Certain Fractional Integral Operators and the Generalized Incomplete Hypergeometric Functions

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Abstract

In this paper, we apply a certain general pair of operators of fractional integration involving Appell’s function $F_3$ in their kernel to the generalized incomplete hypergeometric functions $\gamma_q[z]$ and $\Gamma_q[z]$, which were introduced and studied systematically by Srivastava et al. in the year 2012. Some interesting special cases and consequences of our main results are also considered.

Keywords: Gamma function; Incomplete Gamma functions; Decomposition formula; Incomplete Pochhammer symbols; Generalized incomplete hypergeometric functions; Fractional integral operators

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1. Introductions and Definitions

Throughout the present investigation, we shall (as usual) denote by $\mathbb{R}$ and $\mathbb{C}$ the sets of real and complex numbers, respectively. In terms of the familiar (Euler’s) Gamma function $\Gamma(z)$ which is defined, for $z \in \mathbb{C} \setminus \mathbb{Z}_0^-$, by

$$
\Gamma(z) = \begin{cases} 
\int_0^\infty t^{z-1} e^{-t} \, dt & (\Re(z) > 0) \\
\frac{\Gamma(z+n)}{n-1 \prod_{j=0}^{n-1} (z+j)} & (z \in \mathbb{C} \setminus \mathbb{Z}_0^-; \, n \in \mathbb{N}),
\end{cases}
$$

(1.1)

the Pochhammer symbol $\lambda_\nu$ ($\lambda, \nu \in \mathbb{C}$) is given, in general, by

$$
\lambda_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 
1 & (\nu = 0; \, \lambda \in \mathbb{C} \setminus \{0\}) \\
\lambda(\lambda+1) \cdots (\lambda+n-1) & (\nu \in \mathbb{N}; \, \lambda \in \mathbb{C}),
\end{cases}
$$

(1.2)

it being assumed conventionally that $(0)_0 := 1$ and understood tacitly that the $\Gamma$-quotient exists (see, for details, (Srivastava and Manocha, 1984, p. 21 et seq.)).

The closely-related incomplete Gamma functions $\gamma(z, \kappa)$ and $\Gamma(z, \kappa)$ defined, respectively, by

$$
\gamma(z, \kappa) := \int_0^\infty t^{z-1} e^{-t} \, dt \quad (\Re(z) > 0; \, \kappa \geq 0)
$$

(1.3)

and

$$
\Gamma(z, \kappa) := \int_\kappa^\infty t^{z-1} e^{-t} \, dt \quad (\kappa \geq 0; \, \Re(z) > 0 \text{ when } \kappa = 0),
$$

(1.4)

are known to satisfy the following decomposition formula:

$$
\gamma(z, \kappa) + \Gamma(z, \kappa) = \Gamma(z) \quad (\Re(z) > 0).
$$

(1.5)

The function $\Gamma(z)$ given by (1.1), and its incomplete versions $\gamma(z, \kappa)$ and $\Gamma(z, \kappa)$ given by (1.3) and (1.4), respectively, are known to play important and useful rôles in the study of the analytic solutions of a variety of problems in diverse areas of science and engineering (see, for example, (Abramowitz and (Editors), 1972), (Andrews, 1985), (Chaudhry and Zubair, 2001), (A. Erdélyi and Tricomi, 1953), (N. L. Johnson and Balakrishnan, 1995), (A. A. Kilbas and Trujillo, 2006), (Luke, 1975), (W. Magnus and Soni, 1966), (K. B. Oldham and Spanier, 2009), (F. W. J. Olver and Clark, 2010), (Srivastava and Choi, 2001), (Srivastava and Choi, 2012), (Srivastava and Karlsson, 1985), (Srivastava and Kashyap, 1982), (Temme, 1996), (Watson, 1944) and (Whittaker and Watson, 1973); see also (H. M. Srivastava and Agarwal, 2012) and especially the references cited therein).

