission policy, flow of triggers/viruses and the maintenance activities correspond to the orbit, the server, the Type1 and Type2 service, the starting failure repair process, the feedback policy, the arrival of negative customers and the vacation policy respectively.

This model finds other practical applications in Verteiler Ensprintz Pumps manufacturing, in computer networking systems, manufacturing systems and communication systems etc., For example, in the process of cell transfer, if the interference of a virus causes an information element transmission failure, then some kinds of virus can be seen as negative customers. In computer networking systems, if the virus enters a node, one or more files may be infected. A virus may originate from outside the network, e.g., through a floppy disk, or by an electronic mail, production lines, in the operational model of WWW server for HTTP requests, call centers, inventory and production, maintenance and quality control in industrial organizations etc.

3. System Analysis

In this section, we first develop the steady state difference-differential equations for the retrial system by treating the elapsed retrial times, the elapsed service times, the elapsed vacation times and the elapsed repair times as supplementary variables. Then we derive the probability generating function (PGF) for the server states, the PGF for the number of customers in the system and orbit by using the supplementary variable method.

In steady state, we assume that $R(0) = 0$, $R(\infty) = 1$, $S_i(0) = 0$, $S_i(\infty)= 1$, $V_j(0) = 0$, $V_j(\infty) = 1$, $H(0)=0$, $H(\infty) = 1$ (for i =1, 2 and $j = 1, 2, \ldots, J$) are continuous at $x = 0$ and $G_i(0) = 0$, $G_i(\infty) = 1$ (for i =1, 2) are continuous at $x = 0$ and $y = 0$. So that the function $a(x)$, $\mu_i(x)$, $\gamma(x)$, $\eta(x)$ and $\xi_i(y)$ are the conditional completion rates for retrial, service on both types, on vacation and repair on both types respectively (for $i = 1, 2$).

$$a(x)dx = \frac{dR(x)}{1-R(x)},\ \mu_i(x)dx = \frac{dS_i(x)}{1-S_i(x)},$$

$$\gamma(x)dx = \frac{dV(x)}{1-V(x)},\ \eta(x)dx = \frac{dH(x)}{1-H(x)} \text{ and } \xi_i(y)dy = \frac{dG_i(y)}{1-G_i(y)}.$$

In addition, let $R^0(t),\ S_i^0(t),\ V_j^{\ 0}(t),\ H^0(t)$ and $G_i^0(t)$ be the elapsed retrial times, service times on both types, vacation times, repair times on starting failure server or repair times on both types (for i=1,2 and j = 1, 2, 3, 4, …, J+3) respectively at time t. We also note that the states of the system at time t can be described by the bivariate Markov process $\{C(t), N(t);\ t \ge 0\}$ where $C(t)$ denotes the server state (0,1, 2, 3, 4, …, J+3) depending on the server is idle, busy on FTS or STS, repair on starting failure server, repair on FTS or STS and 1st vacation, 2nd vacation, …, Jth vacation. $N(t)$ denotes the number of customers in the orbit. If $C(t)$ = 0 and $N(t)$ > 0, then $R^0(t)$ represent the elapsed retrial time, if $C(t)$ = 1 and $N(t) \ge 0$ then $S_i^0(t)$ corresponding to the elapsed time of the customer being served on FTS (STS) (for i = 1, 2). If $C(t)$ = 2 and $N(t) \ge 0$, then $H^0(t)$ corresponding to the elapsed time of the failure server being repaired. If $C(t)$=3 and $N(t) \ge 0$, then $G_i^0(t)$ corresponding to the elapsed time of the server being repaired on FTS (STS) (for i = 1, 2). If $C(t)$ = 4 and $N(t) \ge 0$, then $V_1^{\ 0}(t)$ corresponding to the elapsed 1st vacation time. If $C(t)$ = j+4 and $N(t) \ge 0$, then $V_j^0(t)$ corresponding to the elapsed jst vacation time.

Let $\{t_n;\ n = 1, 2, ...\}$ be the sequence of epochs at which either a type 1 or type 2 service completion occurs, a vacation period ends or a repair period ends. The sequence of random vectors $Z_n = \{C(t_n+), N(t_n+)\}$ forms a Markov chain which is embedded in the retrial queueing system. It follows from Appendix that $\{Z_n;\ n \in N\}$ is ergodic if and only if $\rho < 1$, then the system will be stable, where $\rho = X^{[1]}(1 - R^*(\lambda)) + \varpi$ and

$$\varpi = \left\{ \begin{array}{l} \alpha\Big(\theta - (\theta - p)\Big(p_1 S_1^*(\delta) + p_2 S_2^*(\delta)\Big)\Big) + \bar{\alpha}\Big(\lambda b X^{[1]} h^{(1)} + 1\Big) \\ + \dfrac{\alpha\lambda b X^{[1]}}{\delta}\Big(p_1\Big(1 + \alpha_1 g_1^{(1)}\Big)\Big(1 - S_1^*(\delta)\Big) + p_2\Big(1 + \alpha_2 g_2^{(1)}\Big)\Big(1 - S_2^*(\delta)\Big)\Big) \end{array} \right\}.$$

For the process $\{N(t),\ t \ge 0\}$ we define the probabilities $P_0(t) = P\{C(t) = 0,\ N(t) = 0\}$ and the probability densities for $(i = (1,2),\ t \ge 0,\ (x, y) \ge 0$ and $n \ge 0)$

$$\begin{aligned} P_n(x,t)dx &= P\Big\{C(t) = 0, N(t) = n,\ x \le R^0(t) < x + dx\Big\}; \\ \Pi_{i,n}(x,t)dx &= P\Big\{C(t) = 1, N(t) = n,\ x \le S_i^{\ 0}(t) < x + dx\Big\}; \\ Q_n(x,t)dx &= P\Big\{C(t) = 2,\ N(t) = n,\ x \le H^0(t) < x + dx\Big\}; \\ R_{i,n}(x,y,t)dy &= P\Big\{C(t) = 3,\ N(t) = n,\ y \le G_i^{\ 0}(t) < y + dy \big/ S_i^{\ 0}(t) = x\Big\}; \\ \Omega_{j,n}(x,t)dx &= P\Big\{C(t) = j + 3,\ N(t) = n,\ x \le V_j^{\ 0}(t) < x + dx\Big\}, \text{for } (1 \le j \le J). \end{aligned}$$

The following probabilities are used in subsequent sections:

$P_0(t)$ is the probability that the system is empty at time t. $P_n(x,t)$ is the probability that at time t there are exactly n customers in the orbit and the elapsed retrial time of the test customer undergoing retrial lying in between x and x+dx. $\Pi_{i,n}(x,t), \mathrm{i} = 1,2,$, is the probability that at time t there are exactly n customers in the orbit and the elapsed service time of the test customer undergoing service lying in between x and x+dx in their respective types. $Q_n(x,t)$ is the probability that at time t there are exactly n customers in the orbit and the elapsed repair time of server lying in between x and $x+dx$ on the failure server. $R_{i,n}(x,y,t), \mathrm{i}=1,2,$ is the probability that at time t there are exactly n customers in the orbit, the elapsed service time of the test customer undergoing service is x and the elapsed repair time of server lying in between y and $y+dy$ in their respective types. $\Omega_{j,n}(x,t), j=1,2,\cdots,J,$ is the probability that at time t there are exactly n customers in the orbit and the elapsed vacation time of the vacation lying in between x and $x+dx$.

We assume that the stability condition is fulfilled so that we can set for $t \geq 0$, $x \geq 0$, $n \geq 1$ and (i = 1, 2 and j = 1,2, …, J)

$$P_0 = \lim_{t\to\infty} P_0(t),\ P_n(x) = \lim_{t\to\infty} P_n(x,t),\ \prod_{i,n}(x) = \lim_{t\to\infty} \prod_{i,n}(x,t),\ Q_n(x) = \lim_{t\to\infty} Q_n(x,t),$$
$$\Omega_{j,n}(x) = \lim_{t\to\infty} \Omega_{j,n}(x,t) \text{ and } R_{i,n}(x,y) = \lim_{t\to\infty} R_{i,n}(x,y,t).$$

3.1. The steady-state equations

By the method of supplementary variable technique, we obtain the following system of equations that govern the dynamics of the system behavior.

