Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 8, Issue 2 (December 2013), pp. 717 – 732

Reducibility of Eulerian Graphs and Digraphs

Akram B. Attar
Department of Mathematics
Faculty of Education for Pure Sciences
University of Thi-Qar
Thi-Qar, Iraq
akramattar70@yahoo.com

Received: July 20, 2013; Accepted: July 20, 2013

Abstract

In this paper the concept of reducibility in graph theory is discussed, and the deletable vertex (edge) in graph (digraph) is defined. The class of graphs (digraphs) $\Re$ is called vertex (edge) reducible if for any $G \in \Re$ either $\Re$ is the trivial graph (null graph) or it contains a vertex (edge) v such that $G - v \in \Re$. We introduce some classes of graphs (digraphs) which are reducible and others which are not. The vertex reducibility and edge reducibility of Eulerian graphs and Eulerian digraphs have also been studied.

Keywords: Eulerian graphs; digraphs

AMS-MSC (2010) No.: 05C45, 05C20

1. Introduction

The concept of reducibility is amply discussed for some classes of lattices by Bordalo and Monjardet (1996). In fact they proved that the class of pseudo complemented lattices as well as the class of semi-modular lattices is reducible. Kharat and Waphare (2001) identified some classes of postes, which are reducible. Further, they have introduced a concept of reducibility number for posets. We discuss some analogous concepts in graphs. Akram (2007) introduced the concept of contractibility number of graphs and Akram (2009) included the concept of vertex extension of graphs. Borse and Waphare (2008) defined a non–separating cycle of a graph G as a cycle for which $G - V(C)$ is connected. Attar (2012) defined the edge removable cycle as follows let $\Im$ be a class of graphs (digraphs), satisfying some property, the cycle C in G is called edge removable if $G - E(C) \in \Im$.

For the undefined concepts and terminology the reader is referred to Wilson (1978), Clark (1991), Harary (1969), West (1999) and Tutte (1984).

Definition 1.1.

Let $\Re$ be a class of graphs satisfying certain property P , and $G \in \Re$. A vertex(edge) v in G is called deletable with respect to $\Re$, if $G - v \in \Re$. In general, a set S of vertices (edges) is called *deletable* with respect to $\Re$, if $G - S \in \Re$. Generally, if $|S| = k$ then we say that S is a $k-$deletable set.

Definition 1.2.

Let $\Re$ be a class of graphs satisfying a certain property P . The class $\Re$ is called *vertex* (edge) reducible if for any $G \in \Re$ either $\Re$ is the trivial graph (null graph) or it contains a vertex (edge) v such that $G - v \in \Re$.

The following results provide some reducible classes.

***Proposition* 1.1.**

1. The class of trees is vertex reducible, but not edge reducible.

2. The class of connected graphs is vertex reducible.

Proof:

The proposition follows from the well-known fact that every non-trivial connected graph contains a vertex which is not a cut vertex [see Harary (1969)].

***Proposition* 1.2.**

1. The class of bipartite graphs is vertex reducible and edge reducible.

2. The class of complete graphs is vertex reducible, but not edge reducible.

Proof:

Obvious.

***Proposition* 1.3.**

The classes of Hamiltonian graphs, regular graphs, Eulerian graphs are neither edge reducible nor vertex reducible.

Proof:

The proof follows from the fact that neither an edge nor a vertex of a cycle is deletable.

Definitions 1.3.

Let $\Re$ be a class of graphs and, $G \in \Re$ be non-trivial (non-null). The vertex (edge) reducibility number of G with respect to $\Re$ is the smallest positive integer m, if exists, such that G contains a deletable set S of vertices (edges) of cardinality m. We write $m = v - red_{\Re}(G)[e - red_{\Re}(G)]$. If such a number does not exist for G, then we say that the corresponding reducibility number is ∞.

One can immediately note that a class $\Re$ is reducible if and only if its reducibility number is 1 for every non-trivial graph $G \in \Re$.

In this paper we provide characterizations for the vertex and edge reducibility number of Eulerian graphs and Eulerian digraphs. We require the following concepts and results.