In view of the great potential for applications in a wide variety of fields, Srivastava et al. (H. M. Srivastava and Agarwal, 2012) introduced and studied systematically the following family
of generalized incomplete hypergeometric functions (H. M. Srivastava and Agarwal, 2012, p. 675, Equations (4.1) and (4.2)):

\[ p^\gamma_q \begin{bmatrix} (a_1, \kappa), a_2, \cdots, a_p; z \\ b_1, \cdots, b_q; \end{bmatrix} := \sum_{n=0}^{\infty} \frac{(a_1; \kappa)_n a_2 \cdots a_p z^n}{(b_1)_n \cdots (b_q)_n n!} \]  \hspace{1cm} (1.6)

and

\[ p^\Gamma_q \begin{bmatrix} (a_1, \kappa), a_2, \cdots, a_p; z \\ b_1, \cdots, b_q; \end{bmatrix} := \sum_{n=0}^{\infty} \frac{(a_1; \kappa)_n a_2 \cdots a_p z^n}{(b_1)_n \cdots (b_q)_n n!}, \]  \hspace{1cm} (1.7)

where, in terms of the incomplete Gamma functions \( \gamma(z, \kappa) \) and \( \Gamma(z, \kappa) \) defined by (1.3) and (1.4), respectively, the \textit{incomplete} Pochhammer symbols

\[ (\lambda; \kappa)_\nu \quad \text{and} \quad [\lambda; x]_\nu \quad (\lambda, \nu \in \mathbb{C}; x \geq 0) \]

are defined as follows:

\[ (\lambda; \kappa)_\nu := \frac{\gamma(\lambda + \nu, \kappa)}{\Gamma(\lambda)} \quad (\lambda, \nu \in \mathbb{C}; \kappa \geq 0) \]  \hspace{1cm} (1.8)

and

\[ [\lambda; \kappa]_\nu := \frac{\Gamma(\lambda + \nu, \kappa)}{\Gamma(\lambda)} \quad (\lambda, \nu \in \mathbb{C}; \kappa \geq 0). \]  \hspace{1cm} (1.9)

so that, obviously, these incomplete Pochhammer symbols \( (\lambda; \kappa)_\nu \) and \( [\lambda; \kappa]_\nu \) satisfy the following decomposition formula:

\[ (\lambda; \kappa)_\nu + [\lambda; \kappa]_\nu = (\lambda)_\nu \quad (\lambda, \nu \in \mathbb{C}; \kappa \geq 0), \]  \hspace{1cm} (1.10)

where \( (\lambda)_\nu \) is the Pochhammer symbol given by (1.2).

\textbf{Remark 1.} As already pointed out by Srivastava et al. (H. M. Srivastava and Agarwal, 2012, p. 675, Remark 7), since

\[ |(\lambda; \kappa)_n| \leq |(\lambda)_n| \quad \text{and} \quad |[\lambda; \kappa]_n| \leq |(\lambda)_n| \quad (n \in \mathbb{N}_0; \lambda \in \mathbb{C}; \kappa \geq 0), \]  \hspace{1cm} (1.11)

the precise (sufficient) conditions under which the infinite series in the definitions (1.6) and (1.7) would converge absolutely can be derived from those that are well-documented in the case of the generalized hypergeometric function \( {}_pF_q (p, q \in \mathbb{N}_0) \) (see, for details, (Rainville, 1971, pp. 72–73) and (Srivastava and Karlsson, 1985, p. 20); see also (Bailey, 1964), (Carlson, 1977), (Luke, 1975) and (Slater, 1966)). Indeed, in their special case when \( \kappa = 0 \), both \( p^\gamma_q (p, q \in \mathbb{N}_0) \) and \( p^\Gamma_q (p, q \in \mathbb{N}_0) \) would reduce immediately to the widely- and extensively-investigated generalized hypergeometric function \( {}_pF_q (p, q \in \mathbb{N}_0) \). Furthermore, as an immediate consequence
of the definitions (1.6) and (1.7), we have the following decomposition formula:

\[ p^\gamma_q \left[ \left( a_1, \kappa, a_2, \ldots, a_p; b_1, \ldots, b_q; z \right) \right] + p^\Gamma_q \left[ \left( a_1, \kappa, a_2, \ldots, a_p; b_1, \ldots, b_q; z \right) \right] = p^F_q \left[ \left( a_1, \ldots, a_p; b_1, \ldots, b_q; z \right) \right] \]  \tag{1.12}

in terms of the familiar generalized hypergeometric function \( p^F_q \) (\( p, q \in \mathbb{N}_0 \)).

