Exact Soliton Solutions for Second-Order Benjamin-Ono Equation

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Received: July 15, 2010; Accepted: February 3, 2011

Abstract

The homogeneous balance method is proposed for seeking the travelling wave solutions of the second-order Benjamin-Ono equation. Many exact traveling wave solutions of second-order Benjamin-Ono equation, which contain soliton like and periodic-like solutions are successfully obtained. This method is straightforward and concise, and it may also be applied to other nonlinear evolution equations.

Keywords: Homogeneous balance method; second-order Benjamin-Ono equation; Riccati equation; Soliton-like solution; Periodic-like solution.

MSC 2000: 34B15; 47E05; 35G25.

1. Introduction

It is well known that the nonlinear partial differential equations (NPDEs) are widely used to describe complex phenomena in various fields of sciences, such as physics, biology, chemistry, etc. Exact solutions of these equations are therefore very important and significant in the nonlinear sciences.


The aim of this paper is to find exact soliton solutions of the second-order Benjamin-Ono Equation, using the homogeneous balance method.
2. Second-Order Benjamin-Ono Equation

For the second-order Benjamin-Ono equation [Hereman et al. (1986)]:

\[ u_{tt} + \alpha (u^2)_{xx} + \beta u_{xxxx} = 0, \]
\( (1) \)

where \( \alpha \) and \( \beta \) are nonnegative constants, Let us consider the traveling wave solutions

\[ u(x,t) = u(\xi), \quad \xi = k(x + lt) + \xi_0, \]
\( (2) \)

Where \( k, l \) and \( \xi_0 \) are constants. Then Eq. (1) becomes

\[ l^2 u'' + 2\alpha (u')^2 + 2\alpha uu'' + \beta k^2 u''' = 0. \]
\( (3) \)

We now seek the solutions of Eq. (3) in the form

\[ u = \sum_{i=0}^{m} q_i \phi^i, \]
\( (4) \)

where \( q_i \) are constants to be determined later and \( \phi \) satisfy the following Riccati equation

\[ \phi' = a\phi^2 + b\phi + c \]
\( (5) \)

where \( a,b \) and \( c \) are constants.
Balancing the highest order derivative term with nonlinear term in Eq. (3) gives leading order \( m = 2 \). We may therefore choose

\[ u = q_0 + q_1 \phi + q_2 \phi^2, \]
\( (6) \)

where \( q_0, q_1 \) and \( q_2 \) are constants to be determined and \( \phi \) satisfy Eq. (5).
Substituting (6) and (5) into Eq. (3), we have
\[ l^2u'' + 2\alpha(u')^2 + 2\alpha uu'' + \beta k^2 u'''' = l^2 (q_0 + q_1\phi + q_2\phi^2) \]

\[ + 2\alpha((q_0 + q_1\phi + q_2\phi^2)')^2 + 2\alpha(q_0 + q_1\phi + q_2\phi^2)(q_0 + q_1\phi) \]

\[ + q_2\phi^2'' + \beta k^2 (q_0 + q_1\phi + q_2\phi^2)'''' = \{20\alpha a^2 q_2 + 120\beta k^2 a^4 q_2\}\phi^6 \]

\[ + \{24\alpha a^2 q_4 q_2 + 24\beta k^2 a^4 q_1 + 336\beta k^2 a^3 b q_2 + 36\alpha a b q_2^2\}\phi^5 \]

\[ + \{6\alpha a^2 q_1^2 + 60\beta k^2 a^3 b q_1 + (240\beta k^2 c a^3 + 330\beta k^2 b^2 a^2 + 6l^2 a^2)q_2 \]

\[ + 12\alpha a^2 q_0 q_2 + 42\alpha a b q_2 + (32\alpha a c + 16\alpha b^2)q_2^3\}\phi^4 + \{10\alpha a b q_1^2 + \]

\[ (40\beta k^2 c a^3 + 2l^2 a^2 + 50\beta k^2 b^2 a^2)q_1 + 4\alpha a^2 q_0 q_1 + (36\alpha c a + 18\alpha b^2)q_1 q_2 \]

\[ + (10l^2 a b + 440\beta k^2 b c a^2 + 130\beta k^2 b^3 a)q_2 + 20\alpha a b q_0 q_2 + 28\alpha b c q_2^2\}\phi^3 \]

\[ + \{(8\alpha c a + 4\alpha b^2)q_1^2 + (60\beta k^2 b c a^2 + 15\beta k^2 b^3 a + 3l^2 b a)q_1 + 6\alpha b a q_0 q_1 \]

