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A Characterization of Skew Normal Distribution by Truncated Moment

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Abstract

A probability distribution can be characterized through various methods. This paper discusses a new characterization of skew normal distribution by truncated moment. It is hoped that the findings of the paper will be useful for researchers in different fields of applied sciences.

Keywords: Characterization, skew normal, truncated moment

AMS Subject Classifications: 62E10, 62E15

1. Introduction

Before a particular probability distribution model is applied to fit the real world data, it is necessary to confirm whether the given probability distribution satisfies the underlying requirements by its characterization. Thus, characterization of a probability distribution plays an important role in probability and statistics. A probability distribution can be characterized through various methods (see, for example, Ahsanullah et al. (2014) among others). In recent years, there has been a great deal of interest in the characterization of probability distributions by truncated moments. For example, the development of the general theory of the characterization of probability distributions by truncated moment began with the work of Galambos and Kotz (1978). Further development in this area continued with the contributions of many other authors and researchers, among them Kotz and Shanbhag (1980), Glänzel $(1987, 1990)$, and Glänzel et al. (1984), are notable. However, most of these characterizations are based on a simple relationship between two different moments truncated from the left at the same point.

Many authors have also studied characterizations of the skew normal distribution (SND). For example, Gupta et al. (2004) studied the characterization results for the skew normal distribution based on quadratic statistics. For detailed derivations of these results the interested readers are referred to Gupta et al. (2004), and references therein. See also Arnold and Lin (2004), where the authors have shown that the skew normal distributions and their limits are exactly the distributions of order statistics of bivariate normally distributed variables. Further, using generalized skew normal distributions, Arnold and Lin (2004) have been able to characterize the distributions of random variables whose squares obey the chi square distribution with one degree of freedom. For more on characterizations, we refer the interested readers to Ahsanullah et al. (2014), among others.

It appears from the literature that no attention has been paid to the characterizations of the skew normal distribution using truncated moment. As pointed out by Glänzel (1987), these characterizations may also serve as a basis for parameter estimation.

In this paper, we present a new characterization of the skew normal distribution using the truncated moment by considering a product of reverse hazard rate and another function of the truncated point.

The organization of this paper is as follows. Section 2 discusses the skew normal distribution (SND) and some of its properties. In Section 3, characterization of the skew normal distribution by truncated moment is presented. The concluding remarks are provided in Section 4.

2. Distributional Properties of a Skew Normal Distribution

This section discusses the skew normal distribution (SND) and some of its distributional properties.

Definition:

A continuous random variable X is said to have a skew normal distribution with parameters (μ, σ, λ), denoted by $Y \sim SN(\mu, \sigma^2, \lambda)$, if its probability density function and cumulative distribution function are, respectively, given by

$$f\,(x,\,\mu,\,\sigma,\,\lambda) = 2\phi\left(\frac{x-\mu}{\sigma}\right)\,\Phi\left(\lambda\left(\frac{x-\mu}{\sigma}\right)\right), \tag{2.1}$$

and

$$F\,(x,\,\mu,\,\sigma,\,\lambda) = \Phi\left(\frac{x-\mu}{\sigma}\right) - 2T\left(\frac{x-\mu}{\sigma},\,\lambda\right), \tag{2.2}$$

where $-\infty < x < \infty$, $-\infty < \mu < \infty, \sigma > 0$ and $-\infty < \lambda < \infty$, are to be referred as the location, scale and shape parameters, respectively; $\phi(u)$ and $\Phi(\lambda u)$ denote the probability density function and cumulative distribution function of the standard normal distribution, respectively; and $T(u,\,\lambda)$ denotes Owen's (1956) T function as given by

$$T(u,\,\lambda) = \frac{1}{2\pi}\int_0^{\lambda} \frac{e^{-(1/2)u^2(1+x^2)}}{1+x^2}\,dx.$$