The above-mentioned detailed and systematic investigation by Srivastava et al. (H. M. Srivastava and Agarwal, 2012) was indeed motivated largely by the demonstrated potential for applications of the generalized incomplete hypergeometric functions \( p^\gamma_q \) and \( p^\Gamma_q \) and their special cases in many diverse areas of mathematical, physical, engineering and statistical sciences (see, for details, (H. M. Srivastava and Agarwal, 2012) and the references cited therein). Several further properties of each of these generalized incomplete hypergeometric functions and some classes of incomplete hypergeometric polynomials associated with them can be found in the subsequent developments presented in (for example) (Srivastava, 2013b), (Srivastava, 2013a) and (Srivastava and Cho, 2012). Moreover, by using the incomplete Pochhammer symbols given by (1.8) and (1.9), the corresponding incomplete versions of Appell’s two-variable hypergeometric function \( F_2 \) were considered recently in (Çetinkaya, 2013). In the present sequel to these recent works, we propose to derive several image formulas for the generalized incomplete hypergeometric functions \( p^\gamma_q \) and \( p^\Gamma_q \) by applying a certain general pair of fractional integral operators involving Appell’s two-variable hypergeometric function \( F_3 \), which we introduce in Section 2 below. We also consider some interesting special cases and consequences of our main results.

2. Operators of Fractional Integration and Their Applications

In view of their importance and popularity in recent years, the theory of operators of fractional calculus has been developed widely and extensively (see, for example, each of the research monographs (A. Erdélyi and Tricomi, 1954, Chapter 13), (A. A. Kilbas and Trujillo, 2006), (Kiryakova, 1993), (McBride, 1979), (Miller and Ross, 1993), (Oldham and Spanier, 1974), (Podlubny, 1999) and (S. G. Samko and Marichev, 1993); see also (Srivastava and Saxena, 2001)). Here, in this section, we recall a general pair of fractional integral operators which involve in the kernel Appell’s two-variable hypergeometric function \( F_3 \) defined by (see (Appell and de Fériet, 1926, p. 14))

\[ F_3(\alpha, \alpha', \beta, \beta'; \omega; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\omega)_{m+n}} \frac{x^m y^n}{m! n!} \]  \tag{2.1}

\( (\max\{|x|,|y|\} < 1) \).
Indeed, for
\[ x > 0 \quad \text{and} \quad \alpha, \alpha', \beta, \beta', \omega \in \mathbb{C} \quad (\Re(\omega) > 0), \]
these general operators of fractional integration with the \( F_3 \) kernel are defined by
\[
\left( I_{0, x}^{\alpha, \alpha', \beta, \beta', \omega} f \right)(x) := \frac{x^{-\alpha}}{\Gamma(\omega)} \int_0^x (x - t)^{\omega - 1} t^{-\alpha'} F_3 \left( \alpha, \alpha', \beta, \beta'; \omega; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) f(t) \, dt \tag{2.2}
\]
and
\[
\left( I_{x, \infty}^{\alpha, \alpha', \beta, \beta', \omega} f \right)(x) := \frac{x^{-\alpha'}}{\Gamma(\omega)} \int_x^\infty (t - x)^{\omega - 1} t^{-\alpha} F_3 \left( \alpha, \alpha', \beta, \beta'; \omega; 1 - \frac{x}{t}, 1 - \frac{t}{x} \right) f(t) \, dt, \tag{2.3}
\]
where the function \( f(t) \) is so constrained that the defining integrals in (2.2) and (2.3) exist.

