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On Some Hadamard-Type Inequalıtıes for $(r,m)-$ Convex Functıons

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Abstract

In this paper, we define a new class of convex functions which is called $(r,m)-$ convex functions. We also prove some Hadamard's type inequalities based on this new definition.

Keywords: $r-$ convex; Hadamard's inequality; $m-$convex, $(r,m)-$ convex

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1. Introduction

The following definition is well known in the literature: a function $f : I \to \mathbb{R}, I \subseteq \mathbb{R}$, is said to be convex on I if the inequality

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$. Geometrically, this means that if P, Q and R are three distinct points on the graph of f with Q between P and R, then Q is on or below chord PR. Let $f : I \subseteq \mathrm{R} \to \mathrm{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. Then the following double inequality holds for convex functions:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}.$$

This inequality is well known in the literature as Hadamard's inequality. Pearce et al. (1998) generalized this inequality to $r-$convex positive function f which is defined on an interval $[a,b]$, for all $x, y \in [a,b]$ and $\lambda \in [0,1]$;

$$f(\lambda x+(1-\lambda)y) \le \begin{cases} \left(\lambda[f(x)]^r+(1-\lambda)[f(y)]^r\right)^{\frac{1}{r}}, & \text{if } r \ne 0, \\ [f(x)]^{\lambda}[f(y)]^{1-\lambda}, & \text{if } r = 0, \end{cases}$$

Clearly $0-$ convex functions are simply $\log-$ convex functions and $1-$ convex functions are ordinary convex functions. Another inequality which is well known in the literature as Minkowski Inequality is stated as follows;

Let

$$p \ge 1, \quad 0 < \int_a^b f(x)^p dx < \infty, \text{ and } 0 < \int_a^b g(x)^p dx < \infty.$$

Then,

$$\left(\int_a^b (f(x)+g(x))^p dx\right)^{\frac{1}{p}} \le \left(\int_a^b f(x)^p dx\right)^{\frac{1}{p}} + \left(\int_a^b g(x)^p dx\right)^{\frac{1}{p}}. \tag{1}$$

Definition 1.

A function $f : I \to [0,\infty)$ is said to be log-convex or multiplicatively convex if $\log f$ is convex, or, equivalently, if for all $x, y \in I$ and $t \in [0,1]$ one has the inequality:

$$f(tx+(1-t)y) \le [f(x)]^t[f(y)]^{1-t}, \tag{2}$$

[Pečarić et al. (1992)].

We note that a $\log-$ convex function is convex, but the converse may not necessarily be true.

Ngoc et al. (2009) established following theorems for $r-$convex functions:

Theorem 1.

Let $f\ :[a,b]\to(0,\infty)$ be $r-$ convex function on $[a,b]$ with $a<b$. Then the following inequality holds for $0<r\le 1$:

$$\frac{1}{b-a}\int_a^b f(x)dx \le \left(\frac{r}{r+1}\right)^{\frac{1}{r}}\left([f(a)]^r+[f(b)]^r\right)^{\frac{1}{r}}. \tag{3}$$

Theorem 2.

Let $f,g\ :[a,b]\to(0,\infty)$ be $r-$convex and $s-$convex functions respectively on $[a,b]$ with $a<b$. Then, the following inequality holds for $0<r$,

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)g(x)dx &\le \frac{1}{2}\left(\frac{r}{r+2}\right)^{\frac{2}{r}}\left([f(a)]^r+[f(b)]^r\right)^{\frac{2}{r}}\\ &\quad+\frac{1}{2}\left(\frac{s}{s+2}\right)^{\frac{2}{s}}\left([g(a)]^s+[g(b)]^s\right)^{\frac{2}{s}}.\end{aligned} \tag{4}$$

Theorem 3.

Let $f,g\ :[a,b]\to(0,\infty)$ be $r-$convex and $s-$convex functions respectively on $[a,b]$ with $a<b$. Then the following inequality holds if $r>1$, and $\frac{1}{r}+\frac{1}{s}=1$:

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le \left(\frac{[f(a)]^r+[f(b)]^r}{2}\right)^{\frac{1}{r}}\left(\frac{[g(a)]^s+[g(b)]^s}{2}\right)^{\frac{1}{s}}. \tag{5}$$

Gill et al. (1997) proved the following inequality for $r-$convex functions.

Theorem 4.