Definitions 1.4. Clark (1991)

The neighborhood $N(v)$ of the vertex v in a graph G consists of the set of vertices adjacent to v. If U is a nonempty subset of the vertex set V of G then the sub graph $G[U]$ (or simply $[U]$) of G induced by U is defined to be the graph having vertex set U and edge set consisting of those edges of G that have both ends in U. Similarly, if F is a nonempty subset of the edge set E of G then the sub graph $G[F]$ (or simply $[F]$) of G induced by F is the graph whose vertex set is the set of ends of edges in F and whose edge set is F.

Let H be a sub graph of a graph G. A vertex of attachment of H in G is a vertex of H that is incident with some edge of G which is not an edge of H. We write $W(G,H)$ for the set of vertices of attachment of H in G. A sub graph H of a graph G is said to be detached in G if it has no vertices of attachment in G.

If v is a vertex of a digraph D, then its in-degree $d^-(v)$ is the number of arcs in D of the form (w,v) and its out-degree or score $d^+(v)$ is the number of arcs in D of the form (v,w). We can define the induced sub digraph analogous to the induced sub graph.

Let D be a digraph. Then the directed walk in D is a finite sequence $W = v_0 a_1 v_1 ... a_k v_k$ whose terms are alternately vertices and arcs such that for $i = 1,2,...k$, the arc a_i has origin v_{i-1} and terminus v_i. A closed walk has the same first and last vertices, and a spanning walk contains all the vertices. The concepts directed trails, directed paths, and directed cycles have meaning similar to the corresponding known concepts in graphs. A semi path has the same definition of directed path, but each arc x_i may be either $v_{i-1}v_i$ or $v_i v_{i-1}$.

A vertex v of the digraph D is said to be reachable from a vertex u if there is a directed path in D from u to v. A digraph D is called strongly connected or strong if, every two vertices v and w, are mutually reachable; and it is weakly connected, or weak, if every two vertices are joined by a semi path. A strong component of a digraph is a maximal strong sub digraph; and a weak component is a maximal weak sub digraph.

An Eulerian trail in a digraph D is a closed spanning walk in which each arc of D occurs exactly once. A digraph is Eulerian if it has such a trail.

Note that an Eulerian digraph is strongly connected. Good (1996) (see also Welson (1978) characterized Eulerian digraphs as follows.

Theorem 1.1.

A weak digraph D is Eulerian if and only if every vertex of D has equal in-degree and out-degree.

Let H be a sub digraph of a digraph D. *A vertex of attachment* of H in D is a vertex in H that dominates or is dominated by some vertex of D that is not a vertex of H. We write $W(D,H)$ for the set of vertices of attachment of H in D.

A sub digraph H of a digraph D is said to be detached in D if it has no vertices of attachment in D.

2. Vertex Reducibility of Eulerian Graphs and Digraphs

In this section the vertex reducibility number for Eulerian graphs and Eulerian digraphs has been studied. We need the following concept of complementary sub graph.

Definition 2.1. Tutte (1984)

Let H be a sub graph of a graph G. Then there is a sub graph H^c of G such that $E(H^c) = E(G) - E(H)$ and $V(H^c) = (V(G) - V(H)) \bigcup W(G, H)$. We call H^c the complementary sub graph to H in G.

Firstly, we prove some required lemmas and then using these lemmas to characterize the vertex reducibility number of Eulerian graphs.

Lemma 2.1.

Let G be a graph and $U \subseteq V(G)$. Then the complementary sub graph to $G - U$ is the subgraph whose vertex set is $U \bigcup N(U)$ and edge set is $\{e \in E(G) : e$ is incident with a vertex of $U\}$.