$$\lambda b P_0 = \int_0^\infty \Omega_{J,0}(x)\gamma(x)dx\,, \tag{3.1}$$

$$\frac{dP_n(x)}{dx} + [\lambda + a(x)]P_n(x) = 0,\ n \geq 1\ , \tag{3.2}$$

$$\frac{d\Pi_{i,0}(x)}{dx} + [\lambda + \delta + \alpha_i + \mu_i(x)]\Pi_{i,0}(x) = \lambda(1-b)\Pi_{i,0}(x) + \int_0^\infty \xi_i(y)R_{i,0}(x,y)dy,\ n=0,\ \text{for}(i=1,2)\ , \tag{3.3}$$

$$\begin{aligned}\frac{d\Pi_{i,n}(x)}{dx} + [\lambda + \delta + \alpha_i + \mu_i(x)]\Pi_{i,n}(x) &= \lambda(1-b)\Pi_{i,n}(x) + \lambda b\sum_{k=1}^{n}\chi_k\Pi_{i,n-k}(x)\\ &+ \int_0^\infty \xi_i(y)R_{i,n}(x,y)dy,\ n\geq 1,\ \text{for}\,(i=1,2)\end{aligned}\ , \tag{3.4}$$

$$\frac{dQ_0(x)}{dx} + [\lambda + \eta(x)]Q_0(x) = \lambda(1-b)Q_{i,0}(x),\ n=0, \tag{3.5}$$

$$\frac{dQ_n(x)}{dx} + [\lambda + \eta(x)]Q_n(x) = \lambda(1-b)Q_{i,n}(x) + \lambda b\sum_{k=1}^{n}\chi_k Q_{i,n-k}(x),\ n\geq 1\,, \tag{3.6}$$

$$\frac{dR_{i,0}(x,y)}{dy} + [\lambda + \xi_i(y)]R_{i,0}(x,y) = \lambda(1-b)R_{i,0}(x,y),\ n=0,\ \text{for}\,(i=1,2)\ , \tag{3.7}$$

$$\frac{dR_{i,n}(x,y)}{dy} + [\lambda + \xi_i(y)]R_{i,n}(x,y) = \lambda(1-b)R_{i,n}(x,y) + \lambda\sum_{k=1}^{n}\chi_k R_{i,n-k}(x,y),\ \ n\geq 1,\ \text{for}\,(i=1,2), \tag{3.8}$$

$$\frac{d\Omega_{j,0}(x)}{dx} + [\lambda + \gamma(x)]\Omega_{j,0}(x) = \lambda(1-b)\Omega_{j,0}(x),\ n=0,\ \text{for}\,(j=1,2,...,J), \tag{3.9}$$

$$\frac{d\Omega_{j,n}(x)}{dx} + [\lambda + \gamma(x)]\Omega_{j,n}(x) = \lambda(1-b)\Omega_{j,n}(x) + \lambda b\sum_{k=1}^{n}\chi_k\Omega_{j,n-k}(x),\ \ n\geq 1,\ \text{for}\,(j=1,2,...,J)\ . \tag{3.10}$$

The steady-state boundary conditions at x = 0 and y = 0 are

$$
\begin{aligned}
P_n(0) = \sum_{j=1}^{J}\int_0^\infty \Omega_{j,n}(x)\gamma(x)dx + \int_0^\infty Q_n(x)\eta(x)dx + \theta\delta\left(\int_0^\infty \Pi_{1,n-1}(x)dx + \int_0^\infty \Pi_{2,n-1}(x)dx\right) \\
+(1-\theta)\delta\left(\int_0^\infty \Pi_{1,n}(x)dx + \int_0^\infty \Pi_{2,n}(x)dx\right) + q\left(\int_0^\infty \Pi_{1,n}(x)\mu_1(x)dx + \int_0^\infty \Pi_{2,n}(x)\mu_2(x)dx\right) \\
+ p\left(\int_0^\infty \Pi_{1,n-1}(x)\mu_1(x)dx + \int_0^\infty \Pi_{2,n-1}(x)\mu_2(x)dx\right), n \ge 1,
\end{aligned} \tag{3.11}
$$

$$
\Pi_{i,0}(0) = \alpha\, p_i\left\{\int_0^\infty P_1(x)a(x)dx + \lambda b\chi_1 P_0\right\},\ n=0,\ \text{for}\,(i\text{=}1,2), \tag{3.12}
$$

$$
\Pi_{i,n}(0) = \alpha\, p_i\left\{\int_0^\infty P_{n+1}(x)a(x)dx + \lambda\sum_{k=1}^{n}\chi_k\int_0^\infty P_{n-k+1}(x)dx + \lambda b\chi_{n+1}P_0\right\},\ n \ge 1,\ \ \text{for}\,(i\text{=}1,2), \tag{3.13}
$$

$$
Q_n(0) = \bar{\alpha}\left\{\int_0^\infty P_n(x)a(x)dx + \lambda\sum_{k=1}^{n}\chi_k\int_0^\infty P_{n-k}(x)dx + \lambda b\chi_n P_0\right\},\ n \ge 2, \tag{3.14}
$$

$$
\Omega_{1,n}(0) = q\left(\int_0^\infty \Pi_{1,0}(x)\mu_1(x)dx + \int_0^\infty \Pi_{2,0}(x)\mu_2(x)dx\right) + (1-\theta)\delta\left(\int_0^\infty \Pi_{1,0}(x)dx + \int_0^\infty \Pi_{2,0}(x)dx\right),\ n=0\,, \tag{3.15}
$$

$$
\Omega_{j,n}(0) = \begin{cases} \int_0^\infty \Omega_{j-1,n}(x)\gamma(x)dx,\ n=0, & j=2,3...,J, \\ 0, & n \ge 1, \end{cases} \tag{3.16}
$$

$$
R_{i,n}(x,0) = \alpha_i\Pi_{i,n}(x),\ \ n \ge 1,\ \text{for}\,(i=1,2)\ . \tag{3.17}
$$

The normalizing condition is

$$
\begin{aligned}
P_0 + \sum_{n=1}^{\infty}\left(\int_0^\infty P_n(x)dx + \int_0^\infty Q_n(x)dx\right) + \sum_{n=0}^{\infty}\sum_{j=1}^{J}\int_0^\infty \Omega_{j,n}(x)dx \\
+ \sum_{n=0}^{\infty}\sum_{i=1}^{2}\left(\int_0^\infty \Pi_{i,n}(x)dx + \int_0^\infty\int_0^\infty R_{i,n}(x,y)dxdy\right) = 1.
\end{aligned} \tag{3.18}
$$

3.2. The steady-state solution

The probability generating function technique is used here to obtain the steady state solution of the retrial queueing model. To solve the above equations, we define the generating functions for $|z| \le 1$, for (i =1, 2) as follows:

$$P(x,z)=\sum_{n=1}^{\infty}P_n(x)z^n;\ P(0,z)=\sum_{n=1}^{\infty}P_n(0)z^n;\ \Pi_i(x,z)=\sum_{n=0}^{\infty}\Pi_{i,n}(x)z^n;\Pi_i(0,z)=\sum_{n=0}^{\infty}\Pi_{i,n}(0)z^n;Q(x,z)=\sum_{n=1}^{\infty}Q_n(x)z^n;$$
$$Q(0,z)=\sum_{n=1}^{\infty}Q_n(0)z^n;\ \Omega_j(x,z)=\sum_{n=0}^{\infty}\Omega_{j,n}(x)z^n;\ \Omega_j(0,z)=\sum_{n=0}^{\infty}\Omega_{j,n}(0)z^n;R_i(x,y,z)=\sum_{n=0}^{\infty}R_{i,n}(x,y)z^n;$$
$$R_i(x,0,z)=\sum_{n=0}^{\infty}R_{i,n}(x,0)z^n \text{ and } X(z)=\sum_{n=1}^{\infty}\chi_n z^n.$$

Multiplying the steady-state equation and steady-state boundary conditions (3.1) - (3.17) by z^n and summing over n, n = 0,1, 2..., for i = 1,2 and j = 1,2,…J, we will have:

$$\frac{\partial P(x,z)}{\partial x}+[\lambda+a(x)]P(x,z)=0\ , \tag{3.19}$$

$$\frac{\partial \Pi_i(x,z)}{\partial x}+[\lambda b(1-X(z))+\delta+\alpha_i+\mu_i(x)]\Pi_i(x,z)=\int_0^{\infty}\xi_i(y)R_i(x,y,z)dy, \tag{3.20}$$

$$\frac{\partial Q(x,z)}{\partial x}+[\lambda b(1-X(z))+\eta(x)]Q(x,z)=0\ , \tag{3.21}$$

$$\frac{\partial R_i(x,y,z)}{\partial y}+[\lambda b(1-X(z))+\xi_i(y)]R_i(x,y,z)=0\ , \tag{3.22}$$

$$\frac{\partial \Omega_j(x,z)}{\partial x}+[\lambda b(1-X(z))+\gamma(x)]\Omega_j(x,z)=0\ , \tag{3.23}$$

$$\begin{aligned}P(0,z)=&\sum_{j=1}^{J}\int_0^{\infty}\Omega_j(x,z)\gamma(x)dx+\delta\left(1-\theta+\theta z\right)\left(\int_0^{\infty}\Pi_1(x,z)dx+\int_0^{\infty}\Pi_2(x,z)dx\right)+\int_0^{\infty}Q(x,z)\eta(x)dx\\&+\left(pz+q\right)\left(\int_0^{\infty}\Pi_1(x,z)\mu_1(x)dx+\int_0^{\infty}\Pi_2(x,z)\mu_2(x)dx\right)-\sum_{j=1}^{J}\Omega_{j,0}(0)-\lambda bP_0,n\geq 1,\end{aligned} \tag{3.24}$$

$$\Pi_i(0,z)=\alpha p_i\left\{\frac{1}{z}\int_0^{\infty}P(x,z)a(x)dx+\frac{\lambda X(z)}{z}\int_0^{\infty}P(x,z)dx+\frac{\lambda bX(z)P_0}{z}\right\},\ \text{for}\,(i\text{=}1,2), \tag{3.25}$$

$$Q(0,z)=\bar{\alpha}\left\{\int_0^{\infty}P(x,z)a(x)dx+\lambda X(z)\int_0^{\infty}P(x,z)dx+\lambda bX(z)P_0\right\}, \tag{3.26}$$

$$R_i(x,0,z)=\alpha_i\Pi_i(x,z)\,. \tag{3.27}$$

Solving the partial differential equations (3.19)-(3.23), it follows that:

$$P(x,z)=P(0,z)[1-R(x)]\exp\{-\lambda x\}\ , \tag{3.28}$$

$$\Pi_i(x,z)=\Pi_i(0,z)[1-S_i(x)]\exp\{-A_i(z)x\},\ \text{for}\,(i\text{=}1,2), \tag{3.29}$$

$$Q(x,z)=Q(0,z)[1-H(x)]\exp\{-b(z)x\}, \tag{3.30}$$

$$\Omega_j(x,z)=\Omega_j(0,z)[1-V(x)]\exp\{-b(z)x\},\text{for}\,(j=1,2,...,J), \tag{3.31}$$

and

$$R_i(x,y,z)=R_i(x,0,z)[1-G_i(y)]\exp\{-b(z)y\},\ \text{for}\,(i\text{=}1,2), \tag{3.32}$$

where

$$A_i(z) = \lambda b(1 - X(z)) + \delta + \alpha_i\left(1 - G_i^*\left(b(z)\right)\right) \text{ and } b(z) = \lambda b\left(1 - X(z)\right).$$