Suppose f is a positive $r-$convex function on $[a,b]$. Then,

$$\frac{1}{b-a}\int_a^b f(t)dt \le L_r\left(f(a),f(b)\right). \tag{6}$$

If f is a positive $r-$concave function, then the inequality is reversed.

For related results on $r-$convexity see [Yang and Hwang (2001), Gill et al. (1997) and Ngoc et al. (2009)]. Toader (1985) defined $m-$convex functions, as follows:

Definition 2.

The function $f\ :\ [0,b]\to \mathrm{R},\ b>0,$ is said to be $m-$convex, where $m\in[0,1],$ if we have

$$f\left(tx+m(1-t)y\right)\leq tf(x)+m(1-t)f(y),$$

for all $x,y\in[0,b]$ and $t\in[0,1].$ We say that f is $m-$concave if $-f$ is $m-$convex.

We refer to the papers [Bakula et al. (2006); Bakula et al. (2007); Özdemir et al. (2010) and Toader (1988)] involving inequalities for $m-$convex functions. Dragomir and Toader (1993) proved the following inequality for $m-$convex functions.

Theorem 5.

Let $f\ :\ [0,\infty)\to \mathrm{R}$ be a $m-$convex function with $m\in(0,1]$. If $0\leq a<b<\infty$ and $f\in L_1[a,b],$ then one has the inequality:

$$\frac{1}{b-a}\int_a^b f(x)dx\leq \min\left\{\frac{f(a)+mf\left(\frac{b}{m}\right)}{2},\frac{f(b)+mf\left(\frac{a}{m}\right)}{2}\right\}. \tag{7}$$

Dragomir (2002) proved some Hadamard-type inequalities for $m-$convex functions as follows.

Theorem 6.

Let $f\ :\ [0,\infty)\to \mathrm{R}$ be a $m-$convex function with $m\in(0,1]$. If $0\leq a<b<\infty$ and $f\in L_1[a,b],$ then one has the inequality:

$$\begin{aligned} f\left(\frac{a+b}{2}\right)&\leq \frac{1}{b-a}\int_a^b \frac{f(x)+mf\left(\frac{x}{m}\right)}{2}dx\\ &\leq \frac{m+1}{4}\left[\frac{f(a)+f(b)}{2}+m\frac{f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)}{2}\right]. \end{aligned} \tag{8}$$

Theorem 7.

Let $f\ :\ [0,\infty)\to \mathrm{R}$ be a $m-$ convex function with $m\in(0,1]$. If $f\in L_1[am,b]$ where $0\leq a<b<\infty,$ then one has the inequality:

$$\frac{1}{m+1}\left[\int_a^{mb} f(x)dx + \frac{mb-a}{b-ma}\int_{ma}^{b} f(x)dx\right] \le (mb-a)\frac{f(a)+f(b)}{2}. \tag{9}$$

2. Main Results

We will start with the following definition.

Definition 3.

A positive function f is $(r,m)-$ convex on $[a,b]\subset[0,b]$ if for all $x,y\in[a,b]$, $m\in[0,1]$ and $\lambda\in[0,1]$

$$f(\lambda x+m(1-\lambda)y)\le\begin{cases}\left(\lambda f^r(x)+m(1-\lambda)f^r(y)\right)^{\frac{1}{r}}, if\ r\neq 0\\ f^{\lambda}(x)f^{1-\lambda}(y) \qquad ,if\ r=0\end{cases}.$$

This definition of $(r,m)-$convexity naturally complements the concept of $(r,m)-$concavity in which the inequality is reversed.

***Remark* 1.**

We have that $(0,1)-$ convex functions are simply log$-$ convex functions and $(1,1)-$ convex functions are ordinary convex functions on $[a,b]\subset[0,b]$.

***Remark* 2.**

We have that $(r,1)-$ convex functions are $r-$convex functions.

***Remark* 3.**

We have that $(1,m)-$ convex functions are $m-$convex functions.

Now, we will prove some inequalities based on above definition and remarks.

Theorem 8.

Suppose that f is a $(r,m)-$convex function on $[a,b]\subset[0,b]$. Then, we have the inequality;

$$\frac{1}{b-ma}\int_{ma}^{b} f(t)dt \le L_r\left(m^{\frac{1}{r}}f(a), f(b)\right), \tag{10}$$

for $r\neq 0$. If f is a $(r,m)-$concave function, then the inequality is reversed.