\[ + 30a b c q_2 + (8l^2 ac + 4l^2 b^2 + 136\beta k^2 c^2 a^2 + 232\beta k^2 c b^2 a + 16\beta k^2 b^4)q_2 \]

\[ + (16\alpha a c + 8b^2 \alpha)q_0 q_2 + 12a c q_2^2\}\phi^2 + \{6\alpha c b q_1^2 + (16\beta k^2 c^2 a^2 + 2a l^2 c \]

\[ + 22\beta k^2 b^2 c + \beta k^2 b^4 + l^2 b^2)q_1 + 12\alpha c^2 q_0 q_2 + (4\alpha a c + 2\alpha b^2)q_0 q_1 \]

\[ + 12a b c q_0 q_2 + (6l^2 b c + 120\beta k^2 b c a^2 + 30\beta k^2 b^3 c)q_2\}\phi + 2\alpha c^2 q_1^2 \]

\[ + 2a b c q_0 q_1 + (8\beta k^2 a c^2 b + \beta k^2 b^3 c + l^2 b c)q_1 + (16\beta k^2 c^3 a \]

\[ + 2l^2 c^2 + 14\beta k^2 b^2 c^2)q_2 + 4\alpha c^2 q_0 q_2 = 0. \]

Setting the coefficients \( \phi^i \) \((i = 0, 1, 2, 3, 4, 5, 6)\) to zero yields the following set of algebraic equations:

\[ 20\alpha a^2 q_2^2 + 120\beta k^2 a^4 q_2 = 0, \]

\[ 24\alpha a^2 q_4 q_2 + 24\beta k^2 a^4 q_1 + 336\beta k^2 a^3 b q_2 + 36\alpha a b q_2^2 = 0, \]

\[ 6\alpha a^2 q_1^2 + 60\beta k^2 a^3 b q_1 + (240\beta k^2 c a^3 + 330\beta k^2 b^2 a^2 + 6l^2 a^2)q_2 \]

\[ + 12\alpha a^2 q_0 q_2 + 42\alpha a b q_2 + (32\alpha a c + 16\alpha b^2)q_2^3 = 0, \]
For which, with the aid of Maple, we obtain the following solution of the above set of algebraic equations:

\[ q_0 = -\frac{\beta k^2 b^2 + 8\beta k^2 cb + 1}{2\alpha}, \quad q_1 = -\frac{6\beta k^2 ab}{\alpha}, \quad q_2 = -\frac{6\beta k^2 a^2}{\alpha}. \]  

For the Riccati equation (5), we can solve it by using the homogeneous balance method as follows:

**Case: I. Let** \( \phi = \sum_{i=0}^{m} b_i \tanh^i \xi. \) Balancing \( \phi' \) with \( \phi^2 \) leads to

\[ \phi = b_0 + b_1 \tanh \xi. \]  

Substituting (9) into (5), we obtain the following solution of (5):

\[ \phi = -\frac{1}{2a}(b + 2 \tanh \xi), \quad ac = \frac{b^2}{4} - 1. \]  

From (8), (10) and (6), we have the following traveling wave solution of second-order Benjamin-Ono equation (1):

\[ 10\alpha abq_1^2 + (40\beta k^2 ca^3 + 2l^2 a^2 + 50\beta k^2 b^2 a^2)q_1 + 4\alpha a^2 q_0 q_1 + (36\alpha ca + 18ab^2)q_2 + (10l^2 ab + 440\beta k^2 bca^2 + 130\beta k^2 b^3 a)q_2 + 20\alpha abq_0 q_2 + 28abcq_2^2 = 0, \]

\[ (8\alpha ac + 4\alpha b^2)q_1^2 + (60\beta k^2 bca^2 + 15\beta k^2 b^3 a + 3l^2 ba)q_1 + 6\alpha baq_0 q_1 + 30\alpha bcaq_2 + (8l^2 ac + 4l^2 b^2 + 136\beta k^2 c^2 a^2 + 232\beta k^2 c^2 b^2 a + 16\beta k^2 b^4)q_2 + (16\alpha ac + 8b^2 \alpha)q_0 q_2 + 12ac^2 q_2^2 = 0, \]

\[ 6\alpha cbq_1^2 + (16\beta k^2 c^2 a^2 + 2al^2 c)q_1 + 22\beta ak^2 b^2 c + \beta k^2 b^4 + l^2 b^2)q_1 + 12\alpha c^2 q_0 q_2 + (4\alpha ac + 2\alpha b^2)q_0 q_1 + 12abcq_0 q_2 + (6l^2 bc + 120\beta k^2 b^2 c^2 a + 30\beta k^2 b^3 c)q_2 = 0, \]