The operators or integral transforms in (2.2) and (2.3) were introduced by Marichev (Marichev, 1974) as Mellin type convolution operators with the Appell function \( F_3 \) in their kernel. These operators were rediscovered and studied by Saigo (Saigo, 1996) as generalizations of the so-called Saigo fractional integral operators (see also (Kiryakova, 2006) and (Srivastava and Saigo, 1987)). Such further properties as (for example) their relations with the Mellin transform and with the hypergeometric operators (or the Saigo fractional integral operators), together with their decompositional, operational and other properties in the McBride space \( F_{p, \mu} \) (see (McBride, 1979)) were studied by Saigo and Maeda (Saigo and Maeda, 1998) (see also some recent investigations on the subject of fractional calculus in (Agarwal, 2012a; Agarwal, 2012b; Agarwal and Jain, 2011; J. A. T. Machado and Mainardi, 2010; J. A. T. Machado and Mainardi, 2011; S. D. Purohit and Kalla, 2011)).

**Remark 2.** The Appell function \( F_3 \) involved in the definitions (2.2) and (2.3) satisfies a system of two linear partial differential equations of the second order and reduces to the Gauss hypergeometric function \( _2F_1 \) as follows (see (Appell and de Fériet, 1926, p. 25, Eq. (35)) and (Srivastava and Karlsson, 1985, p. 301, Eq. 9.4(87))):

\[
F_3(\alpha, \omega - \alpha, \beta, \omega - \beta; \omega; x, y) = _2F_1 \left[ \begin{array}{c} \alpha, \beta; \\ \omega; \\ x + y - xy \end{array} \right]. \tag{2.4}
\]

Moreover, it is easily observed that

\[
F_3(\alpha, 0, \beta, \beta'; \omega; x, y) = F_3(\alpha, \alpha', 0; \omega; x, y) = _2F_1 \left[ \begin{array}{c} \alpha, \beta; \\ \omega; \\ x \end{array} \right] \tag{2.5}
\]
and
are easy consequences of the definitions in ((Saigo and Maeda, 1998, p. 394)):

\[ F_3(0, \alpha', \beta, \beta'; \omega; x, y) = F_3(\alpha, \alpha', 0, \beta'; \omega; x, y) = _2F_1 \left[ \begin{array}{c} \alpha', \beta'; \\ \omega' \end{array} \right]. \quad (2.6) \]

In view of the obvious reduction formula (2.5), the general operators reduce to the aforementioned Saigo operators \( T_{0,x}^{\alpha,\beta,\omega} \) and \( T_{x,\infty}^{\alpha,\beta,\omega} \) defined by (see, for details, (Saigo, 1996); see also (Kiryakova, 2006) and (Srivastava and Saigo, 1987) and the references cited therein)

\[
T_{0,x}^{\alpha,\beta,\omega}(f)(x) := \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \ _2F_1 \left( \alpha + \beta, -\omega; 1 - \frac{t}{x} \right) f(t) dt
\]

and

\[
T_{x,\infty}^{\alpha,\beta,\omega}(f)(x) := \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} \ _2F_1 \left( \alpha + \beta, -\omega; 1 - \frac{x}{t} \right) f(t) dt,
\]

respectively. In fact, we have the following relationships:

\[
T_{0,x}^{\alpha,\beta,\omega}(f)(x) = (T_{0,x}^{\omega,\alpha-\omega,-\beta} f)(x) \quad (\omega \in \mathbb{C})
\]

and

\[
T_{x,\infty}^{\alpha,\beta,\omega}(f)(x) = (T_{x,\infty}^{\omega,\alpha-\omega,-\beta} f)(x) \quad (\omega \in \mathbb{C}).
\]