Proof :

Let $r \neq \{0,-1\}$. First assume that $m^{\frac{1}{r}} f(a) \neq f(b)$. By the definition of $(r,m)-$ convexity, we can write

$$\int_{ma}^{b} f(t)dt = (b-ma)\int_0^1 f\big(sb+m(1-s)a\big)ds$$
$$\leq (b-ma)\int_0^1 \left[sf^r(b)+m(1-s)f^r(a)\right]^{\frac{1}{r}} ds$$
$$= (b-ma)\left(\frac{r}{r+1}\right)\frac{[f(b)]^{r+1}-\left[m^{\frac{1}{r}} f(a)\right]^{r+1}}{f^r(b)-mf^r(a)}.$$

Using the fact that

$$L_r\left(m^{\frac{1}{r}} f(a), f(b)\right)=\left(\frac{r}{r+1}\right)\frac{[f(b)]^{r+1}-\left[m^{\frac{1}{r}} f(a)\right]^{r+1}}{f^r(b)-mf^r(a)},$$

we obtain the desired result. Similarly, for $m^{\frac{1}{r}} f(a) = f(b)$, we have

$$\int_{ma}^{b} f(t)dt \leq (b-ma)\int_0^1 \left(s\left[m^{\frac{1}{r}} f(a)\right]^r+(1-s)\left[m^{\frac{1}{r}} f(a)\right]^r\right)^{\frac{1}{r}} ds$$
$$=(b-ma)\int_0^1 m^{\frac{1}{r}}\left(sf^r(a)+(1-s)f^r(a)\right)^{\frac{1}{r}} ds$$
$$=(b-ma)L_r\left(m^{\frac{1}{r}} f(a), m^{\frac{1}{r}} f(a)\right).$$

Finally, let $r=-1$, for $m^{\frac{1}{r}} f(a) \neq f(b)$, we have

$$\int_{ma}^{b} f(t)dt \leq (b-ma)\int_0^1 \left[sf^{-1}(b)+m(1-s)f^{-1}(a)\right]^{-1} ds.$$

Computing the right hand side of the above inequality, we get

$$\int_{ma}^{b} f(t)dt \leq (b-ma)L_{-1}\left(\frac{f(a)}{m}, f(b)\right).$$

The proof of the other case such as $m^{\frac{1}{r}} f(a) = f(b)$, may be obtained in a similar way.

***Remark* 4.**

In Theorem 8, if we choose $m=1$, we have the inequality (6).

Theorem 9.

Let $f:[a,b]\subset[0,b]\to(0,\infty)$ be $(r,m)-$ convex function on $[a,b]$ with $a<b$. Then, the following inequality holds:

$$\frac{1}{b-ma}\int_{ma}^{b} f(x)dx \leq \left(\frac{r}{r+1}\right)\left[f^{r}(a)+mf^{r}(b)\right]^{\frac{1}{r}}, \tag{11}$$

for $0<r\leq 1$.

Proof:

Since f is $(r,m)-$convex function and $r>0$, we can write

$$f(ta+m(1-t)b) \leq \left(t[f(a)]^{r}+m(1-t)[f(b)]^{r}\right)^{\frac{1}{r}}$$

for all $t,m\in[0,1]$. It is easy to observe that

$$\begin{aligned}\frac{1}{b-ma}\int_{ma}^{b} f(x)dx &= \int_{0}^{1} f(ta+m(1-t)b)dt\\ &\leq \int_{0}^{1}\left(t[f(a)]^{r}+m(1-t)[f(b)]^{r}\right)^{\frac{1}{r}}dt.\end{aligned}$$

Using the inequality (1), we get

$$\begin{aligned}\int_{0}^{1}\left(t[f(a)]^{r}+m(1-t)[f(b)]^{r}\right)^{\frac{1}{r}}dt &\leq \left[\left(\int_{0}^{1} t^{\frac{1}{r}} f(a)dt\right)^{r}+\left(\int_{0}^{1}(1-t)^{\frac{1}{r}} m^{\frac{1}{r}} f(b)dt\right)^{r}\right]^{\frac{1}{r}}\\ &= \left[\left(\frac{r}{r+1}\right)^{r}\left(f^{r}(a)+mf^{r}(b)\right)\right]^{\frac{1}{r}}\\ &= \left(\frac{r}{r+1}\right)\left[f^{r}(a)+mf^{r}(b)\right]^{\frac{1}{r}}.\end{aligned}$$

Thus,

$$\frac{1}{b-ma}\int_{ma}^{b} f(x)dx \le \left(\frac{r}{r+1}\right)\left[f^{r}(a)+mf^{r}(b)\right]^{\frac{1}{r}},$$

which completes the proof.