\[ 2\alpha c^2 q_1^2 + 2\alpha bcaq_0 q_1 + (8\beta k^2 c^2 b + \beta k^2 b^3 c + l^2 b c)q_1 + (16\beta k^2 c^3 a + 2l^2 c^2 + 14\beta k^2 b^2 c^2)q_2 + 4\alpha c^2 q_0 q_2 = 0. \]
\[ u(x,t) = \frac{2\beta k^2 b^2 - 8\beta k^2 ca - l^2}{2\alpha} - \frac{6k^2 \beta}{\alpha} \tanh^2[k(x + lt) + \xi_0], \quad (11) \]

where

\[ ac = \frac{b^2}{4} - 1. \]

Similarly, let \( \phi = \sum b_i \coth \xi_i \), then we obtain the following new traveling wave soliton solutions of second-order Benjamin-Ono equation (1):

\[ u(x,t) = \frac{2\beta k^2 b^2 - 8\beta k^2 ca - l^2}{2\alpha} - \frac{6k^2 \beta}{\alpha} \coth^2[k(x + lt) + \xi_0], \quad (12) \]

where

\[ ac = \frac{b^2}{4} - 1. \]

**Case: II.** From (Zhao, X. Q.; Tang, D. B.; 2002), when \( a=1, \ b=0 \), the Riccati Eq. (5) has the following solutions:

\[ \phi = -\sqrt{c} \tanh(-\sqrt{c} \xi), \quad c < 0, \]

\[ \phi = -\frac{1}{\xi}, \quad c = 0, \quad (13) \]

\[ \phi = \sqrt{c} \tan(\sqrt{c} \xi), \quad c > 0. \]

From (6), (8) and (13), we have the following traveling wave solutions of second-order Benjamin-Ono equation (1):

When \( c < 0 \), we have

\[ u(x,t) = -\frac{\beta k^2 b^2 + 8\beta k^2 ca + l^2}{2\alpha} + \frac{6\beta k^2 ab \sqrt{-c}}{\alpha} \tanh[\sqrt{-c}(k(x + lt) + \xi_0)] + \frac{6\beta k^2 a^2 c}{\alpha} \tanh^2[\sqrt{-c}(k(x + lt) + \xi_0)]. \quad (14) \]

When \( c = 0 \), we have
\[ u(x,t) = -\frac{\beta k^2 b^2 + 8\beta k^2 c a + l^2}{2\alpha} + \frac{6\beta k^2 ab}{\alpha (k (x + lt) + \xi_0)} - \frac{6\beta k^2 a^2}{\alpha (k (x + lt) + \xi_0)^2}. \]  

When \( c > 0 \), we have

\[ u(x,t) = -\frac{\beta k^2 b^2 + 8\beta k^2 c a + l^2}{2\alpha} - \frac{6\beta k^2 ab \sqrt{c}}{\alpha \tan[\sqrt{c} (k (x + lt) + \xi_0)]} - \frac{6\beta k^2 a^2 c}{\alpha \tan^2[\sqrt{c} (k (x + lt) + \xi_0)]}. \]  

**Case: III.** We suppose that the Riccati equation (5) has the following solutions of the form

\[ \phi = A_0 + \sum_{i=1}^{n} (A_i f^i + B_i f^{i-1} g), \]  

with

\[ f = \frac{1}{\cosh \xi + r}, \quad g = \frac{\sinh \xi}{\cosh \xi + r}, \]

which satisfy

\[ f' = -f g, \quad g' = 1 - g^2 - rf, \]

\[ g^2 = 1 - 2rf + (r^2 - 1)f^2. \]

Balancing \( \phi' \) with \( \phi^2 \) leads to

\[ \phi = A_0 + A_0 f + B_0 g. \]  

Substituting (18) into (5), collecting the coefficient of the same power

\[ f^i g^j (\xi) (i = 0, 1, 2; j = 0, 1) \]

and setting each of the obtained coefficients to zero yield the following set of algebra equations

\[ aA_0^2 + aB_1^2 + bA_0 + c = 0, \]

\[ 2aA_0 A_1 - 2arB_1^2 - rB_1 + bA_1 = 0, \]
\[ aA_1^2 + a(r^2 - 1)B_1^2 + (r^2 - 1)B_1 = 0, \]
\[ 2aA_0B_1 + bB_1 = 0, \]
\[ 2aA_1B_1 + A_1 = 0, \]

which have solutions
\[ A_0 = -\frac{b}{2a}, \quad A_1 = \pm \sqrt{\frac{(r^2 - 1)}{4a^2}}, \quad B_1 = -\frac{1}{2a}, \quad c = \frac{b^2 - 1}{4a}. \quad (19) \]