In particular, if in the above definitions $\mu = 0, \sigma = 1$, then we have a standard skew normal distribution, denoted by $X_\lambda \sim SN(\lambda)$, with probability density function as given by

$$f_X(x\,;\,\lambda) = 2\,\phi(x)\Phi(\lambda\,x), -\infty < x < \infty, \tag{2.3}$$

where

$$\phi(x) = \left(\frac{1}{\sqrt{2\pi}}\right)e^{-\frac{1}{2}x^2}$$

and

$$\Phi(\lambda\,x) = \int_{-\infty}^{\lambda\,x}\phi(t)\,dt$$

denote the probability density function and cumulative distribution function of the standard normal distribution, respectively. The continuous random variable X_λ is said to have a skew normal distribution since the family of distributions it represents includes the standard $N(0,1)$ distribution as a special case, but in general with members having skewed density. This is also evident from the fact that $X_\lambda{}^2 \sim \chi_1^2$ (Chi-square distribution with one degree of freedom) for all values of the parameter λ. As pointed out by Azzalini (1985), the skew normal density function $f_X(x;\lambda)$ has the following characteristics:

a) $SN(0) = N(0,1)$.

b) If $X_\lambda \sim SN(\lambda)$, then $-X_\lambda \sim SN(-\lambda)$.

c) If $\lambda \to \pm\infty$, and $Z \sim N(0,1)$, then $SN(\lambda) \to \pm|Z| \sim HN(0,1)$, that is, $SN(\lambda)$ tends to the half-normal distribution.

d) If $X_\lambda \sim SN(\lambda)$, then $X_\lambda{}^2 \sim \chi_1^2$.

e) The MGF of X_λ is given by $M_\lambda(t) = 2e^{\frac{t^2}{2}}\,\Phi(\delta t),\ t \in \Re$, where $\delta = \dfrac{\lambda}{\sqrt{1+\lambda^2}}$.

f) It is easy to see that $E(X_\lambda) = \delta\left(\sqrt{\dfrac{2}{\pi}}\right)$, and $Var(X_\lambda) = \dfrac{\pi - 2\delta^2}{\pi}$.

g) The characteristic function of X_λ is given by $\psi_\lambda(t) = e^{\frac{-t^2}{2}}\left[1 + i\,h(\delta t)\right],\ t \in \Re$, where

$$h(x) = \left(\sqrt{\frac{2}{\pi}}\right)\int_0^x e^{\frac{y^2}{2}}\,dy, \text{ and } h(-x) = -h(x) \text{ for } x \ge 0.$$

h) By introducing the following linear transformation $Y_\lambda = \mu + \sigma X_\lambda$, that is,
$X_\lambda = \dfrac{Y_\lambda - \mu}{\sigma}$,

where $\mu \ge 0, \sigma > 0$, we obtain the skew normal distribution with the probability density function given by (2.1). Some characteristic values of the random variable Y_λ are as follows:

Mean: $E(Y_\lambda) = \mu + \left(\sigma\delta\sqrt{\dfrac{2}{\pi}}\right)$,

Variance: $Var(Y_\lambda) = \dfrac{\sigma^2(\pi - 2\delta^2)}{\pi}$, and

Skewness: $\gamma_1 = \left(\dfrac{4-\pi}{2}\right)\dfrac{[E(X_\lambda)]^3}{[Var(X_\lambda)]^{\frac{3}{2}}} = \dfrac{4-\pi}{2}(\delta\sqrt{2/\pi}))^3(1 - \dfrac{2\delta^2}{\pi})^{-3/2}$,

where $\delta = \dfrac{\lambda}{\sqrt{1+\lambda^2}}$.

Note: The skewness is limited in the interval $(-1, 1)$.

Kurtosis: $\gamma_2 = 2\,(\pi - 3)\dfrac{[E(X_\lambda)]^4}{[Var(X_\lambda)]^2} = 2(\pi - 3)(\delta\sqrt{2/\pi})^4 (1 - \dfrac{2\delta^2}{\pi})^{-2}$.

$$MGF = 2e^{\lambda t + \frac{\sigma^2 t^2}{2}}\,\Phi(\lambda\sigma t).$$

The shape of the skew normal probability density function given by (2.1) depends on the values of the parameter λ. For some values of the parameters (μ, σ, λ), the shapes of the pdf of SN(λ) (2.1) are provided in Figures 1 and 2 below.