Corollary 1.

In Theorem 9, if we choose a $(1,m)-$ convex function on $[a,b]$ with $a<b$. Then, we have the following inequality;

$$\frac{1}{b-ma}\int_{ma}^{b} f(x)dx \le \frac{f(a)+mf(b)}{2}.$$

Corollary 2.

In Theorem 9, if we choose an $(r,1)-$ convex function on $[a,b]$ with $a<b$. Then, we have the following inequality;

$$\frac{1}{b-a}\int_{a}^{b} f(x)dx \le \left(\frac{r}{r+1}\right)\left[f^{r}(a)+f^{r}(b)\right]^{\frac{1}{r}}.$$

Remark 5.

In Theorem 9, if we choose a $(1,1)-$ convex function on $[a,b]\subset[0,b]$ with $a<b$. Then, we have the right hand side of Hadamard's inequality.

Theorem 10.

Let $f,g\ :[a,b]\subset[0,b]\to(0,\infty)$ be $(r_1,m)-$ convex and $(r_2,m)-$ convex function on $[a,b]$ with $a<b$. Then, the following inequality holds;

$$\frac{1}{b-ma}\int_{ma}^{b} f(x)g(x)dx \le \frac{1}{2}\left(\left[f(a)\right]^{r_1}+m\left[f(b)\right]^{r_1}\right)^{\frac{1}{r_1}}\left(\left[g(a)\right]^{r_2}+m\left[g(b)\right]^{r_2}\right)^{\frac{1}{r_2}}, \tag{12}$$

for $r_1>1$ and $\frac{1}{r_1}+\frac{1}{r_2}=1$.

Proof:

Since f is $(r_1,m)-$ convex function and g is $(r_2,m)-$ convex function, we have

$$f(ta+m(1-t)b) \le \left(t\left[f(a)\right]^{r_1}+m(1-t)\left[f(b)\right]^{r_1}\right)^{\frac{1}{r_1}}$$

and

$$g(ta+m(1-t)b)\leq\left(t\left[g(a)\right]^{r_2}+m(1-t)\left[g(b)\right]^{r_2}\right)^{\frac{1}{r_2}},$$

for all $t,m\in[0,1]$. Since f and g are non-negative funcions, hence

$$\begin{aligned}&f(ta+m(1-t)b)g(ta+m(1-t)b)\\&\qquad\leq\left(t\left[f(a)\right]^{r_1}+m(1-t)\left[f(b)\right]^{r_1}\right)^{\frac{1}{r_1}}\left(t\left[g(a)\right]^{r_2}+m(1-t)\left[g(b)\right]^{r_2}\right)^{\frac{1}{r_2}}.\end{aligned}$$

Integrating both sides of the above inequality over $[0,1]$ with respect to t, we obtain

$$\begin{aligned}&\int_0^1\left[f(ta+m(1-t)b)g(ta+m(1-t)b)\right]dt\\&\qquad\leq\int_0^1\left[\left(t\left[f(a)\right]^{r_1}+m(1-t)\left[f(b)\right]^{r_1}\right)^{\frac{1}{r_1}}\left(t\left[g(a)\right]^{r_2}+m(1-t)\left[g(b)\right]^{r_2}\right)^{\frac{1}{r_2}}\right]dt.\end{aligned}$$

By applying Hölder's inequality, we have

$$\begin{aligned}&\int_0^1\left[\left(t\left[f(a)\right]^{r_1}+m(1-t)\left[f(b)\right]^{r_1}\right)^{\frac{1}{r_1}}\left(t\left[g(a)\right]^{r_2}+m(1-t)\left[g(b)\right]^{r_2}\right)^{\frac{1}{r_2}}\right]dt\\&\qquad\leq\left[\int_0^1\left(t\left[f(a)\right]^{r_1}+m(1-t)\left[f(b)\right]^{r_1}dt\right)\right]^{\frac{1}{r_1}}\left[\int_0^1\left(t\left[g(a)\right]^{r_2}+m(1-t)\left[g(b)\right]^{r_2}dt\right)\right]^{\frac{1}{r_2}}\\&\qquad=\frac{1}{2}\left(\left[f(a)\right]^{r_1}+m\left[f(b)\right]^{r_1}\right)^{\frac{1}{r_1}}\left(\left[g(a)\right]^{r_2}+m\left[g(b)\right]^{r_2}\right)^{\frac{1}{r_2}}.\end{aligned}$$