Proof:

Let H be the complementary subgraph to $G - U$. By Definition 2.1,

$$V(H) = \{V(G) - V(G - U)\} \bigcup W(G, G - U) = U \bigcup W(G, G - U).$$

We have

$$W(G, G - U) = N(U) - U.$$

Hence,

$$V(H) = U \bigcup N(U).$$

Further,

$$E(H) = E(G) - E(G - U).$$

Let $e \in E(H) = E(G) - E(G-U)$. We have $e \in E(G)$ and $e \notin E(G-U)$. This implies that at least one of the end vertices of e is in U. Hence, $E(H) = \{e \in E(G) : e$ is incident with a vertex of $U\}$.

Lemma 2.2.

Let G be a graph and U be a nonempty subset of $V(G)$. Let H be the complementary subgraph to $G-U$. Then a component H_1 of H is the complementary subgraph to the subgraph $G-(V(H_1)\bigcap U)$.

Proof:

Since H is a complementary sub graph to $G-U$, we have

$$V(H) = \{V(G) - V(G-U)\} \bigcup W(G, G-U) = U \bigcup W(G, G-U)$$

and

$$E(H) = E(G) - E(G-U).$$

To prove a component H_1 of H is the complementary subgraph to the subgraph $G-\{V(H_1)\bigcap U\}$ we have to prove

$$\begin{aligned} V(H_1) &= \{V(G) - (V(G) - (V(H_1)\bigcap U))\} \bigcup W(G, G-\{V(H_1)\bigcap U\}) \\ &= \{V(H_1)\bigcap U\} \bigcup W(G, G-\{V(H_1)\bigcap U\}) \end{aligned} \tag{1}$$

and

$$E(H_1) = E(G) - E(G-\{V(H_1)\bigcap U\}). \tag{2}$$

Since H_1 is a component in H, $V(H_1) \subseteq V(H)$ and $E(H_1) \subseteq E(H)$. As

$$V(H) = U \bigcup W(G, G-U), V(H_1) \subseteq U \bigcup W(G, G-U).$$

Hence,

$$V(H_1) = \{V(H_1)\bigcap U\} \bigcup \{W(G, G-U) \bigcap V(H_1)\}.$$

We prove that

$$W(G, G-U) \cap V(H_1) = W(G, G-\{V(H_1) \cap U\}).$$

Let

$$x \in W(G, G-U) \cap V(H_1).$$

We have $x \in W(G, G-U)$ and $x \in V(H_1)$. Since $x \in W(G, G-U)$, we have $x \in V(G-U)$ and there is an edge $e_1 \in E(G)$ incident with x such that its other end vertex $x_1 \notin V(G-U)$. By Lemma 2.1, we get $V(H) = U \cup N(U)$ and, hence,

$$x \in (U \cup N(U)) \cap V(H_1) = (U \cap V(H_1)) \cup (N(U) \cap V(H_1)).$$

Now using the fact that $x \in V(G-U) = V(G) - U$, we get that $x \notin U$ and, hence, $x \in N(U) \cap V(H_1)$. The vertex $x_1 \in U$ and it is adjacent to x. As $x_1 \in U$, by Lemma 2.1, the edge $e_1 \in E(H)$. Using the fact that H_1 is a component of H we get that $x_1 \in V(H_1)$. Therefore $x_1 \in V(H_1) \cap U$. Thus, $x \in W(G, G-\{V(H_1) \cap U\})$.

On the other hand, suppose $x \in W(G, G-\{V(H_1) \cap U\})$. Hence, $x \notin V(H_1) \cap U$ and there is an edge e incident with x whose other end vertex $x_0 \in V(H_1) \cap U$. We have $x_0 \in V(H_1)$ and $x_0 \in U$. Since $x_0 \in U$, by Lemma 2.1, e is an edge of H. As H_1 is a component of H and $x_0 \in V(H_1)$ we have $x \in V(H_1)$. Recall that $x \notin V(H_1) \cap U$ and hence $x \notin U$. Thus x is a vertex of $G-U$, e is incident with x and its other end $x_0 \in U$. Therefore, $x \in W(G, G-U)$. Thus, $x \in W(G, G-U) \cap V(H_1)$, as required.