In our investigation, we shall make use of each of the following known image formulas which are easy consequences of the definitions in ((Saigo and Maeda, 1998, p. 394)):

\[
\left( I_{0,x}^{\alpha,\alpha',\beta,\beta',\omega} t^{\rho-1} \right)(x) = \frac{\Gamma(\rho)\Gamma(\rho + \omega - \alpha - \alpha' - \beta)\Gamma(\rho - \alpha' + \beta')}{\Gamma(\rho + \beta')\Gamma(\rho + \omega - \alpha - \alpha')\Gamma(\rho + \omega - \alpha' - \beta)} x^{\rho + \omega - \alpha - \alpha' - 1} \quad (\Re(\omega) > 0; \ \Re(\rho) > \max \{0, \Re(\alpha + \alpha' + \beta - \omega), \Re(\alpha' - \beta')\})
\]

and

\[
\left( I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\omega} t^{\rho-1} \right)(x) = \frac{\Gamma(1 - \rho - \beta)\Gamma(1 - \rho - \omega + \alpha + \alpha')\Gamma(1 - \rho + \alpha + \beta' - \omega)}{\Gamma(1 - \rho)\Gamma(1 - \rho + \alpha + \alpha' + \beta' - \omega)\Gamma(1 - \rho + \alpha - \beta)} x^{\rho + \omega - \alpha - \alpha' - 1} \quad (\Re(\omega) > 0; \ 0 < \Re(\rho) < 1 + \min \{\Re(-\beta), \Re(\alpha + \alpha' - \omega), \Re(\alpha + \beta' + \omega)\})
\]

We now state and prove our main fractional integral formulas involving each of the generalized incomplete hypergeometric functions \( {}_p\gamma_q \) and \( {}_p\Gamma_q \) defined by (1.6) and (1.7).
Theorem 1. Let \( x > 0 \) and \( \kappa \geq 0 \). Suppose also that the parameters \( \alpha, \alpha', \beta, \beta', \mu, \omega, \rho \in \mathbb{C} \) are constrained by

\[
\Re(\omega) > 0 \quad \text{and} \quad \Re(\rho) > \max\{0, \Re(\alpha + \alpha' + \beta - \omega), \Re(\alpha' - \beta')\}.
\]

Then each of the following fractional integral formulas holds true:

\[
\left( I_{0, x}^{[\alpha, \alpha', \beta, \beta', \omega]} \left[ t^{\rho-1} p \gamma_q(\mu t) \right] \right)(x) = x^{\rho + \omega - \alpha - \alpha' - 1} \frac{\Gamma(\rho) \Gamma(\rho + \beta' - \alpha') \Gamma(\rho + \omega - \alpha - \beta - \alpha')}{\Gamma(\rho + \beta') \Gamma(\rho + \omega - \alpha - \alpha') \Gamma(\rho + \omega - \beta - \alpha')} \left[ (a_1, \kappa), a_2, \ldots, a_p, \rho, \rho + \beta' - \alpha', \rho + \omega - \alpha - \beta - \alpha'; \mu x \right]_{n+3} \tag{2.13}
\]

and

\[
\left( I_{x, \infty}^{[\alpha, \alpha', \beta, \beta', \omega]} \left[ t^{\rho-1} p \Gamma_q(\mu t) \right] \right)(x) = x^{\rho + \omega - \alpha - \alpha' - 1} \frac{\Gamma(\rho) \Gamma(\rho + \beta' - \alpha') \Gamma(\rho + \omega - \alpha - \beta - \alpha')}{\Gamma(\rho + \beta') \Gamma(\rho + \omega - \alpha - \alpha') \Gamma(\rho + \omega - \beta - \alpha')} \left[ (a_1, \kappa), a_2, \ldots, a_p, \rho, \rho + \beta' - \alpha', \rho + \omega - \alpha - \beta - \alpha'; \mu x \right]_{n+3}, \tag{2.14}
\]

provided that each member of the assertions (2.13) and (2.14) exists.