From (3.9) we obtain,

$$\Omega_{j,0}(x) = \Omega_{j,0}(0)[1 - V(x)]e^{-\lambda bx},\ j = 1, 2, ..., J. \tag{3.33}$$

Multiplying with equation (3.33) by $\gamma(x)$ on both sides for $j = J$ and integrating with respect to x from 0 to ∞, then from (3.1) we have:

$$\Omega_{J,0}(0) = \frac{\lambda b P_0}{V^*(\lambda b)}. \tag{3.34}$$

From equation (3.34) and solving (3.16), (3.33) over the range j = J-1, J-2,…,1, after some simplifications, we will have:

$$\Omega_{j,0}(0) = \frac{\lambda b P_0}{[V^*(\lambda b)]^{J-j+1}},\ j = 1, 2, ..., J - 1. \tag{3.35}$$

From (3.16), (3.34) and (3.35), we obtain

$$\Omega_j(0, z) = \frac{\lambda b P_0}{[V^*(\lambda b)]^{J-j+1}},\ j = 1, 2, ..., J. \tag{3.36}$$

Integrating the equation (3.33) from 0 to ∞ and using (3.34) and (3.35) again, we finally obtain

$$\Omega_{j,0}(0, z) = \frac{P_0\left(1 - V^*(\lambda b)\right)}{[V^*(\lambda b)]^{J-j+1}},\ j = 1, 2, ..., J. \tag{3.37}$$

Note that $\Omega_{j,0}$ represents the steady-state probability that no customer appears while the server is on the j^{th} vacation.

Let us define Ω_0 as the probability that no customer appears in the system while the server is on vacation. Then,

$$\Omega_0 = \frac{P_0\left(1 - [V^*(\lambda)]^J\right)}{b[V^*(\lambda)]^J}. \tag{3.38}$$

Inserting equations (3.29)-(3.31) and (3.36) in (3.37), we will have

$$\begin{aligned} P(0, z) = {} & \lambda b P_0\left(N(z) - 1\right) + Q(0, z)H^*\left(b(z)\right) + (pz + q)\left(\Pi_1(0, z)S_1^*\left(A_1(z)\right) + \Pi_2(0, z)S_1^*\left(A_1(z)\right)\right) \\ & + \frac{\delta(1 - \theta + \theta z)}{A_1(z)A_2(z)}\left(A_2(z)\Pi_1(0, z)\left(1 - S_1^*\left(A_1(z)\right)\right) + A_1(z)\Pi_2(0, z)\left(1 - S_2^*\left(A_2(z)\right)\right)\right), \end{aligned} \tag{3.39}$$

where

$$N(z)=\frac{1-[V^*(\lambda b)]^J}{[V^*(\lambda b)]^J\left(1-V^*(\lambda b)\right)}\left[V^*\left(\lambda b(1-X(z))\right)-1\right].$$

Inserting equation (3.28), (3.29) in (3.25) and make some manipulation, finally we will have,

$$\Pi_i(0,z)=\alpha p_i\left\{\frac{P(0,z)}{z}\left[R^*(\lambda)+X(z)(1-R^*(\lambda))\right]+\frac{\lambda bX(z)}{z}P_0\right\},\ \text{for } (i=1,2)\ . \tag{3.40}$$

Inserting equation (3.28), (3.29) in (3.26) and simplifying, we get,

$$Q(0,z)=\bar{\alpha}\left\{P(0,z)\left[R^*(\lambda)+X(z)(1-R^*(\lambda))\right]+\lambda bX(z)P_0\right\}, \tag{3.41}$$

Using the equation (3.29) in (3.27), gives

$$R_i(x,0,z)=\alpha_i\left(\Pi_i(0,z)\left[1-S_i(x)\right]\exp\left\{-A_i(z)x\right\}\right),\ \text{for } (i=1,2)\ . \tag{3.42}$$

Using (3.40)-(3.41) in (3.39), yeilds

$$P(0,z)=\frac{Nr(z)}{Dr(z)}, \tag{3.43}$$

$$Nr(z)=\lambda bP_0\left[\begin{array}{r}\left(z\left(N(z)-1\right)+X(z)\left\{\alpha(pz+q)\left(p_1S_1^*\left(A_1(z)\right)+p_2S_2^*\left(A_2(z)\right)\right)+\bar{\alpha}zH^*\left(b(z)\right)\right\}\right)A_1(z)A_2(z)\\ +X(z)\left\{\alpha\delta(1-\theta+\theta z)\left[p_1A_2(z)\left(1-S_1^*\left[A_1(z)\right]\right)+p_2A_1(z)\left(1-S_2^*\left[A_2(z)\right]\right)\right]\right\}\end{array}\right],$$

and

$$Dr(z)=\left\{\begin{array}{r}\left(z-\left\{\alpha(pz+q)\left(p_1S_1^*\left(A_1(z)\right)+p_2S_2^*\left(A_2(z)\right)\right)+\bar{\alpha}zH^*\left(b(z)\right)\right\}\left(R^*(\lambda)+X(z)(1-R^*(\lambda))\right)\right)A_1(z)A_2(z)\\ -\left(R^*(\lambda)+X(z)(1-R^*(\lambda))\right)\left\{\alpha\delta(1-\theta+\theta z)\left[p_1A_2(z)\left(1-S_1^*\left[A_1(z)\right]\right)+p_2A_1(z)\left(1-S_2^*\left[A_2(z)\right]\right)\right]\right\}\end{array}\right\}.$$

Also: Using (3.43) in (3.40), gives

$$\Pi_i(0,z)=\alpha\lambda bP_0p_i\left\{\left[\left(N(z)-1\right)\left(R^*(\lambda)+X(z)(1-R^*(\lambda))\right)+X(z)\right]A_1(z)A_2(z)\right\}\Big/Dr(z),\text{for } (i=1,2). \tag{3.44}$$

Using (3.43) in (3.41), gives

$$Q(0,z)=\bar{\alpha}z\lambda bP_0\left\{\left[\left(N(z)-1\right)\left(R^*(\lambda)+X(z)(1-R^*(\lambda))\right)+X(z)\right]A_1(z)A_2(z)\right\}\Big/Dr(z). \tag{3.45}$$

Using (3.44) in (3.42), gives

$$R_i(x,0,z)=\alpha\alpha_i\lambda bP_0p_i\left\{\begin{array}{r}\left[\left(N(z)-1\right)\left(R^*(\lambda)+X(z)(1-R^*(\lambda))\right)+X(z)\right]A_1(z)A_2(z)\\ \times\left(1-S_i(x)\right)e^{-A_i(z)x}\end{array}\right\}\Bigg/Dr(z),\text{for } (i=1,2). \tag{3.46}$$

Finally Using (3.43)-(3.46) in (3.28)-(3.32), then we will have the probability generating functions

$$P(x,z), \Pi_1(x,z),\ \ \Pi_2(x,z),\ \ H(x,z),\ \ R_1(x,y,z) \text{ and } R_2(x,y,z)..$$

Next we are interested in investigating the marginal orbit size distributions due to system state of the server.

Theorem 3.1.

Under the stability condition $\rho < 1$, the joint distributions of the number of customers in the system when server being idle, busy on both types, on vacation, under repair on starting failure server and under repair on both types (for i =1, 2) are given by

$$P(z) = \frac{Nr(z)}{Dr(z)}, \tag{3.47}$$

$$Nr(z) = b\left(1 - R^*(\lambda)\right) P_0 \left[\begin{array}{r} \left(z\left(N(z)-1\right) + X(z)\left\{\alpha(pz+q)\left(p_1 S_1^*\left(A_1(z)\right) + p_2 S_2^*\left(A_2(z)\right)\right) + \bar{\alpha} z H^*\left(b(z)\right)\right\}\right) A_1(z) A_2(z) \\ + X(z)\left\{\alpha\delta(1-\theta+\theta z)\left[p_1 A_2(z)\left(1 - S_1^*\left[A_1(z)\right]\right) + p_2 A_1(z)\left(1 - S_2^*\left[A_2(z)\right]\right)\right]\right\} \end{array} \right],$$

$$Dr(z) = \left\{ \begin{array}{l} \left(z - \left\{\alpha(pz+q)\left(p_1 S_1^*\left(A_1(z)\right) + p_2 S_2^*\left(A_2(z)\right)\right) + \bar{\alpha} z H^*\left(b(z)\right)\right\}\left(R^*(\lambda) + X(z)(1 - R^*(\lambda))\right)\right) A_1(z) A_2(z) \\ \quad -\left(R^*(\lambda) + X(z)(1 - R^*(\lambda))\right)\left\{\alpha\delta(1-\theta+\theta z)\left[p_1 A_2(z)\left(1 - S_1^*\left[A_1(z)\right]\right) + p_2 A_1(z)\left(1 - S_2^*\left[A_2(z)\right]\right)\right]\right\} \end{array} \right\},$$