(2). Let $e = uv \in E(H_1)$. Then, both $u, v \in V(H_1)$. Since $V(H_1) \subseteq V(H)$, $u, v \in V(H)$. By Lemma 2.1, at least one of u, v is in U. Therefore, at least one of u, v is in $V(H_1) \cap U$. Hence, e is not an edge of $G-\{V(H_1) \cap U\}$.

On the other hand, if $e = uv$ is an edge not in $G-\{V(H_1) \cap U\}$, then either $u \in V(H_1) \cap U$ or $v \in V(H_1) \cap U$. Suppose $u \in V(H_1) \cap U$. By Lemma 2.1, it follows that e is an edge of H. Since H_1 is a component of H, e is an edge of H_1.

Lemma 2.3:

Let G be a graph having no odd vertex, and H be a sub graph of G. Then, H has no odd vertex if and only if H^c has no odd vertex.

Proof:

Let G be a graph having no odd vertex, and let H be a sub graph of G For any vertex v in H^c we have, $d_G(v) = d_{H^c}(v)$ if $v \notin V(H)$, and $d_G(v) = d_H(v) + d_{H^c}(v)$ if $v \in V(H)$. Since G has no odd vertex, it follows that, if every vertex of H is even. Then, every vertex of H^c is even. The converse follows by using similar arguments.

Here is a stipulated characterization for vertex reducibility number of eulerian graphs.

Theorem 2.2.

Let $\Im$ be the class of Eulerian graphs and $G \in \Im$. Then $v - red_{\Im}(G) = k$, if and only if k is the smallest number such that there exists a set of vertices U of cardinality k with $H = G - U$ is connected and H^c is Eulerian.

Proof:

Suppose $v - red_{\Im}(G) = k$. There exists U, a subset of cardinality k of $V(G)$ such that $H = G - U$ is Eulerian, and U is a smallest such set. Since H is Eulerian we have H is connected. By Lemma 2.3, each vertex in H^c has even degree. To prove Euleriannas of H^c, it is enough to prove that H^c is connected.

Suppose H^c is not connected, and H_1 is a component of H^c such that $\phi \neq V(H_1) \bigcap U = S$. By Lemma 2.3, H_1 is the complementary sub graph to $G - S$. We obtain a contradiction to minimality of k by proving that $G - S$ is Eulerian and $|S| < k$. Note that if $S = U$ then $H_1 = H^c$, a contradiction to our assumption that H^c is not connected. Hence $|S| < k$.Since H^c has no odd vertex, the component H_1 has no odd vertex and hence, by Lemma 2.3, $G - S$ has no odd vertex. It remains to prove that $G - S$ is connected. If possible, suppose H_2 is a component of $G - S$ which is disjoint from the component of $G - S$ that contains H. We prove that H_2 is detached in G. Suppose on the contrary that e is an edge in G with end vertices x, y such that $x \in V(H_2))$ and $y \notin V(H_2)$. Since H_2 is disjoint from H it follows that $y \in S$. Then, $x \in V(H_2) \subseteq U$ and $y \in S \subset V(H_1) \bigcap U$ and hence e is an edge in H^c. As H_1 is a component of H^c, we get that $x, y \in V(H_1)$ and therefore $x \in V(H_1) \bigcap S$, a contradiction to $x \in V(H_2) \subseteq V(G) - S$. Thus H_2 is detached in G and hence H_2 is a proper component of G, which is impossible. We conclude that $G - S$ is connected.

To prove the smallestness of k, suppose U_1 is a set of vertices in G such that $G-U_1$ is connected and the complementary sub graph to i $G-U_1$ is Eulerian. If $|U_1| < k = |U|$ then, as $G-U_1$ is connected and the complementary sub graph to i $G-U_1$ is Eulerian, by Lemma 2.3, $G-U_1$ is Eulerian, a contradiction to $v-red_{\Im}(G)=k$.