Proof: For convenience, we denote by \( \Omega(x) \) the left-hand side of the first assertion (2.13) of Theorem 1. Then, by applying the definition (1.6) and changing the order of integration and summation, we get

\[
\Omega(x) := \left( I_{0, x}^{[\alpha, \alpha', \beta, \beta', \omega]} \left[ t^{\rho-1} p \gamma_q(\mu t) \right] \right)(x) = \left( I_{0, x}^{[\alpha, \alpha', \beta, \beta', \omega]} \left[ t^{\rho-1} \sum_{n=0}^{\infty} \frac{\Gamma(\rho + \beta' - \alpha') \Gamma(\rho + \omega - \alpha - \beta - \alpha')}{\Gamma(\rho + \beta') \Gamma(\rho + \omega - \alpha - \alpha') \Gamma(\rho + \omega - \beta - \alpha')} \cdot \frac{(a_1; \kappa)_n (a_2)_n \cdots (a_p)_n \cdot (\mu t)^n}{n!} \right] \right)(x) = \left( I_{0, x}^{[\alpha, \alpha', \beta, \beta', \omega]} \left[ t^{\rho+n-1} \right] \right)(x), \tag{2.15}
\]

where the inversion of the order of integration and summation can be justified, under the conditions given with Theorem 1, by the absolute convergence of the integral involved and the uniform convergence of the series involved.

Since \( n \in \mathbb{N}_0 \), we can make use of the definition (2.11) with \( \rho \) replaced by \( \rho + n \) \( (n \in \mathbb{N}_0) \). We thus find from (2.15) that

\[
\Omega(x) = x^{\rho + \omega - \alpha - \alpha' - 1} \sum_{n=0}^{\infty} \frac{(a_1; \kappa)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \cdot \frac{\Gamma(\rho + n) \Gamma(\rho + \beta' - \alpha' + n) \Gamma(\rho + \omega - \alpha - \beta - \alpha' + n)}{\Gamma(\rho + \beta' + n) \Gamma(\rho + \omega - \alpha - \alpha' + n) \Gamma(\rho + \omega - \beta - \alpha' + n)} \left[ (a_1; \kappa)_n (a_2)_n \cdots (a_p)_n \cdot (\mu x)^n \right]_{n+3} \tag{2.16}
\]
By interpreting the last member of (2.16) by means of the definition (1.6), we obtain the right-hand side of (2.13). Similarly, we can derive the second assertion (2.14) of Theorem 1 by applying the definitions (1.7) and (2.12). This completes the proof of Theorem 1.

**Theorem 2.** Let \( x > 0 \) and \( \kappa \geq 0 \). Suppose also that the parameters \( \alpha, \alpha', \beta, \beta', \mu, \omega, \rho \in \mathbb{C} \) are constrained by
\[
\Re(\omega) > 0 \quad \text{and} \quad 0 < \Re(\rho) < 1 + \min \{ \Re(-\beta), \Re(\alpha + \alpha' - \omega), \Re(\alpha + \beta' - \omega) \}.
\]
Then each of the following fractional integral formulas holds true:
\[
\left( I_{0,x}^{(\alpha,\alpha',\beta,\beta',\omega)} \left[ t^{p-1} p^\gamma_q \left( \frac{\mu}{t} \right) \right] (x) \right) = x^{\rho+\omega+\alpha-\alpha'-1} 
\]
\[
\quad \frac{\Gamma(1-\rho-\beta)\Gamma(1-\rho-\omega+\alpha+\alpha')\Gamma(1-\rho-\omega+\alpha+\beta')}{\Gamma(1-\rho)\Gamma(1-\rho-\omega+\alpha+\alpha'+\beta')\Gamma(1-\rho+\alpha-\beta)} 
\]
\[
\cdot \mu \sum_{a_1, \cdots, a_p} (a_1, \kappa, a_2, \cdots, a_p, 1 - \rho - \beta, 1 - \rho - \omega + \alpha + \alpha', 1 - \rho - \omega + \alpha + \beta') 
\]
\[
\quad b_1, \cdots, b_q, 1 - \rho, 1 - \rho - \omega + \alpha + \alpha' + \beta', 1 - \rho + \alpha - \beta; \frac{\mu}{x} \tag{2.17}
\]
and
\[
\left( I_{\infty,x}^{(\alpha,\alpha',\beta,\beta',\omega)} \left[ t^{p-1} p^\gamma_q \left( \frac{\mu}{t} \right) \right] (x) \right) = x^{\rho+\omega+\alpha-\alpha'-1} 
\]
\[
\quad \frac{\Gamma(1-\rho-\beta)\Gamma(1-\rho-\omega+\alpha+\alpha')\Gamma(1-\rho-\omega+\alpha+\beta')}{\Gamma(1-\rho)\Gamma(1-\rho-\omega+\alpha+\alpha'+\beta')\Gamma(1-\rho+\alpha-\beta)} 
\]
\[
\cdot \mu \sum_{a_1, \cdots, a_p} (a_1, \kappa, a_2, \cdots, a_p, 1 - \rho - \beta, 1 - \rho - \omega + \alpha + \alpha', 1 - \rho - \omega + \alpha + \beta') 
\]
\[
\quad b_1, \cdots, b_q, 1 - \rho, 1 - \rho - \omega + \alpha + \alpha' + \beta', 1 - \rho + \alpha - \beta; \frac{\mu}{x} \tag{2.18}
\]
provided that each member of the assertions (2.17) and (2.18) exists.