$$\Pi_1(z) = \alpha\lambda b P_0 p_1 \left(1 - S_1^*\left(A_1(z)\right)\right)\left\{\left[\left(N(z)-1\right)\left(R^*(\lambda) + X(z)(1 - R^*(\lambda))\right) + X(z)\right] A_2(z)\right\} \Big/ Dr(z), \tag{3.48}$$

$$\Pi_2(z) = \alpha\lambda b P_0 p_2 \left(1 - S_2^*\left(A_2(z)\right)\right)\left\{\left[\left(N(z)-1\right)\left(R^*(\lambda) + X(z)(1 - R^*(\lambda))\right) + X(z)\right] A_1(z)\right\} \Big/ Dr(z), \tag{3.49}$$

$$Q(z) = \bar{\alpha} z \lambda b P_0 \left(1 - H^*\left(b(z)\right)\right)\left\{\left[\left(N(z)-1\right)\left(R^*(\lambda) + X(z)(1 - R^*(\lambda))\right) + X(z)\right] A_1(z) A_2(z)\right\} \Big/ b(z) Dr(z), \tag{3.50}$$

$$R_1(z) = \alpha\,\alpha_1 \lambda b P_0 p_1 \left\{ \begin{array}{r} \left[\left(N(z)-1\right)\left(R^*(\lambda) + X(z)(1 - R^*(\lambda))\right) + X(z)\right] A_2(z) \\ \times\left(1 - S_1^*\left(A_1(z)\right)\right)\left(1 - G_1^*\left(b(z)\right)\right) \end{array} \right\} \Bigg/ b(z) Dr(z), \tag{3.51}$$

$$R_2(z) = \alpha\,\alpha_2 \lambda b P_0 p_2 \left\{ \begin{array}{r} \left[\left(N(z)-1\right)\left(R^*(\lambda) + X(z)(1 - R^*(\lambda))\right) + X(z)\right] A_1(z) \\ \times\left(1 - S_2^*\left(A_2(z)\right)\right)\left(1 - G_2^*\left(b(z)\right)\right) \end{array} \right\} \Bigg/ b(z) Dr(z), \tag{3.52}$$

$$\Omega_j(z) = \frac{P_0\left(\left[V^*\left(\lambda b(1 - X(z))\right)\right] - 1\right)}{\left(X(z)-1\right)[V^*(\lambda b)]^{J-j+1}},\ j = 1, 2, ..., J, \tag{3.53}$$

where

$$P_0 = \frac{1}{\beta}\left\{1 - X^{[1]}(1 - R^*(\lambda)) - \varpi\right\}, \tag{3.54}$$

$$\beta = \left\{\begin{aligned} & X^{[1]}(1-R^*(\lambda))(b-1)\left(\frac{N'(1)}{X^{[1]}} + \bar{\alpha}\lambda b h^{(1)} + \frac{\alpha\lambda b}{\delta}\left(p_1\left(1+\alpha_1 g_1^{(1)}\right)\left(1-S_1^*(\delta)\right) + p_2\left(1+\alpha_2 g_2^{(1)}\right)\left(1-S_2^*(\delta)\right)\right)+1\right) \\ & \qquad - \alpha\left(\theta - 1 - (\theta - p)\left(p_1 S_1^*(\delta) + p_2 S_2^*(\delta)\right)\right)\left(\frac{N'(1)}{X^{[1]}} + b(1-R^*(\lambda)) + 1\right) \end{aligned}\right\},$$

and

$$N'(1) = \frac{\left\{1 - \left[V^*(\lambda b)\right]^J\right\}\lambda b X^{[1]} v^{(1)}}{\left[V^*(\lambda b)\right]^J \left(1 - V^*(\lambda b)\right)}, A_i(z) = b(z) + \delta + \alpha_i\left[1 - G_i^*\left(b(z)\right)\right] \text{ and } b(z) = \lambda b\left(1 - X(z)\right).$$

Proof:

Integrating the above equations (3.28) - (3.31) with respect to x and define the partial probability generating functions as,

$$P(z) = \int_0^\infty P(x,z)dx,\ \Pi_i(z) = \int_0^\infty \Pi_i(x,z)dx,\ Q(z) = \int_0^\infty Q(x,z)dx,\ \Omega_j(z) = \int_0^\infty \Omega_j(x,z)dx \quad \text{for } (i = 1,2).$$

Integrating the above equations (3.32) with respect to x and y define the partial probability generating functions as,

$$R_i(x,z) = \int_0^\infty R_i(x,y,z)dy,\ R_i(z) = \int_0^\infty R_i(x,z)dx \ \text{ for } (i = 1,2).$$

Since, the only unknown is P_0 the probability that the server is idle when no customer in the orbit and it can be determined using the normalizing condition (j = 1, 2, …, J). Thus, by setting z = 1 in (3.47) – (3.53) and applying the L 'Hospitals' rule whenever necessary and we get

$$P_0 + P(1) + \Pi_1(1) + \Pi_2(1) + Q(1) + R_1(1) + R_2(1) + \sum_{j=1}^{J}\Omega_j(1) = 1.$$

Theorem 3.2.

Under the stability condition $\rho < 1$, probability generating function of number of customers in the system and orbit size distribution at stationary point of time is

$$K_s(z) = P_0\left(\frac{Nr1(z)}{Dr1(z)} + \frac{Nr2(z)}{Dr1(z)} + \frac{Nr3(z)}{Dr1(z)}\right), \tag{3.55}$$

$$Nr1(z) = z\left\{\begin{array}{l}\left\{\alpha\left[1-\left(p_1S_1^*\left(A_1(z)\right)+p_2S_2^*\left(A_2(z)\right)\right)\right]+\bar{\alpha}z\left(1-H^*\left(b(z)\right)\right)\right\}A_1(z)A_2(z)\\ \qquad -\left\{\alpha\delta\left[p_1A_2(z)\left(1-S_1^*\left[A_1(z)\right]\right)+p_2A_1(z)\left(1-S_2^*\left[A_2(z)\right]\right)\right]\right\}\end{array}\right\}$$
$$\times\left\{\left(N(z)-1\right)\left(R^*(\lambda)+X(z)(1-R^*(\lambda))\right)+X(z)\right\},$$

$$Nr2(z) = -N(z)\left\{\begin{array}{l}\left(\begin{array}{l}zA_1(z)A_2(z)-\left\{\alpha(pz+q)\left(p_1S_1^*\left(A_1(z)\right)+p_2S_2^*\left(A_2(z)\right)\right)+\bar{\alpha}zH^*\left(b(z)\right)\right\}\\ \times\left(R^*(\lambda)+X(z)(1-R^*(\lambda))\right)\end{array}\right)\\ -\left(R^*(\lambda)+X(z)(1-R^*(\lambda))\right)\left\{\alpha\delta(1-\theta+\theta z)\left[\begin{array}{l}p_1A_2(z)\left(1-S_1^*\left[A_1(z)\right]\right)\\ +p_2A_1(z)\left(1-S_2^*\left[A_2(z)\right]\right)\end{array}\right]\right\}\end{array}\right\},$$

$$Nr3(z) = \left(1-X(z)\right)\left\{\begin{array}{l}zA_1(z)A_2(z)\left(1+b(1-R^*(\lambda))(N(z)-1)\right)-\left(R^*(\lambda)+X(z)(1-b)(1-R^*(\lambda))\right)\\ \times\left(\left\{\alpha(pz+q)\left(p_1S_1^*\left(A_1(z)\right)+p_2S_2^*\left(A_2(z)\right)\right)+\bar{\alpha}zH^*\left(b(z)\right)\right\}A_1(z)A_2(z)\right.\\ \left.+\left\{\alpha\delta(1-\theta+\theta z)\left[p_1A_2(z)\left(1-S_1^*\left[A_1(z)\right]\right)+p_2A_1(z)\left(1-S_2^*\left[A_2(z)\right]\right)\right]\right\}\right)\end{array}\right\},$$

$$Dr1(z) = \left(1-X(z)\right)\left\{\begin{array}{l}\left(\begin{array}{l}zA_1(z)A_2(z)-\left\{\alpha(pz+q)\left(p_1S_1^*\left(A_1(z)\right)+p_2S_2^*\left(A_2(z)\right)\right)+\bar{\alpha}zH^*\left(b(z)\right)\right\}\\ \times\left(R^*(\lambda)+X(z)(1-R^*(\lambda))\right)A_1(z)A_2(z)\end{array}\right)\\ -\left(R^*(\lambda)+X(z)(1-R^*(\lambda))\right)\left\{\alpha\delta(1-\theta+\theta z)\left[\begin{array}{l}p_1A_2(z)\left(1-S_1^*\left[A_1(z)\right]\right)\\ +p_2A_1(z)\left(1-S_2^*\left[A_2(z)\right]\right)\end{array}\right]\right\}\end{array}\right\},$$

$$K_o(z) = P_0\left(\frac{Nr4(z)}{Dr1(z)}+\frac{Nr2(z)}{Dr1(z)}+\frac{Nr3(z)}{Dr1(z)}\right), \tag{3.56}$$

and

$$Nr4(z) = \left\{\begin{array}{l}\left\{\alpha\left[1-\left(p_1S_1^*\left(A_1(z)\right)+p_2S_2^*\left(A_2(z)\right)\right)\right]+\bar{\alpha}z\left(1-H^*\left(b(z)\right)\right)\right\}A_1(z)A_2(z)\\ \qquad -\left\{\alpha\delta\left[p_1A_2(z)\left(1-S_1^*\left[A_1(z)\right]\right)+p_2A_1(z)\left(1-S_2^*\left[A_2(z)\right]\right)\right]\right\}\end{array}\right\}$$
$$\times\left\{\left(N(z)-1\right)\left(R^*(\lambda)+X(z)(1-R^*(\lambda))\right)+X(z)\right\},$$

where P_0 is given in equation (3.54).