Conversely, suppose k is the smallest number such that there exists $U \subseteq V(G)$ of cardinality k with $H = G-U$ is connected and H^c is eulerian. By Lemma 2.2, H is Eulerian. Hence, $v-red_{\Im}(G) \leq k$. Assume that $v-red_{\Im}(G) = n < k$. Let $U_1 \subseteq V(G)$ be a set such that $|U_1| = n$ and $G-U_1$ is Eulerian then, as proved in the previous part, we have $G-U_1$ is connected and the complementary sub graph to $G-U_1$ is Eulerian, which is a contradiction to the choice of k. Hence, $v-red_{\Im}(G)=k$.

Corollary 2.1.

Let $\Im$ be the class of Eulerian graphs and $G \in \Im$. Then $v-red_{\Im}(G)=2$, if and only if G contains two vertices u, v such that

1. $G-\{u,v\}$ is connected, and

2. $N(u)=N(v)$ where $N(u), N(v)$ are the neighbors of u and v respectively.

Proof:

The condition (2) of the corollary says that the complementary sub graph to $H = G-\{u,v\}$ is Eulerian. In fact, every vertex of H^c other than u and v has degree 2. The proof follows from Theorem 2.1.

Corollary 2.2.

Let $\Im$ be the class of Eulerian graphs. Let H be any complete graph with odd number of vertices. Then, $v-red_{\Im}(H)=2$.

We introduce the concept of complementary sub digraph analogous to Definition 2.1.

Definition 2.2.

Let H be a sub digraph of a digraph D. Then there is a sub digraph H^c of D such that

$$A(H^c) = A(D) - A(H) \text{ and } V(H^c) = (V(D) - V(H)) \bigcup W(D, H).$$

We call H^c the *complementary sub digraph* to H in D.

Now we prove some lemmas with the help of which we characterize the vertex reducibility number of Eulerian digraphs.

Lemma 2.4.

Let D be a digraph and $D-U$. Then the complementary sub digraph to $D-U$ is the sub digraph whose vertex set is $U \bigcup \{u \in V(D) : (v,u) or (u,v) \in A(D)$ for some $u \in U\}$ and arc set is $\{(u,v) \in A(D) :$ either $u \in U$ or $v \in U\}$.

Proof:

Let H be the complementary sub digraph to $D-U$. By Definition 2.2,

$$V(H) = \{V(D) - V(D-U)\} \bigcup W(D, D-U) = U \bigcup W(D, D-U).$$

We have $W(D, D-U) = \{v \in V(D) : (v,u) or (u,v) \in A(D)$ for some $u \in U\}$. Thus,

$$V(H) = U \bigcup \{v \in V(D) : (v,u) or (u,v) \in A(D), \text{ for some } u \in U\} \text{ and}$$
$$A(H) = A(D) - A(D-U).$$

Let $a \in A(H) = A(D) - A(D-U)$. We have $a \in A(D)$ and $a \notin A(D-U)$. Therefore, at least one of the end vertices of a is in U. Hence, $A(H) = \{(u,v) \in A(D) :$ either $u \in U$ or $v \in U\}$.

Lemma 2.5:

Let D be a digraph and U be a nonempty subset of $V(D)$. Let H be the complementary sub digraph to $D-U$. Then a component H_1 of H is a complementary sub digraph to the sub digraph $D-(V(H_1) \bigcap U$.

The proof is similar to the proof of Lemma 2.2.

Lemma 2.6.

Let D be a digraph having in-degree equal to out-degree for each vertex, and H be a sub digraph of D. Then in-degree and out-degree are equal for each vertex in H if and only if in-degree and out-degree are equal for each vertex in H^c.

Proof:

Let v be a vertex in H^c, then $d_D^+(v) = d_{H^c}^+(v)$ and $d_D^-(v) = d_{H^c}^-(v)$, if $v \notin V(H)$, and $d_D^+(v) = d_{H^c}^+(v) + d_D^+(H)$ and $d_D^-(v) = d_{H^c}^-(v) + d_D^-(H)$, if $v \in V(H)$. Since in-degree and out-degree are equal for each vertex of D, it follows that if every vertex of H has in-degree equal to out-degree then every vertex of H^c has in-degree and out-degree equal.