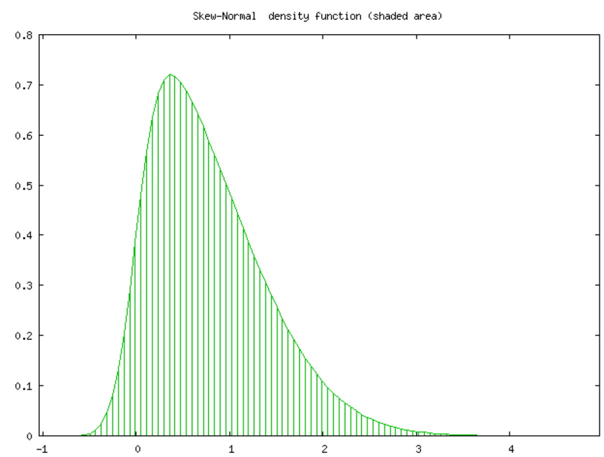

Figure 1: Plot of the pdf of SN(λ) for $\mu = 0, \sigma = 1, \lambda = 5$

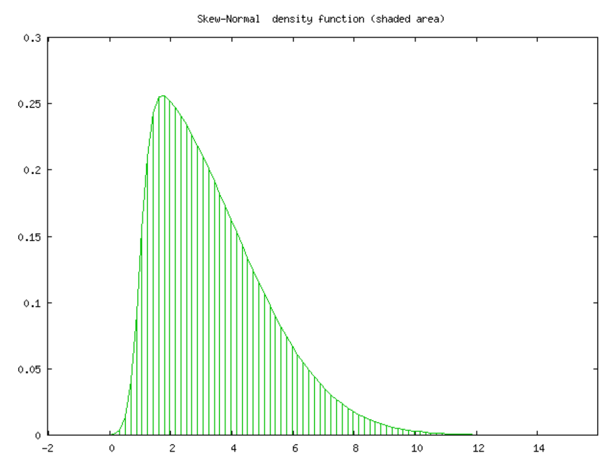

Figure 2: Plot of the pdf of SN(λ) for $\mu = 1, \sigma = 3, \lambda = 10$

The skew normal distribution represents a parametric class of probability distributions, reflecting varying degrees of skewness, which includes the standard normal distribution as a special case. The term skew normal distribution (SND) was introduced by Azzalini (1985, 1986), who gave a systematic treatment of this distribution. The skewness parameter involved in this class makes it possible for probabilistic modeling of the data obtained from skewed population. The skew normal distributions are also useful in the study of the robustness and as priors in Bayesian analysis of the data. For recent developments on the skew normal distribution, the interested readers are referred to Pewsey (2000), Gupta et al. (2002), Nadarajah & Kotz (2003), Dalla Valle (2004), Genton (2004), Gupta & Gupta (2004), Azzalini (2006), Nadarajah & Kotz (2006), Chakraborty and Hazarika (2011), Azzalini and Regoli (2012), and very recently Ahsanullah et al. (2014) among others.

3. A New Characterization of the Skew Normal Distribution

In what follows, we will present a new characterization of the skew normal distribution in a different direction. We shall do this by using truncated moment. For this, we need the following assumptions and Lemma (Lemma 3.1).

Let the random variable X be a random variable having absolutely continuous (with respect to Lebesgue measure) cumulative distribution function (cdf) F(x) and the probability density function (pdf) f(x). We assume $\alpha = \inf\{x|F(x) > 0\}$ and $\beta = \sup\{x \mid F(x) < 1\}$. We define

$$\eta(x) = \frac{f(x)}{F(x)},$$

and $g(x)$ is a differentiable function with respect to x for all real $x \in (\alpha, \beta)$.

Lemma 3.1.

Suppose that X has an absolutely continuous (with respect to Lebesgue measure) cdf $F(x)$,with corresponding pdf $f(x)$ and $E(X|X \leq x)$ exists for all real $x \in (\alpha, \beta)$. Then

$$E(X|X \leq x) = g(x)\,\eta(x),$$

where $g(x)$ is a differentiable function, and $\eta(x) = \frac{f(x)}{F(x)}$, for all real x $\in (\alpha, \beta)$, if

$$f(x) = ce^{\int_{\alpha}^{x} \frac{u - g'(u)}{g(u)} du},$$

where c is determined such that $\int_{-\infty}^{\infty} f(x)dx = 1$.

(Note: Since cdf $F(x)$ is absolutely continuous (with respect to Lebesgue measure), then by Radon-Nikodym Theorem the pdf $f(x)$ exists and, hence $\int_{-\infty}^{x} \frac{u - g'(u)}{g(u)} du$ exists).