Proof:

The probability generating function of the number of customer in the system $K_s(z)$ and the probability generating function of the number of customer in the orbit $K_o(z)$ are obtained by using

$$K_s(z) = P_0 + P(z) + z\left(Q(z) + \sum_{i=1}^{2}\left(\Pi_i(z) + R_i(z)\right)\right) + \sum_{j=1}^{J}\Omega_j(z) \text{ and } K_o(z)$$
$$= P_0 + P(z) + Q(z) + \sum_{i=1}^{2}\left(\Pi_i(z) + R_i(z)\right) + \sum_{j=1}^{J}\Omega_j(z).$$

Substituting (3.47) – (3.54) in the above results, the equations (3.55) and (3.56) can then be obtained by direct calculation.

4. Performance Measures

In this section, we obtain some interesting probabilities, when the system is in different states. We also derive the system performance measures. Since our results are numerically validated; Note that (3.54) gives the steady-state probability that the server is idle but available in the system. It follows from (3.47)-(3.53) that the probabilities of the server state are as follows in Theorem 4.1.

Theorem 4.1.

If the system satisfies the stability condition $\rho < 1$, then we will have the following probabilities:

(i) Let P denotes the steady-state probability that the server is idle during the retrial time, then

$$P = P(1) = \frac{1}{\beta}\left\{b\left(1 - R^*(\lambda)\right)\left\{N'(1) + X^{[1]} - 1 + \varpi\right\}\right\}.$$

(ii) Let Π_1 denote the steady-state probability that the server is busy on first type service with positive customer, then

$$\Pi_1 = \Pi_1(1) = \frac{1}{\beta}\left(\frac{1 - S_1^*(\delta)}{\delta}\right)\left\{\alpha\lambda b p_1\left[N'(1) + X^{[1]}R^*(\lambda))\right]\right\}.$$

(iii) Let Π_2 denote the steady-state probability that the server is busy on second type service with positive customer, then

$$\Pi_2 = \Pi_2(1) = \frac{1}{\beta}\left(\frac{1 - S_2^*(\delta)}{\delta}\right)\left\{\alpha\lambda b p_2\left[N'(1) + X^{[1]}R^*(\lambda))\right]\right\}.$$

(iv) Let F_{Loss} denote the frequency of the customer loss due to arrival of negative customers, then

$$F_{Loss} = \delta(1-\theta)\left[\Pi_1 + \Pi_2\right] = \frac{1}{\beta}\left\{\alpha\lambda b(1-\theta)\left(N'(1) + X^{[1]}R^*(\lambda)\right)\left[p_1\left(1 - S_1^*(\delta)\right) + p_2\left(1 - S_2^*(\delta)\right)\right]\right\}.$$

(v) Let Q denote the steady-state probability that the server is on starting failure, then

$$Q = Q(1) = \frac{1}{\beta}\left\{\bar{\alpha}\lambda b p_1 h^{(1)}\left[N'(1) + X^{[1]}R^*(\lambda))\right]\right\}.$$

(vi) Let Ω denote the steady-state probability that the server is on vacation, then

$$\Omega = \sum_{j=1}^{J}\Omega_j(1) = \frac{1}{\beta}\left\{\frac{N'(1)}{X^{[1]}}\left(1 - X^{[1]}\left[1 - R^*(\lambda)\right] - \varpi\right)\right\}.$$

(vii) Let R_1 denote the steady-state probability that the server is under repair time on first type service, then

$$R_1 = R_1(1) = \frac{1}{\beta}\left(\frac{1 - S_1^*(\delta)}{\delta}\right)\left\{\alpha\alpha_1\lambda b p_1 g_1^{(1)}\left[N'(1) + X^{[1]}R^*(\lambda))\right]\right\}.$$

(viii) Let R_2 denote the steady-state probability that the server is under repair time on second type service, then

$$R_2 = R_2(1) = \frac{1}{\beta}\left(\frac{1 - S_2^*(\delta)}{\delta}\right)\left\{\alpha\alpha_2\lambda b p_2 g_2^{(1)}\left[N'(1) + X^{[1]}R^*(\lambda))\right]\right\}.$$

Proof:

The stated properties follow by direct calculation.

Theorem 4.2.

Let L_s, L_q, W_s and W_q be the mean number of customers in the system, the mean number of customers in the orbit, average time a customer spends in the system and average time a customer spends in the orbit respectively, then under the stability condition, we have

$$L_q = P_0\left[\frac{Nr_q'''(1)Dr_q''(1) - Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right],$$

$$Nr''(1) = -2\delta^2\left\{\begin{array}{l}\left(1 - R^*(\lambda)\right)\left\{N'(1)\left(b - X^{[1]}\right) + \left(X^{[1]}\right)^2(b-1)\left(1 - \bar{\alpha}\lambda b h^{(1)}\right)\right\} \\ \quad + \alpha\left\{N'(1) + X^{[1]}\left(1 - b\left(1 - R^*(\lambda)\right)\right)\right\}\left(1 - \theta + (\theta - p)\left(p_1 S_1^*(\delta) + p_2 S_2^*(\delta)\right)\right)\end{array}\right\}$$
$$- 2\alpha\delta\left(p_1 A_1'(1)\left(1 - S_1^*(\delta)\right) + p_2 A_2'(1)\left(1 - S_2^*(\delta)\right)\right),$$

$$Nr'''(1) = -6\alpha X^{[1]}(1-b)\left(1 - R^*(\lambda)\right)A_1'(1)A_2'(1)\left(1 - \left(p_1 S_1^*(\delta) + p_2 S_2^*(\delta)\right)\right)$$
$$- 3\delta^2 N''(1)\left(1 - X^{[1]}(1-b)(1 - R^*(\lambda)) - \alpha\left(\theta - (\theta - p)\left(p_1 S_1^*(\delta) + p_2 S_2^*(\delta)\right)\right)\right)$$

$$-3N'(1)\delta^2\left[\begin{array}{l}(1-R^*(\lambda))(\alpha-(1-b))\left(2X^{[1]}+X^{[2]}\right)-2\alpha\left(p+X^{[1]}(1-R^*(\lambda))\right)\\ \qquad\times\left(p_1S_1^{*'}(\delta)A_1'(1)+p_2S_2^{*'}(\delta)A_2'(1)\right)-\alpha\left(1-R^*(\lambda)\right)\left(p_1S_1^*(\delta)+p_2S_2^*(\delta)\right)\\ \qquad\times\left(2pX^{[1]}+X^{[2]}\right)\end{array}\right]$$

$$-3N'(1)\delta\left\{\begin{array}{l}2\left(A_1'(1)+A_2'(1)\right)\left[\begin{array}{l}\left(b-\bar{\alpha}X^{[1]}\right)(1-R^*(\lambda))\\ -\alpha\left(1-\left(p+X^{[1]}(1-R^*(\lambda))\right)\left(p_1S_1^*(\delta)+p_2S_2^*(\delta)\right)\right)\end{array}\right]\\ \qquad+2\alpha X^{[1]}\left(1-R^*(\lambda)\right)\left(p_1A_1'(1)\left(1-S_1^*(\delta)\right)+p_2A_2'(1)\left(1-S_2^*(\delta)\right)\right)\end{array}\right\}$$

$$+3\alpha\delta X^{[1]}\left(1-b\left(1-R^*(\lambda)\right)\right)\left\{p_1A_2''(1)\left(1-S_1^*(\delta)\right)+p_2A_1''(1)\left(1-S_2^*(\delta)\right)\right\}$$

$$+6\alpha\delta\theta X^{[1]}\left(1-b\left(1-R^*(\lambda)\right)\right)\left(p_1A_2'(1)\left(1-S_1^*(\delta)\right)+p_2A_1'(1)\left(1-S_2^*(\delta)\right)\right)$$

$$+6\alpha\delta X^{[1]}\left(1-b\left(1-R^*(\lambda)\right)\right)\left(A_1'(1)+A_2'(1)-A_1'(1)A_2'(1)\right)\left(p_1S_1^*(\delta)+p_2S_2^*(\delta)\right)$$

$$-2\delta\left(A_1'(1)+A_2'(1)\right)\left(3X^{[1]}(1-b)\left(1-R^*(\lambda)\right)-1\right)$$

$$+3\delta X^{[1]}(1-b)\left(1-R^*(\lambda)\right)\left(\begin{array}{l}\left(X^{[1]}+\alpha\left(p_1S_1^{*'}(\delta)A_1'(1)+p_2S_2^{*'}(\delta)A_2'(1)\right)+\bar{\alpha}\lambda bh^{(1)}X^{[1]}\right)\\ -\alpha\left(A_1''(1)+A_2''(1)\right)\left(1-\left(p_1S_1^*(\delta)+p_2S_2^*(\delta)\right)\right)\end{array}\right)$$