The converse part follows by using the similar arguments.

The following result characterizes the vertex reducibility of Eulerian digraphs.

Theorem 2.3.

Let $\Im$ be the class of Eulerian digraphs and $D \in \Im$. Then $v - red_{\Im}(D) = k$ if and only if k is the smallest number such that there exists a set of vertices U of cardinality k with $H = D - U$ is strongly connected and H^c is Eulerian.

Proof:

Let D be an Eulerian digraph and $v - red_{\Im}(D) = k$. Therefore, there exists $U \subseteq V(D)$ of cardinality k such that $H = D - U$ is Eulerian and k is a smallest such number. Since H is Eulerian, H is strongly connected. We prove H^c is Eulerian. By Theorem 1.1 and Lemma 2.6, it is enough to prove that H^c is weakly connected. Assume that H^c is not weakly connected. Let H_1 be a weak component of H^c such that $\phi \neq V(H_1) \bigcap U = S$. By Lemma 2.5, H_1 is the complementary sub digraph to $D - S$.

We obtain a contradiction to the minimality of k by proving that $D - S$ is Eulerian and $|S| < k$. Note that if $S = U$ then $H_1 = H^c$, a contradiction to our assumption that H^c is not weakly connected. By Lemma 2.6, every vertex in $D - S$ has in-degree equal to out-degree. We prove that $D - S$ is weakly connected. If possible suppose H_2 is a weakly component of $D - S$ which is disjoint from the weak component of $D-S$ which contains sub digraph $D-S$. We prove that H_2 is detached in D. Suppose that a is an arc in D with end vertices x, y such that $x \in V(H_2)$ and $y \notin V(H_2)$. Since H_2 is disjoint from H, $y \in S$.

Therefore, $x \in V(H_2) \subseteq U$ and $y \in S \subset V(H_1) \bigcap U$. Hence a is an arc in H^c. As H_1 is a weak component of H_1, we get that $x, y \in V(H_1)$ and therefore $x \in V(H_1) \bigcap S$ a contradiction to $x \in V(H_2) \subseteq V(D) - S$. Thus H_2 is detached in D and hence H_2 is a proper weak component of D, which is impossible. Hence $D-S$ is connected.

To prove the smallestness of k, suppose U_1 is a set of vertices in D such that $D-U_1$ is strongly connected and the complementary sub digraph to $D-U_1$ is Eulerian. If $|U_1| < k = |U|$ then, as $D-U_1$ is strongly connected and the complementary sub digraph to $D-U_1$ is Eulerian, by Lemma 2.6, $D-U_1$ is Eulerian, a contradiction to $v-red_{\Im}(D) = k$.

Conversely, suppose that k is the smallest number such that there exists $U \subseteq V(D)$ of cardinality k with $H = D-U$ is strongly connected and the complementary sub digraph H^c is Eulerian. By Lemma 2.6, we have H is Eulerian. Hence, $v-red_{\Im}(D) \leq k$. Assume that $v-red_{\Im}(D) = n < k$. Let $U_1 \subseteq V(D)$ be a set such that $|U_1| = n$ and $D-U_1$ is Eulerian then as proved in the first part, $D-U_1$ is strongly connected and the complementary sub digraph to $D-U_1$ is Eulerian, which is a contradiction to smallestness of k. Hence, $v-red_{\Im}(D) = k$.

Corollary 2.3.

Let $\Im$ be the class of Eulerian digraphs, then $v-red_{\Im}(D) = 2$ if and only if there exist two vertices u, v such that

1. $D_1 = D - \{u, v\}$ is strongly connected; and

2. u dominates w, if and only if w dominates v; and v dominates w, if and only if w dominates u.

Proof:

Observe that if u, v satisfy conditions (1) and (2) then the complementary sub digraph to $D - \{U, V\}$ is Eulerian. The proof follows from Theorem 2.3.

3. Edge Reducibility of Eulerian Graphs and Digraphs

We characterize edge reducibility number of Eulerian graphs.

Theorem 3.1.