By using the fact that

$$\frac{1}{b-ma}\int_{ma}^{b}f(x)g(x)dx=\int_0^1\left[f(ta+m(1-t)b)g(ta+m(1-t)b)\right]dt.$$

We obtain the desired result.

Corollary 3.

In Theorem 10, if we choose $m=1$, $r_1=r_2=2$ and $f(x)=g(x)$, we have the following inequality;

$$\frac{1}{b-a}\int_a^b f^2(x)dx\leq\frac{1}{2}\left[f^2(a)+f^2(b)\right].$$

Corollary 4.

In Theorem 10, if we choose $m=1$ and $r_1=r_2=2,$ we have the following inequality;

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le \sqrt{\frac{f^2(a)+f^2(b)}{2}}\sqrt{\frac{g^2(a)+g^2(b)}{2}}.$$

Theorem 11.

Let $f:[a,b]\subset[0,b]\to(0,\infty)$ be a $(r,m)-$ convex function on $[a,b]$ with $m\in(0,1].$ If $0\le a<b<\infty$ and $f\in L_1[a,b],$ then one has the following inequality;

$$\frac{1}{b-a}\int_a^b f(x)dx \le \min\left\{L_r\left(f(a),m^{\frac{1}{r}}f\left(\frac{b}{m}\right)\right),L_r\left(f(b),m^{\frac{1}{r}}f\left(\frac{a}{m}\right)\right)\right\}.$$

Proof:

Since f is $(r,m)-$convex function, we can write

$$f(tx+m(1-t)y)\le\left(t[f(x)]^r+m(1-t)[f(y)]^r\right)^{\frac{1}{r}}$$

for all $x,y\ge 0$ and $r\neq 0,$ which gives

$$f(ta+(1-t)b)\le\left(t[f(a)]^r+m(1-t)\left[f(\frac{b}{m})\right]^r\right)^{\frac{1}{r}} \tag{13}$$

and

$$f(tb+(1-t)a)\le\left(t[f(b)]^r+m(1-t)\left[f(\frac{a}{m})\right]^r\right)^{\frac{1}{r}}, \tag{14}$$

for $t\in[0,1].$ Integrating both sides of (13) over $[0,1]$ with respect to $t,$ we obtain

$$\int_0^1 f(ta+(1-t)b)dt\le\int_0^1\left(t[f(a)]^r+m(1-t)\left[f(\frac{b}{m})\right]^r\right)^{\frac{1}{r}}dt,$$

or

$$\frac{1}{b-a}\int_a^b f(x)dx \le \int_0^1 \left(t[f(a)]^r + m(1-t)\left[f(\frac{b}{m})\right]^r \right)^{\frac{1}{r}} dt.$$

Now, suppose that $r \neq \{0,-1\}$. First assume that $f(a) \neq m^{\frac{1}{r}} f\left(\frac{b}{m}\right)$. Then, we get

$$\frac{1}{b-a}\int_a^b f(x)dx \le L_r\left(f(a), m^{\frac{1}{r}} f\left(\frac{b}{m}\right)\right).$$

Similarly, for $f(a) = m^{\frac{1}{r}} f\left(\frac{b}{m}\right)$, we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)dx &\le \int_0^1 \left(t[f(a)]^r + (1-t)[f(a)]^r \right)^{\frac{1}{r}} dt \\ &= L_r\left(f(a), f(a)\right).\end{aligned}$$

Finally, let $r=-1$, for $f(a) \neq m^{\frac{1}{r}} f\left(\frac{b}{m}\right)$, we have again

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)dx &\le \int_0^1 \left(t[f(a)]^r + m(1-t)\left[f(\frac{b}{m})\right]^r \right)^{\frac{1}{r}} dt \\ &= L_{-1}\left(f(a), m^{-1} f\left(\frac{b}{m}\right)\right).\end{aligned}$$

When $f(a) = f(b)$, the proof is similar. So, we obtain the inequality

$$\frac{1}{b-a}\int_a^b f(x)dx \le L_r\left(f(a), m^{\frac{1}{r}} f\left(\frac{b}{m}\right)\right).$$

Analogously, by integrating both sides of the inequality (14), we obtain

$$\frac{1}{b-a}\int_a^b f(x)dx \le L_r\left(f(b), m^{\frac{1}{r}} f\left(\frac{a}{m}\right)\right),$$

which completes the proof.