From (18), (19) we obtain
\[ \phi = -\frac{1}{2a}(b + \frac{\sinh \xi + \sqrt{(r^2 - 1)}}{\cosh \xi + r}). \quad (20) \]

Also from (6), (8) and (20), we obtain the new solutions of second-order Benjamin-Ono equation (1):
\[ u(x,t) = \frac{2\beta k^2 b^2 - 8\beta k^2 ca - l^2}{2a} + \frac{3k^2 \beta}{2a} \left( \frac{\sinh(k(x + lt) + \xi_0) + \sqrt{(r^2 - 1)}}{\cosh(k(x + lt) + \xi_0 + r)} \right)^2. \quad (21) \]

**Case: IV.** We take \( \phi \) in the Riccati equation (5) being of the form
\[ \phi = e^{p\xi} \rho(z) + p_4(\xi), \quad (22) \]

where
\[ z = e^{p\xi} + p_3, \]

where \( p_1, p_2 \) and \( p_3 \) are constants to be determined.

Substituting (22) into (5) we find that when \( c = \frac{-p_1^2 + b^2}{4a} \), we have
\[ \phi = -\frac{p_1 e^{p\xi}}{a(e^{p\xi} + p_3)} + \frac{p_1 - b}{2a}. \quad (23) \]

If \( p_3 = 1 \) in (23), we have
\[ \phi = -\frac{p_1}{2a} \tanh\left(\frac{1}{2} p_1 \xi\right) - \frac{b}{2a}. \]  

(24)

If \( p_3 = -1 \) in (23), we have

\[ \phi = -\frac{p_1}{2a} \coth\left(\frac{1}{2} p_1 \xi\right) - \frac{b}{2a}. \]  

(25)

From (6), (8) and (23), we obtain the following new traveling wave solutions of second-order Benjamin-Ono equation (1):

\[
\begin{align*}
   u(x,t) &= -\frac{\beta k^2 b^2 + 8\beta k^2 ca + l^2}{2\alpha} \\
   & \quad - \frac{6\beta k^2}{\alpha} \left( \frac{e^{p_1(k(x+b)+\xi_0)}}{e^{p_1(k(x+b)+\xi_0)}} + p_3 \right) - \frac{p_1 - b}{2} \left( \frac{e^{p_1(k(x+b)+\xi_0)}}{e^{p_1(k(x+b)+\xi_0)}} + p_3 \right) - \frac{p_1 + b}{2} 
\end{align*}
\]

(26)

When \( p_3 = 1 \), we obtain the following traveling wave (soliton-like) solutions of second-order Benjamin-Ono equation (1):

\[
\begin{align*}
   u(x,t) &= \frac{2\beta k^2 b^2 - 8\beta k^2 ca - l^2}{2\alpha} - \frac{3k^2 \beta p_1^2}{2\alpha} \tanh^2\left(\frac{p_1}{2}(k(x+lt)+\xi_0)\right) 
\end{align*}
\]

(27)

When \( p_3 = -1 \), we have the following traveling wave (periodic-like) solutions of equal width wave equation (1):

\[
\begin{align*}
   u(x,t) &= \frac{2\beta k^2 b^2 - 8\beta k^2 ca - l^2}{2\alpha} - \frac{3k^2 \beta p_1^2}{2\alpha} \coth^2\left(\frac{p_1}{2}(k(x+lt)+\xi_0)\right) 
\end{align*}
\]

(28)

**Case: V.** We suppose that the Riccati equation (5) have the following solutions of the form:

\[ \phi = A_0 + \sum_{i=1}^{m} \sinh^{i-1}(A_i \sinh w + B_i \cosh w), \]  

(29)

where \( \frac{dw}{d\xi} = \sinh w \) or \( \frac{dw}{d\xi} = \cosh w \). It is easy to find that \( m = 1 \) by balancing \( \phi' \) and \( \phi^2 \).