Let $\Im$ be the class of Eulerian graphs and $G \in \Im$. Then, $e - red_{\Im}(G) = k$ if and only if k is the length of a smallest cycle C in G such that $G - E(C)$ is connected.

Proof:

Suppose that $e - red_{\Im}(G) = k$. Then there exists a set of edges $\{e_1, e_2,..., e_k\}$ such that $G_1 = G - \{e_1, e_2,..., e_k\}$ is Eulerian. Now, we claim that the edge induced subgraph C of $\{e_1, e_2,..., e_k\}$ forms a cycle. We consider the following two cases.

Case 1: C contains a cycle properly. Let $\{e_1, e_2,..., e_n\}$ with $n < k$ be a cycle in C. We have $G - \{e_1, e_2,..., e_k\}$ is Eulerian, a contradiction to $e - red_{\Im}(G) = k$.

Case 2: C does not contain any cycle. Then C is a forest and has an end vertex. It follows that removal of C from G gives a non Eulerian graph which is a contradiction.

Therefore, C is a cycle. The smallest of k follows immediately.

Conversely, assume that k is the length of a smallest cycle C in G such that $G - E(C)$ is connected. We prove that $e - red_{\Im}(G) = k$. As $G - E(C)$ is connected, it follows that $G - E(C)$ is Eulerian. Hence, $e - red_{\Im}(G) \le k$. If $e - red_{\Im}(G) \ne k$, then there exists an edge set $\{f\ , f_2,..., f_n\}$ with $n < k$ such that $G - \{f_1, f_2,..., f_n\}$ is eulerian. By the previous part of the proof, the set $\{f\ , f_2,..., f_n\}$ contains C_1 a cycle of smaller length than k such that $G - E(C_1)$ is connected which is impossible. Hence, $e - red_{\Im}(G) = k$.

Corollary 3.1.

Let G be a non-trivial simple Eulerian graph then the following statements are true.

1. If every cycle in G contains a vertex of degree 2 in G then $e - red_{\Im}(G) = \infty$.

2. If $e-red_{\Im}(G)=\infty$, then G contains a vertex of degree 2.

Proof:

The statement (1) follows from Theorem 3.1. The statement (2) follows by taking $k=1$ in the following result of Mader (1974) and Theorem 3.1.

Theorem 3.2. Mader (1974)

Let G be a k-connected simple graph with minimum degree at least $k+2$. Then, G contains a circuit C such that $G-E(C)$ is k-connected.

Now we try to find the edge reducibility number for line graphs. Consider the set X of edges of a simple graph G with at least one edge as a family of 2-vertices subsets of $V(G)$. The line graph of G, denoted by $L(G)$ is the intersection graph $\Omega(X)$. Thus, the vertices of $L(G)$ are the edges of G with two vertices of $L(G)$ being adjacent whenever the corresponding edges of G are adjacent. If $x=uv$ is an edge of G then the degree of x in $L(G)$ is clearly $d(u)+d(v)-2$. If G is Eulerian then the line graph $L(G)$ is Eulerian Harary (1969).

Theorem 3.3.

Let $\Im$ be the class of Eulerian graphs, $G\in\Im$ be simple and $|V(G)|\neq 3$. Then, $e-red_{\Im}(L(G))=3$ if and only if $\exists v\in V(G)$ such that $d(v)\geq 3$.

Proof:

Assume that $e-red_{\Im}(L(G))=3$. Then, by Theorem 3.1, there exists a cycle C of length 3 in $L(G)$ such that $L(G)-E(C)$ is Eulerian. If there is no $v\in V(G)$ such that $d(v)\geq 3$, then G is a cycle and $L(G)$ is also a cycle (a contradiction). Hence, it is necessary that G contains a vertex with degree greater than or equal to 3.

Conversely, let G be an Eulerian graph containing a vertex v with $d(v)\geq 3$. Since G is Eulerian $d(v)\geq 4$. Thus, the sub graph H of $L(G)$ induced by the edges incident at v in $L(G)$ is complete graph on at least 4 vertices. We select any triangle C in H, and assert that $L(G)-E(C)$ is connected. This assertion is clearly true as any two vertices of C can be joined by a path in $L(G)$ which does not contain any edge of C. Now taking into account that $L(G)$ is simple the result follows by Theorem 3.1.