$$+3\alpha\delta\left(1-R^*(\lambda)\right)\left[X^{[2]}+2X^{[1]}(1-b)\right]\left(p_1A_1'(1)\left(1-S_1^*(\delta)\right)+p_2A_2'(1)\left(1-S_2^*(\delta)\right)\right)$$

$$+3\delta^2\left[\alpha\left(\theta-(\theta-p)\left(p_1S_1^*(\delta)+p_2S_2^*(\delta)\right)\right)+\bar{\alpha}\lambda\delta^2bh^{(1)}X^{[1]}\right]\left(\begin{array}{l}2X^{[1]}(1-b)\left(1-R^*(\lambda)\right)\\ +X^{[2]}\left(1-b\left(1-R^*(\lambda)\right)\right)\end{array}\right)$$

$$+3\delta^2X^{[1]}(1-b)(1-R^*(\lambda))\left(2X^{[2]}+\bar{\alpha}\left(2X^{[1]}+\lambda bX^{[1]}\left(2h^{(1)}+\lambda bX^{[1]}h^{(2)}\right)+\lambda bX^{[2]}h^{(1)}\right)\right)$$

$$-3\alpha\delta^2\left(1-b\left(1-R^*(\lambda)\right)\right)\left\{X^{[2]}+2X^{[1]}(\theta-p)\left(p_1S_1^{*'}(\delta)A_1'(1)+p_2S_2^{*'}(\delta)A_2'(1)\right)\right\}$$

$$-3\delta^2R^*(\lambda)X^{[1]}\left\{\alpha\left(\begin{array}{l}p_1\left(S_1^{*''}(\delta)\left(A_1'(1)\right)^2+S_1^{*'}(\delta)A_1''(1)\right)\\ +p_2\left(S_2^{*''}(\delta)\left(A_2'(1)\right)^2+S_2^{*'}(\delta)A_2''(1)\right)\end{array}\right)+\bar{\alpha}\lambda\delta^2bX^{[2]}h^{(1)}\right\},$$

$$Dr''(1)=-2X^{[1]}\left[1-X^{[1]}(1-R^*(\lambda))-\varpi\right],$$

$$\begin{aligned}Dr'''(1)=&\,3X^{[1]}\alpha\delta^2p_1\left(S_1^{*''}(\delta)\left(A_1'(1)\right)^2+S_1^{*'}(\delta)A_1''(1)\right)\\ &+3X^{[1]}\alpha\delta^2p_2\left(S_2^{*''}(\delta)\left(A_2'(1)\right)^2+S_2^{*'}(\delta)A_2''(1)\right),\\ &+6X^{[1]}\alpha\delta^2\left(p+X^{[1]}(1-R^*(\lambda))\right)\left(p_1S_1^{*'}(\delta)A_1'(1)+p_2S_2^{*'}(\delta)A_2'(1)\right)\\ &+3X^{[1]}\alpha\delta^2\left(1-R^*(\lambda)\right)\left(p_1S_1^*(\delta)+p_2S_2^*(\delta)\right)\left(2pX^{[1]}+X^{[2]}\right)\\ &+3X^{[1]}\bar{\alpha}\delta^2\left\{\begin{array}{l}\lambda bX^{[1]}\left(2h^{(1)}+\lambda bX^{[1]}h^{(2)}\right)+X^{[2]}\left(\lambda bh^{(1)}+(1-R^*(\lambda))\right)\\ +2X^{[1]}(1-R^*(\lambda))\left(1+\lambda bX^{[1]}h^{(1)}\right)\end{array}\right\}\end{aligned}$$

$$-6X^{[1]}\delta\left(A_1'(1)+A_2'(1)\right)\begin{bmatrix} 1-\alpha\begin{pmatrix} p_1S_1^{*'}(\delta)A_1'(1)+p_2S_2^{*'}(\delta)A_2'(1) \\ +\left(p+X^{[1]}(1-R^*(\lambda))\right)\left(p_1S_1^*(\delta)+p_2S_2^*(\delta)\right)\end{pmatrix} \\ -\bar{\alpha}\left(\lambda bX^{[1]}h^{(1)}+X^{[1]}(1-R^*(\lambda))+1\right)\end{bmatrix}$$
$$-3X^{[1]}\alpha\left(1-\left(p_1S_1^*(\delta)+p_2S_2^*(\delta)\right)\right)\left(\delta A_1''(1)A_2''(1)+2A_1'(1)A_2'(1)\right)$$
$$-3X^{[2]}\left[1-X^{[1]}(1-R^*(\lambda))-\varpi\right],$$

$$L_s = P_0\left[\frac{Nr_s'''(1)Dr_q''(1)-Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right],$$

$$Nr_s'''(1)=Nr_q'''(1)+6\delta^2\left[N'(1)+X^{[1]}R^*(\lambda)\right]$$
$$\times\left(\frac{\alpha}{\delta}\left(p_1A_1'(1)\left(1-S_1^*(\delta)\right)+p_2A_1'(1)\left(1-S_2^*(\delta)\right)\right)-\bar{\alpha}\lambda bX^{[1]}h^{(1)}\right),$$

$$W_s=\frac{L_s}{\lambda E(X)},$$

and

$$W_q=\frac{L_q}{\lambda E(X)},$$

where

$$N''(1)=\frac{(\lambda b)^2E(V^2)\left(1-[V^*(\lambda b)]^J\right)}{\left(1-V^*(\lambda b)\right)[V^*(\lambda b)]^J},$$

$$S_i^{*'}(\delta)=\int_0^\infty xe^{-\delta x}dS_i(x),\ S_i^{*''}(\delta)=\int_0^\infty x^2e^{-\delta x}dS_i(x),\ \text{for}(i{=}1{,}2),$$
$$A_1'(1)=\lambda bX^{[1]}\left(1+\alpha_1g_1^{(1)}\right),\ A_2'(1)=\lambda bX^{[1]}\left(1+\alpha_2g_2^{(1)}\right),$$
$$A_1''(1)=\lambda bX^{[2]}\left(1+\alpha_1g_1^{(1)}\right)+\alpha_1g_1^{(2)}\left(\lambda bX^{[1]}\right)^2,$$

and

$$A_2''(1)=\lambda bX^{[2]}\left(1+\alpha_2g_2^{(1)}\right)+\alpha_2g_2^{(2)}\left(\lambda bX^{[1]}\right)^2.$$

Proof:

The mean number of customers in the orbit (L_q) under steady state condition is obtained by differentiating (3.56) with respect to z and evaluating at z = 1, to get

$$L_q = K_o'(1) = \lim_{z\to 1}\frac{d}{dz}K_o(z).$$

The mean number of customers in the system (L_s) under steady state condition is obtained by differentiating (3.55) with respect to z and evaluating at z = 1, giving

$$L_s = K_s'(1) = \lim_{z\to 1}\frac{d}{dz}K_s(z) .$$

The average time a customer spends in the system (W_s) and the average time a customer spends in the queue (W_q) are found by using the Little's formula.

$$L_s = \lambda W_s \text{ and } L_q = \lambda W_q .$$

5. Special Cases

In this section, we analyze briefly some special cases of our model, which are consistent with the existing literature.

5.1. Case 1

Single type; No batch arrival; No Vacation; No balking; No starting failure; No breakdown and Exponential retrial.

Let $Pr[X_k = n] = 1$; $Pr[S_2 = 0] = 1$; $b = \alpha = 1$; $Pr[V = 0] = 1$; $p_1 = 0$ and $\alpha_1 = \alpha_2 = 0$. Our model can be reduced to a single server feedback retrial queueing system with negative customers. The following expression coinsides with the result of Krishnakumar et al. (2013).

$$K_o(z) = (1-\rho)\exp\left\{-\frac{\lambda}{\mu}\int_0^\infty \frac{A(u)}{B(u)}du\right\}\left\{\frac{\lambda\left(1-S_1^*(\lambda(1-z)+\delta)\right)+Dr(z)}{Dr(z)}\right\},$$

$$Dr(z) = \lambda\left(z-\left[(1-r+rz)\right]S_1^*(\lambda(1-z)+\delta)\right)+\delta(1-\theta)+\delta(\theta-r)\left[S_1^*(\lambda(1-z)+\delta)\right],$$

where

$$\rho = \frac{\lambda\left(S_1^*(\delta)-1\right)/\delta}{\left(1-\theta(1-S_1^*(\delta))-rS_1^*(\delta)\right)} .$$

5.2. Case 2

Single type; No batch arrival; No retrial; No feedback; No negative customer; No balking and No breakdown.

Let $p_2 = 0$, $Pr[S_2 = 0] = 1$, $p = \delta = 0$; $b = 1$, $R^*(\lambda) \to 1$ and $\alpha_1 = \alpha_2 = 0$. Then, we get a batch arrival queueing system with balking and modified vacations.

$$K(z)=\frac{(1-\rho)\left\{\left[z-1\right]S_1^*\left[\lambda b(1-X(z)\right]\right\}}{\left\{z-S_1^*\left[\lambda b(1-X(z)\right]\right\}}$$
$$\times\frac{\left(1-[V^*(\lambda b)]^J\right)\left(V^*\left[\lambda b(1-X(z)\right]-1\right)+(X(z)-1)\left(1-[V^*(\lambda b)]^J\right)\left(V^*(\lambda b)\right)^J}{\lambda b v^{(1)}\left(1-[V^*(\lambda b)]^J\right)+\left(V^*(\lambda b)\right)^J\left(1-V^*(\lambda b)\right)\left[X(z)-1\right]}.$$

The above result coincides with the result of Ke (2007).