Proof:

We have

$$\frac{\int_{\alpha}^{x} uf(u)du}{F(x)} = \frac{g(x)f(x)}{F(x)}.$$

Thus,

$$\int_{\alpha}^{x} uf(u)du = g(x)f(x).$$

Differentiating both sides of the equation with respect to x, we obtain

$$xf(x) = g'(x)f(x) + g(x)f'(x).$$

On simplification, we get

$$\frac{f'(x)}{f(x)} = \frac{x - g'(x)}{g(x)}.$$

Integrating the above equation, we obtain

$$f(x) = ce^{\int_{\alpha}^{x} \frac{u - g'(u)}{g(u)} du},$$

where c is determined such that $\int_{-\infty}^{\infty} f(x)dx = 1$. This completes the proof of Lemma 3.1.

We now have the following characterization theorem (Theorem 3.1).

A Characterization Theorem

Theorem 3.1:

Suppose that X has an absolutely continuous (with respect to Lebesgue measure) cdf $F(x)$, pdf $f(x)$, and $E(X|X \le x)$ exists for all x in (α, β). We assume $\alpha = -\infty$, $\beta = \infty$, and $E(X)$ and $f'(x)$ exist for all $x \in (-\infty, \infty)$. Then,

$$E(X|X \le x) = g(x)\ \eta(x),$$

where

$$\eta(x) = \frac{f(x)}{F(x)},$$

$$g(x) = \frac{x(\Phi(x) - 2T(x,\lambda))}{2\phi(x)\Phi(\lambda x)} - \frac{H(x,\lambda)}{2\phi(x)\Phi(\lambda x)},$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},$$

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du,$$

$T(x,\lambda)$ is the Owen (1956) T function as given by

$$T(x,\lambda) = \frac{1}{2\pi} \int_0^{\lambda} \frac{e^{-(1/2)x^2(1+u^2)}}{1+u^2} du$$

and

$$\mathrm{H}(\mathrm{x}, \lambda) = \int_{-\infty}^{x} (\Phi(u) - 2T(u,\lambda)) du,$$

if and only if

$$f(x) = 2\phi(x)\ \Phi(\lambda x),$$

for any real λ.

Proof:

Suppose

$$f(x) = 2\phi(x)\Phi(\lambda x),$$

then

$$g(x) = \frac{\int_{-\infty}^{x} uf(u)du}{f(x)} = \frac{xF(x)}{f(x)} - \frac{\int_{-\infty}^{x} F(x)}{f(x)},$$

$$=\frac{x[\Phi(x)-2T(x,\lambda)]}{\phi(x)\Phi(\lambda x)}-\frac{H(x,\lambda)}{\phi(x)\Phi(\lambda x)}.$$

Suppose that

$$g(x)=\frac{x(\Phi(x)-2T(x,\lambda))}{2\phi(x)\Phi(\lambda x)}-\frac{H(x,\lambda)}{2\phi(x)\Phi(\lambda x)},$$

then

$$g'(x)=x+[\frac{x(\Phi(x)-2T(x,\lambda))}{2\phi(x)\Phi(\lambda x)}-\frac{H(x,\lambda)}{2\phi(x)\Phi(\lambda x)}](x-\frac{\gamma\phi(\gamma x)}{\Phi(\lambda x)})$$

$$=x+g(x)\left(x-\frac{\lambda\varphi(\gamma x)}{\Phi(\lambda x)}\right).$$

Thus,

$$\frac{f'(x)}{f(x)}=\frac{x-g'(x)}{g(x)}=-x+\lambda\frac{\phi(\lambda x)}{\Phi(\lambda X)}.$$

On integrating the above equation, we have

$$f(x)=c\,e^{-(1/2)x^2}\Phi(\lambda x),$$

where

$$1/c=\int_{-\infty}^{\infty}e^{-(1/2)x^2}\Phi(\lambda x)\,dx=\frac{\sqrt{2\pi}}{2}.$$

This completes the proof of Theorem3.1.

4. Concluding Remarks

As pointed out above, before a particular probability distribution model is applied to fit the real world data, it is necessary to confirm whether the given probability distribution satisfies the underlying requirements by its characterization. Thus, characterization of a probability distribution plays an important role in probability and statistics. A probability distribution can be characterized through various methods. This paper considers a new characterization of the skew normal distribution using truncated moment by considering a product of reverse hazard rate and another function of the truncated point. In this regard, some distributional properties of the skew normal distribution are also provided. We believe that the findings of this paper would be useful for the practitioners in various fields of studies and further enhancement of research in distribution theory, and its applications.

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