So we choose

\[ \phi = A_0 + A_i \sinh w + B_i \cosh w, \]  

(30)

when \( \frac{dw}{d\xi} = \sinh w \), we substitute (30) and \( \frac{dw}{d\xi} = \sinh w \), into (5) and set the coefficients of \( \sinh w \) \( \cosh w \) \( (i = 0,1,2; j = 0,1) \) to zero. A set of algebraic equations is obtained as follows:
\[ aA_0^2 + aB_1^2 + bA_0 + c = 0, \]
\[ 2aA_0A_1 + bA_1 = 0, \]
\[ aA_1^2 + aB_1^2 - B_1 = 0, \]
\[ 2aA_0B_1 + bB_1 = 0, \]
\[ 2aA_1B_1 + A_1 = 0, \]

for which, we have the following solutions:

\[ A_0 = -\frac{b}{2a}, \quad A_1 = 0, \quad B_1 = \frac{1}{a}, \tag{31} \]

where

\[ c = \frac{b^2 - 4}{4a}, \]

and

\[ A_0 = -\frac{b}{2a}, \quad A_1 = \pm \sqrt{\frac{1}{2a}}, \quad B_1 = \frac{1}{2a}, \tag{32} \]

where

\[ c = \frac{b^2 - 1}{4a}. \]

From \( \frac{dw}{d\xi} = \sinh w \), we have

\[ \sinh w = -\csc h\xi, \quad \cosh w = -\coth \xi. \tag{33} \]

Also (31) – (33), give

\[ \phi = -\frac{b + 2 \coth \xi}{2a}, \tag{34} \]

where
\[ c = \frac{b^2 - 4}{4a}, \]

and

\[ \phi = -\frac{b \pm \csc h \xi + \coth \xi}{2a}, \quad (35) \]

where

\[ c = \frac{b^2 - 1}{4a}. \]

From (6), (8), (34) and (35) we get the following traveling wave solutions of second-order Benjamin-Ono equation (1):

\[ u(x,t) = \frac{2\beta k^2 b^2 - 8\beta k^2 c a - l^2}{2a} - \frac{6k^2 \beta}{a} \coth^2[k(x + lt) + \xi_0], \quad (36) \]

where

\[ ac = \frac{b^2}{4} - 1. \]

and

\[ u(x,t) = \frac{2\beta k^2 b^2 - 8\beta k^2 c a - l^2}{2a} - \frac{3k^2 \beta}{2a} \left(\coth(k(x + lt) + \xi_0) \pm \csc h(k(x + lt) + \xi_0)\right)^2, \quad (37) \]

where

\[ c = \frac{b^2 - 1}{4a}. \]

Similarly, when \( \frac{dw}{d\xi} = \cosh w \), we obtain the following traveling wave (periodic-like) solutions of second-order Benjamin-Ono equation (1):

\[ u(x,t) = \frac{2\beta k^2 b^2 - 8\beta k^2 c a - l^2}{2a} - \frac{6k^2 \beta}{a} \cot^2[k(x + lt) + \xi_0], \quad (38) \]
where

\[ ac = \frac{b^2}{4} - 1. \]

and

\[
\begin{align*}
\frac{2}{2} = & \frac{2}{2} - \frac{8}{2} (a - 1) \\
& \frac{2}{2} \left( \cot(k(x + lt) + \xi_0) \pm \csc(k(x + lt) + \xi_0) \right)^2,
\end{align*}
\]

(39)

where

\[ c = \frac{b^2 - 1}{4a}. \]

In summary we have used the homogeneous balance method to obtain many traveling wave solutions of second-order Benjamin-Ono equation.

We now summarize the key steps as follows:

**Step 1:** For a given nonlinear evolution equation

\[ F(u, u_t, u_x, u_{xt}, u_{tt}, \ldots) = 0, \]

(40)

we consider its traveling wave solutions \( u(x, t) = u(\xi), \quad \xi = k(x + lt) + \xi_0 \) then Eq. (40) is reduced to an nonlinear ordinary differential equation

\[ Q(u, u', u'', u''' \ldots) = 0, \]

(41)

where a prime denotes \( \frac{d}{d\xi} \).

**Step 2:** For a given ansatz equation (for example, the ansatz equation is \( \phi' = a\phi^2 + b\phi + c \) in this paper), the form of \( u \) is decided and the homogeneous balance method is used on Eq. (41) to find the coefficients of \( u \).

**Step 3:** The homogeneous balance method is used to solve the ansatz equation.

**Step 4:** Finally, the traveling wave solutions of Eq. (40) are obtained by combining steps 2 and 3.
3. Conclusion

The second-order Benjamin-Ono equation is soluble using the homogeneous balance method. The efficiency of this method was demonstrated. New exact solution of the second-order Benjamin-Ono equation was obtained. The solutions obtained may be significant and important for the explanation of some practical physical problems. The method may also be applied to other nonlinear partial differential equations.

REFERENCES