5.3. Case 3

Single type; No batch arrival; No negative arrival and No breakdown.

Let $Pr[X_k = n] = 1$; $p_1 = 1$, $Pr\ [S_2 = 0] = 1$, $\delta = 0$ and $\alpha_1 = \alpha_2 = 0$. Our model can be reduced to a modified vacation for an M/G/1 retrial queueing system with balking and feedback. In this case, the following expression agrees with the result in Ke and Chang (2009a).

$$K_s(z)=P_0\begin{pmatrix} z\left\{1-S_1^*\left[A_1(z)\right]\right\}\left\{\left(N(z)-1\right)R^*(\lambda)+z\left[R^*(\lambda)+N(z)(1-R^*(\lambda))\right]\right\} \\ -N(z)\left\{z-(pz+q)\left(R^*(\lambda)+z(1-R^*(\lambda))\right)S_1^*\left[A_1(z)\right]\right\} \\ +b(1-z)\left\{z\left(R^*(\lambda)+N(z)(1-R^*(\lambda))\right)-(pz+q)R^*(\lambda)S_1^*\left[A_1(z)\right]\right\}\end{pmatrix}$$
$$\times\left\{b(1-z)\left\{z-(pz+q)\left(R^*(\lambda)+z(1-R^*(\lambda))\right)S_1^*\left[A_1(z)\right]\right\}\right\}^{-1}.$$

5.4. Case 4

Single type; No batch arrival; No Vacation; No feedback; No balking; No negative customers and No breakdown.

Let $Pr[X_k = n] = 1$; $Pr\ [S_2 = 0] = 1$; $p = 0$; $r = 0$; $\delta = 0$; $b = 1$; $Pr[V = 0] = 1$ and $\alpha_1 = \alpha_2 = 0$. Then, we get a single server retrial queueing system with general retrial times. The following result coincides with the result of Gomez-Corral (1999).

$$K_s(z)=\frac{\left\{\left[R^*(\lambda)-\lambda E(S_1)\right]\left[z-1\right]S_1^*\left[\lambda-\lambda z\right]\right\}}{\left\{z-\left[R^*(\lambda)+z(1-R^*(\lambda))\right]S_1^*\left[\lambda-\lambda z\right]\right\}},$$

and

$$L_q=\lim_{z\to 1}K_o'(z)=\frac{\lambda^2E(S_1^2)+\ 2\lambda E(S_1)\left[1-R^*(\lambda)\right]}{2\left[R^*(\lambda)-\lambda E(S_1)\right]}.$$

6. Numerical illustration

In this section, we present some numerical examples using MATLAB in order to illustrate the effect of various parameters in the system performance measures. We consider distributions of retrial times, service times, vacation times and repair times as exponential, Erlangian and hyper-exponential. Further we assume that customers are arriving one by one, so $X^{[1]} = 1$, $X^{[2]} = 0$. The arbitrary values to the parameters are chosen as follows $\mu_1 = 8$; $\mu_2 = 10$; $\xi_1 = 6$; $\xi_2 = 8$; $\alpha_1 = 2$; $\alpha_2 = 1$; $\gamma = 5$; $c = 0.8$ so that they satisfy the stability condition. The following tables give the computed values of various characteristics of our model such as: probability that the server is idle, denoted by P_0, the mean orbit size, denoted by L_q, probability that server is idle during retrial time, denoted by P; busy on both types phases, denoted by Π_1, and Π_2, on vacation, denoted by Ω, repair on failure server, denoted by Q, F_{Loss} probability and under repair on both types, denoted by R_1 and $R2$, respectively. Probability density functions for the exponential, Erlang-2stage and hyper-exponential are respectively.

$$f(x) = \phi e^{-\phi x}, x > 0, \quad f(x) = \phi^2 x e^{-\phi x}, x > 0 \text{ and } f(x) = c\phi e^{-\phi x} + (1-c)\phi^2 e^{-\phi^2 x}, x > 0,$$

Table 1 shows that when negative arrival rate (δ) increases, then the probability that the server is idle P_0 increases, the mean orbit size L_q increases and the probability that frequency of customer loss due to arrival of negative customer server (F_{Loss}) also increase. Table 2 shows that when the service loss probability (θ) increases, then the probability that server is idle P_0 decreases, the mean orbit size L_q increases and the probability that server is idle during retrial time P also increases. Table 3 shows that when the successful arrival probability (α) increases, the probability that the server is idle P_0 increases, then the mean orbit size L_q decreases and probability that server is idle during retrial time P also decrease. Table 4 shows that when the number of vacations (J) increases, the probability that server is idle P_0 decreases, then the probability that server is idle during retrial time P increases and probability that server is on vacation Ω also increases. Table 5 shows that when repair rate on FTS (ξ_1) increases, the probability that server is idle P_0 increases, then the mean orbit size L_q decreases and the probability that the server is under repair on FTS R_1 also decreases.

For the effect of the parameters λ, δ, p, α, a, θ, p_1, γ and ξ_1 on the system performance measures in two dimensional graphs are drawn in Figure 1 and 2. Figure 1 shows that the mean orbit size L_q increases with increasing value of the negative arrival rate (δ). Figure 2 shows that the idle probability P_0 decreases by increasing the value of the service loss probability (θ). Three dimensional graphs are illustrated in Figure 3 – Figure 6. In figure 3, the surface displays an upward trend as expected for increasing the value of the arrival rate (λ) and the negative arrival rate (δ) against the mean orbit size L_q. The mean orbit size L_q decreases increasing value of the feedback probability (p) and balking probability (b) in figure 4. The surface displays an upward trend as expected for the increasing the value of the successful service probability (α) and vacation rate (γ) against the idle probability P_0 in figure 5. In figure 6, the mean orbit size L_q decreases with for increasing value of the first type probability (p_1) and repair rate on FTS (ξ_1).

Table 1. The effect of negative arrival probability (δ) on P_0, Lq and F_{Loss}.

Retrial distribution	*Exponential*			*Erlang-2 stage*			*Hyper-Exponential*		
δ	P_0	L_q	F_{Loss}	P_0	L_q	F_{Loss}	P_0	L_q	F_{Loss}
Negative arrival rate									
4.00	0.7988	1.2789	0.0591	0.6885	1.4015	0.0891	0.7908	1.2158	0.0657
5.00	0.8161	1.6095	0.0670	0.7084	1.8551	0.0974	0.8080	1.6184	0.0737
6.00	0.8278	2.0110	0.0736	0.7219	2.5200	0.1038	0.8197	2.2086	0.0802
7.00	0.8363	2.5341	0.0791	0.7317	3.5330	0.1088	0.8281	3.0987	0.0856
8.00	0.8427	3.2411	0.0838	0.7391	5.0698	0.1127	0.8344	4.4235	0.0902

Table 2. The effect of service loss probability (θ) on P_0, Lq and P

Retrial distribution	*Exponential*			*Erlang-2 stage*			*Hyper-Exponential*		
θ	P_0	L_q	P	P_0	L_q	P	P_0	L_q	P
Service loss probability									
0.10	0.7261	0.6403	0.0401	0.6202	0.6584	0.0701	0.7204	0.5891	0.0460
0.20	0.7184	0.6667	0.0425	0.6030	0.7149	0.0776	0.7112	0.6209	0.0492
0.30	0.7101	0.6949	0.0450	0.5834	0.7782	0.0861	0.7013	0.6551	0.0526
0.40	0.7012	0.7249	0.0477	0.5611	0.8494	0.0958	0.6905	0.6919	0.0564
0.50	0.6916	0.7570	0.0507	0.5353	0.9298	0.1070	0.6787	0.7316	0.0605

Table 3. The effect of failure probability ($\bar{\alpha}$) on P_0, Lq and P

Repair distribution	*Exponential*			*Erlang-2 stage*			*Hyper-Exponential*		
α	P_0	L_q	P	P_0	L_q	P	P_0	L_q	P
Successful probability									
0.30	0.6470	1.0183	0.1365	0.4467	4.2181	0.2511	0.6358	1.3797	0.1604
0.40	0.7158	0.6791	0.0963	0.5816	1.6358	0.1711	0.7091	0.8181	0.1123
0.50	0.7549	0.6032	0.0735	0.6550	1.0030	0.1278	0.7502	0.6611	0.0854
0.60	0.7800	0.5913	0.0589	0.7008	0.7807	0.1007	0.776	0.6095	0.0681
0.70	0.7975	0.5984	0.0487	0.7322	0.6900	0.0821	0.7949	0.5937	0.0561

Table 4. The effect of number of vacations (J) on P_0, P and Ω.

Vacation distribution	*Exponential*			*Erlang-2 stage*			*Hyper-Exponential*		
J	P_0	P	Ω	P_0	P	Ω	P_0	P	Ω
Number of vacations									
2.00	0.3961	0.1383	0.0126	0.2095	0.2465	0.0275	0.3851	0.1600	0.0103
3.00	0.3845	0.1390	0.0248	0.1975	0.2494	0.0401	0.3754	0.1608	0.0205
4.00	0.3732	0.1396	0.0367	0.1862	0.2522	0.0519	0.3658	0.1615	0.0305
5.00	0.3622	0.1402	0.0482	0.1755	0.2548	0.0631	0.3564	0.1622	0.0403
6.00	0.3516	0.1409	0.0594	0.1654	0.2572	0.0736	0.3471	0.1630	0.0500

Table 5. The effect of repair rate on FTS (ξ_1) on P_0, Lq and R_1.