Theorem 3.4.

Let $\Im$ be the class of Eulerian digraphs, and $D \in \Im$. Then $a - red_{\Im}(D) = k$ if and only if k is the length of a smallest cycle C in D such that $D - A(C)$ is strongly connected.

Proof:

Suppose that $a - red_{\Im}(D) = k$. Then, there exists a set of arcs $\{a_1, a_2, ..., a_k\}$ such that $D_1 = D - \{a_1, a_2, ..., a_k\}$ is an Eulerian digraph; it is smallest such set.

Let C be the sub digraph formed by $\{a_1, a_2, ..., a_k\}$. It is clear that $D - A(C)$ is strongly connected. We prove that C is a cycle in D. Observe that in-degree and out-degree are equal for every vertex in C. In particular, a strong component of C is Eulerian. Hence, if C is not a cycle then it contains a cycle C_0 properly. As $D_1 = D - A(C)$ is strongly connected, $D_2 = D - A(C_0))$ is also strongly connected, and hence D_2 is Eulerian. This contradicts to our assumption that $a - red_{\Im}(D)$ is k. Therefore C is a cycle in D.

Conversely, assume that k is the length of a smallest cycle C in D such that $D - A(C)$ is strongly connected. We prove that $a - red_{\Im}(D) = k$.

Let $\{a_1, a_2, ..., a_k\}$ be the set of arcs of the cycle C. Therefore, $D - \{a_1, a_2, ..., a_k\}$ is an Eulerian digraph and we get $a - red_{\Im}(D) \leq k$ .If $a - red_{\Im}(D) \neq k$, then there exists a set $\{f_1, f_2, ..., f_n\}$ of arcs in D, with $n < k$ such that $D - \{f_1, f_2, ..., f_n\}$ is an Eulerian digraph. But then $\{f_1, f_2, ..., f_n\}$ forms a cycle as proved in the previous part, which is impossible due to the choice of k. We conclude that $a - red_{\Im}(D) = k$.

4. Conclusions

We conclude that the properties of the graph may be studied by reducing its vertex set. We found that some graphs contain a deletable vertex (edge) and others don't. We found the necessary and sufficient condition for Eulerian graphs to be vertex (edge) reducible. There are many other classes of graphs for example the class of regular graphs, the class of Hamiltonian graphs and others to be discussed in the future.

REFERENCES

Akram. B. Attar (2012). On removable cycles in graphs and digraphs, CJMS. 1(1) (2012), 20-26.

Akram B. Attar (2009). Extensibility of Graphs**,** Journal of Applied Mathematics, Islamic Azad University of Lahijan Vol.6, No.21.

Akram B. Attar (2007). Contractibility of Regular Graphs, J. C. E Al-Mustanseriya Uni. No. 3.

Bordalo and Monjardet (1996). Reducible classes of finite lattices, Order.

Borse Y. M. and Waphare B. N. (2008). On Removable Cycle in Connected Graph, JIMS 18535.iM.

Clark and Holton (1991). A First Look at Graph Theory, World Scientific, London.

Good (1996). Normal recurring decimals, J. Lond. Math. Soc. 13, 379-390.

Harary (1969). Graph Theory, Addison-Wesley, Reading MA.

Kharat and Waphare (2001). Reducibility in finite posets, Europ. J. Combinatorics 22, 197-205.

Mader (1974). Kreuzungsfreie a; b-Wege in endlichen Graphen (German), Abh. Math.Sem. Univ. Hamburg 42, 187–204.

Tutte (1984). *Graph Theory*, Addison-Wesley.

West (1999). *Introduction to Graph Theory, University of Illinois-Urbana*, Prentice Hall of India.

Wilson and Beineke (1978). *Selected Topics in Graph Theory*, Academic Press, London.