**Proof:** The proofs of the fractional integral formulas (2.17) and (2.18) would run parallel to those of (2.13) and (2.14) asserted by Theorem 1. We, therefore, choose to skip the details involved.

3. **Corollaries and Consequences of Theorems 1 and 2**

Upon setting \( \alpha' = 0 \) in Theorems 1 and 2, if we use the relationships (2.9) and (2.10), we can deduce the following interesting corollaries involving the generalized incomplete hypergeometric functions \( p^\gamma_q \) and \( p^\Gamma_q \) defined by (1.6) and (1.7), respectively, and the Saigo fractional integral operators
\[
\left( I_{0,x}^{(\alpha,\beta,\omega)} f \right) (x) \quad \text{and} \quad \left( I_{\infty,x}^{(\alpha,\beta,\omega)} f \right) (x)
\]
defined by (2.7) and (2.8), respectively.
Corollary 1. Let $x > 0$ and $\kappa \geq 0$. Suppose also that the parameters $\alpha, \beta, \mu, \omega, \rho \in \mathbb{C}$ are constrained by
\[ \Re(\omega) > 0 \quad \text{and} \quad \Re(\rho) > \max \{0, \Re(\omega - \alpha - \beta)\}. \]

Then each of the following fractional integral formulas holds true:
\[
\left( I_{0,x}^{(\omega,\alpha - \omega, -\beta)} [t^{\rho - 1} \rho^q (\mu t)] (x) \right) = x^{\rho + \omega - \alpha - 1} \frac{\Gamma(\rho)(\rho + \omega - \alpha - \beta)}{\Gamma(\rho + \omega - \alpha)} \cdot p+2\gamma + 2 
\begin{bmatrix}
(a_1, \kappa), a_2, \cdots, a_p, \rho, \rho + \omega - \alpha - \beta; \\
\mu \cdot b_1, \cdots, b_q, \rho + \omega - \alpha, \rho + \omega - \beta;
\end{bmatrix}
\]  
\[ \tag{3.1} \]

and
\[
\left( I_{x,\infty}^{(\omega,\alpha - \omega, -\beta)} [t^{\rho - 1} \rho^q (\mu t)] (x) \right) = x^{\rho + \omega - \alpha - 1} \frac{\Gamma(\rho)(\rho + \omega - \alpha - \beta)}{\Gamma(\rho + \omega - \alpha)} \cdot p+2\gamma + 2 
\begin{bmatrix}
(a_1, \kappa), a_2, \cdots, a_p, \rho, \rho + \omega - \alpha - \beta; \\
\mu \cdot b_1, \cdots, b_q, \rho + \omega - \alpha, \rho + \omega - \beta;
\end{bmatrix}
\]  
\[ \tag{3.2} \]

provided that each member of the assertions (3.1) and (3.2) exists.