Repair distribution	*Exponential*			*Erlang-2 stage*			*Hyper-Exponential*		
ξ_1	P_0	L_q	R_1	P_0	L_q	R_1	P_0	L_q	R_1
Repair rate on FTS									
6.00	0.4081	1.9758	0.0105	0.2222	2.9702	0.0335	0.3950	2.0571	0.0099
7.00	0.4151	1.9615	0.0090	0.2342	2.9775	0.0287	0.4011	2.0492	0.0085
8.00	0.4204	1.9509	0.0079	0.2432	2.9828	0.0251	0.4056	2.0434	0.0074
9.00	0.4245	1.9427	0.0070	0.2502	2.9869	0.0223	0.4091	2.0389	0.0065

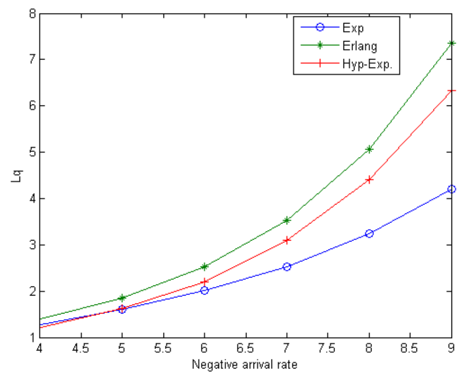

Figure 1. L_q versus δ

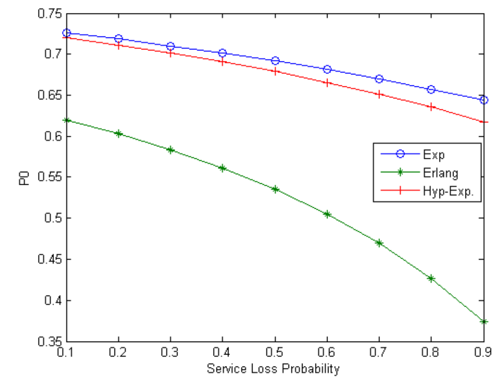

Figure 2. P_0 versus θ

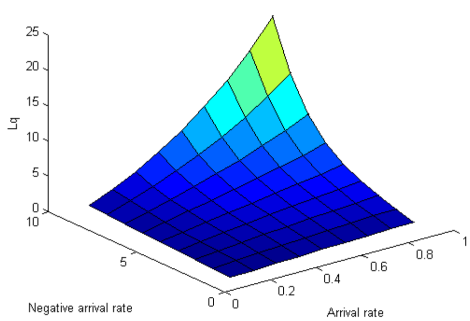

Figure 3. L_q versus λ and δ

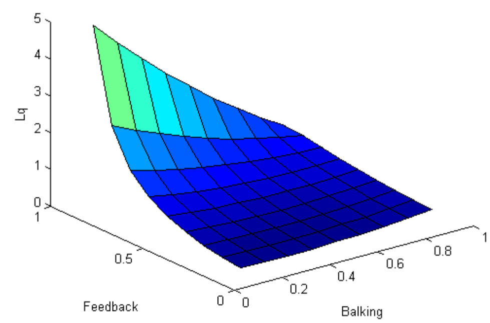

Figure 4. L_q versus p and b

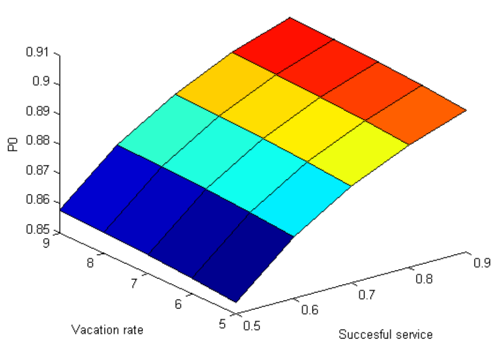

Figure 5. P_0 versus α and γ

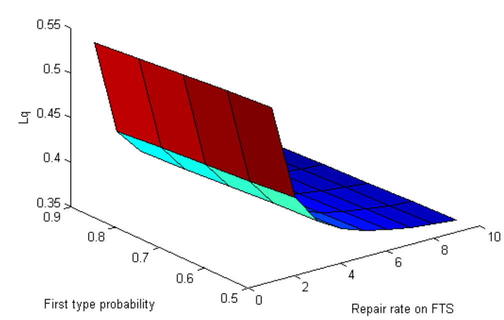

Figure 6. L_q versus p_1 and ξ_1

From the above numerical examples, we can find the influence of the parameters on the performance measures in the system and confirm that the results are coincident with the practical situations.

7. Conclusion

In this paper, we have studied a batch arrival feedback retrial *G*-queueing system with balking under a modified vacation policy and starting failures, where the server provides two types of service. The probability generating functions of the number of customers in the system and orbit are found by using the supplementary variable technique. The explicit expressions for the average queue length of orbit/system and the average waiting time of customer in the system/orbit were obtained, which provided an insight into the system design and management for reducing the waiting time and the queue size of concerned organization under unavoidable techno-economic constraints. The analytical results are validated with the help of numerical illustrations. This model finds potential application in packet switched network to forward the packets within a network for transmission and Simple Mail Transfer Protocol (SMTP) to deliver the messages between mail servers. The novelty of this investigation is the introduction of a feedback retrial queueing system with negative customers, balking and modified vacation where the server is subject to breakdown. Moreover, our model can be considered as generalized version of many existing queueing models equipped with many features. Hopefully, this investigation will be of great help for system managers for making decisions regarding the size of the system and other factors in a well-to-do manner.

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APPENDIX

The embedded Markov chain $\{Z_n;\ n \in N\}$ is ergodic if and only if $\rho < 1$ for our system to be stable, where

$$\rho = X_{[1]}(1 - R^*(\lambda)) + \varpi$$

and

$$\varpi = \left\{ \alpha\Big(\theta - (\theta - p)\Big(p_1 S_1^*(\delta) + p_2 S_2^*(\delta)\Big)\Big) + \bar{\alpha}\Big(\lambda b X_{[1]} h^{(1)} + 1\Big) + \frac{\alpha \lambda b X_{[1]}}{\delta}\Big(p_1\Big(1 + \alpha_1 g_1^{(1)}\Big)\Big(1 - S_1^*(\delta)\Big) + p_2\Big(1 + \alpha_2 g_2^{(1)}\Big)\Big(1 - S_2^*(\delta)\Big)\Big) \right\}.$$

Proof:

From Gomez-Corral (1999), it is not difficult to see that $\{Z_n;\ n \in N\}$ is an irreducible and an aperiodic Markov chain. To prove Ergodicity, we shall use the following Foster's criterion: an irreducible and an aperiodic Markov chain is ergodic if there exists a nonnegative function $f(j)$, $j \in N$ and $\varepsilon > 0$, such that mean drift $\psi_j = E\big[f(z_{n+1}) - f(z_n) / z_n = j\big]$ is finite for all $j \in N$ and $\psi_j \le -\varepsilon$ for all $j \in N$, except perhaps for a finite number j's. In our case, we consider the function $f(j) = j$. Then, we have

$$\psi_j = \begin{cases} \varpi - 1, & j = 0, \\ X_{[1]}(1 - R^*(\lambda)) + \varpi - 1, & j = 1, 2.... \end{cases}$$

Clearly the inequality

$$X_{[1]}(1 - R^*(\lambda)) + \left\{ \begin{aligned} &\alpha\Big(\theta - (\theta - p)\Big(p_1 S_1^*(\delta) + p_2 S_2^*(\delta)\Big)\Big) + \bar{\alpha}\Big(\lambda b X_{[1]} h^{(1)} + 1\Big) \\ &\quad + \frac{\alpha \lambda b X_{[1]}}{\delta}\Big(p_1\Big(1 + \alpha_1 g_1^{(1)}\Big)\Big(1 - S_1^*(\delta)\Big) + p_2\Big(1 + \alpha_2 g_2^{(1)}\Big)\Big(1 - S_2^*(\delta)\Big)\Big) \end{aligned} \right\} < 1$$

is a sufficient condition for ergodicity. The same inequality is also necessary for ergodicity. As noted in Sennot et al. (1983), we can guarantee non-ergodicity, if the Markov chain $\{Z_n;\ n \ge 1\}$ satisfies Kaplan's condition, namely, $\psi_j < \infty$ for all $j \ge 0$ and there exits $j_0 \in$ N such that $\psi_j \ge 0$ for $j \ge j_0$. Notice that, in our case, Kaplan's condition is satisfied because there is a k such that $m_{ij} = 0$ for $j < i - k$ and $i > 0$, where $M = (m_{ij})$ is the one step transition matrix of $\{Z_n;\ n \in N\}$. Then,

$$X_{[1]}(1 - R^*(\lambda)) + \left\{ \begin{aligned} &\alpha\Big(\theta - (\theta - p)\Big(p_1 S_1^*(\delta) + p_2 S_2^*(\delta)\Big)\Big) + \bar{\alpha}\Big(\lambda b X_{[1]} h^{(1)} + 1\Big) \\ &\quad + \frac{\alpha \lambda b X_{[1]}}{\delta}\Big(p_1\Big(1 + \alpha_1 g_1^{(1)}\Big)\Big(1 - S_1^*(\delta)\Big) + p_2\Big(1 + \alpha_2 g_2^{(1)}\Big)\Big(1 - S_2^*(\delta)\Big)\Big) \end{aligned} \right\} \ge 1$$

implies the non-ergodicity of the Markov chain.