***Remark* 6.**

In Theorem 11, if we choose $r=1$, we have the inequality (7).

***Remark* 7.**

In Theorem 11, if we choose $m=1,$ we have the inequality (6).

***Remark* 8.**

In Theorem 11, if we choose $m=r=1,$ we have the right hand side of Hadamard's inequality.

Theorem 12.

Let $f:[a,b]\subset[0,b]\to(0,\infty)$ be a $(r,m)-$ convex function on $[a,b]$ with $m\in(0,1].$ If $f\in L_1[a,b],$ then one has the following inequalities;

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_a^b \frac{f^r(x)+mf^r\left(\frac{x}{m}\right)}{2}dx \\ &\leq \frac{m+1}{4}\left[\frac{f^r(a)+f^r(b)}{2}+m\frac{f^r\left(\frac{a}{m}\right)+f^r\left(\frac{b}{m}\right)}{2}\right]. \end{aligned} \tag{15}$$

Proof:

By the $(r,m)-$convexity of f , we have that

$$f\left(\frac{x+y}{2}\right)\leq\frac{1}{2}\left[f^r(x)+mf^r\left(\frac{y}{m}\right)\right],$$

for all $x,y\in[a,b].$ If we take $x=ta+(1-t)b$ and $y=(1-t)a+tb,$ we deduce

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{2}\left[f^r(ta+(1-t)b)+mf^r\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)\right],$$

for all $t\in[0,1].$ Integrating the result over $[0,1]$ with respect to $t,$ we get

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{2}\left[\int_0^1 f^r(ta+(1-t)b)dt+m\int_0^1 f^r\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)dt\right]. \tag{16}$$

Taking into account that

$$\int_0^1 f^r(ta+(1-t)b)dt=\frac{1}{b-a}\int_a^b f^r(x)dx$$

and

$$\int_0^1 f^r\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)dt=\frac{1}{b-a}\int_a^b f^r\left(\frac{x}{m}\right)dx$$

in (16), we obtain the first inequality of (15).

By the $(r,m)-$ convexity of f , we also have that

$$\frac{1}{2}\left[f(ta+m(1-t)b)+mf\left(t\frac{a}{m}+(1-t)\frac{b}{m}\right)\right]$$
$$\le\frac{1}{2}\left[tf^r(a)+m(1-t)f^r(b)+mtf^r\left(\frac{a}{m}\right)+m^2(1-t)f^r\left(\frac{b}{m}\right)\right],$$

for all $t\in[0,1]$. Integrating the above inequality over $[0,1]$ with respect to $t,$ we get

$$\frac{1}{b-a}\int_a^b\frac{f^r(x)+mf^r(\frac{x}{m})}{2}dx$$
$$\le\frac{1}{2}\left[\frac{f^r(a)+mf^r(b)}{2}+\frac{mf^r(\frac{a}{m})+m^2f^r(\frac{b}{m})}{2}\right].$$

By a similar argument, we can state

$$\frac{1}{b-a}\int_a^b\frac{f^r(x)+mf^r(\frac{x}{m})}{2}dx$$
$$\le\frac{1}{8}\left[f^r(a)+f^r(b)+2m\left(f^r\left(\frac{a}{m}\right)+f^r\left(\frac{b}{m}\right)\right)+m^2\left(f^r\left(\frac{a}{m^2}\right)+f^r\left(\frac{b}{m^2}\right)\right)\right],$$

which completes the proof.

***Remark* 9.**

In theorem 12, if we choose $r=1,$ we have the inequality (8).

***Remark* 10.**

In theorem 12, if we choose $m=r=1,$ we have the Hadamard's inequality.

1. Conclusion

In this paper, a new class of convex functions called $(r,m)-$ convex functions have been defined and some new Hadamard-type inequalities have been obtained.

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