Corollary 2. Let $x > 0$ and $\kappa \geq 0$. Suppose also that the parameters $\alpha, \beta, \mu, \omega, \rho \in \mathbb{C}$ are constrained by
\[ \Re(\omega) > 0 \quad \text{and} \quad 0 < \Re(\rho) < 1 + \min \{\Re(-\beta), \Re(\alpha - \omega)\}. \]

Then each of the following fractional integral formulas holds true:
\[
\left( I_{x,\infty}^{(\omega,\alpha - \omega, -\beta)} [t^{\rho - 1} \rho^q (\mu t)] (x) \right) = x^{\rho + \omega - \alpha - 1} \frac{\Gamma(1 - \rho - \beta)(1 - \rho - \omega + \alpha)}{\Gamma(1 - \rho)(1 - \rho + \alpha - \beta)} \cdot p+2\gamma + 2 
\begin{bmatrix}
(a_1, \kappa), a_2, \cdots, a_p, 1 - \rho - \beta, 1 - \rho - \omega + \alpha; \\
\mu \cdot b_1, \cdots, b_q, 1 - \rho, 1 - \rho + \alpha - \beta;
\end{bmatrix}
\]  
\[ \tag{3.3} \]

and
\[
\left( I_{x,\infty}^{(\omega,\alpha - \omega, -\beta)} [t^{\rho - 1} \rho^q (\mu t)] (x) \right) = x^{\rho + \omega - \alpha - 1} \frac{\Gamma(1 - \rho - \beta)(1 - \rho - \omega + \alpha)}{\Gamma(1 - \rho)(1 - \rho + \alpha - \beta)} \cdot p+2\gamma + 2 
\begin{bmatrix}
(a_1, \kappa), a_2, \cdots, a_p, 1 - \rho - \beta, 1 - \rho - \omega + \alpha; \\
\mu \cdot b_1, \cdots, b_q, 1 - \rho, 1 - \rho + \alpha - \beta;
\end{bmatrix}
\]  
\[ \tag{3.4} \]

provided that each member of the assertions (3.3) and (3.4) exists.

Remark 3. Several further consequences of Corollaries 1 and 2 of this section can easily be derived by setting (for example) $\beta = -\alpha$ or $\beta = 0$. Such interesting consequences of our results would involve the Erdélyi-Kober fractional integral operators $E_{0,x}^{\omega,\eta}$ and $K_{x,\infty}^{\omega,\eta}$, the Riemann-Liouville fractional integral operator $R_{0,x}^{\alpha}$ and the Weyl fractional integral operator $W_{x,\infty}^{\alpha}$. These relatively simpler fractional integral formulas for each of the generalized incomplete
hypergeometric functions $p^\gamma_q$ and $p^\Gamma_q$ defined by (1.6) and (1.7), respectively, can be deduced from Corollaries 1 and 2 by appropriately applying the following relationships (see, for example, (A. M. Mathai and Haubold, 2010)):

\[
(R_0^\alpha f)(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt = (I^\alpha_0 f)(x),
\]

(3.5)

\[
(W^\alpha f)(x) := \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)dt = (I^\alpha_\infty f)(x),
\]

(3.6)

\[
(E^\alpha f)(x) := \frac{x^{-\alpha-\omega}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^\omega f(t)dt = (I^\alpha f_0)(x)
\]

(3.7)

and

\[
(K^\alpha f)(x) := \frac{x^\omega}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\omega} f(t)dt = (I^\alpha f_\infty)(x).
\]

(3.8)

4. Conclusion

We conclude our present investigation by remarking further that the results obtained here are useful in deriving various fractional integral formulas for each of the families of the generalized incomplete hypergeometric functions $p^\gamma_q$ and $p^\Gamma_q$ defined by (1.6) and (1.7), respectively, involving such relatively more familiar fractional integral operators as (for example) the Riemann-Liouville fractional integral operator $R_0^\alpha$ defined in (3.5), the Weyl fractional integral operator $W^\alpha$ defined in (3.6), and the Erdélyi-Kober fractional integral operators $E^\alpha_0$ and $K^\alpha_\infty$ defined in (3.7) and (3.8), respectively.

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