PHYS-4023 Introductory Quantum Mechanics HW # 3

Chapters 4-5 [Quantum Physics 3-rd Ed, Stephen Gasiorowicz]

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Problem 1: Consider the one-dimensional (1D) harmonic oscillators described by the Hamiltonian:

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2} m\omega^2 x^2 \ , \tag{1}$$

where m is the mass of the particle, ω is the angular frequency and $\hat{p}_x$ is the linear momentum operator in the x direction. The allowed energy eigenvalues are: $E_n = \hbar\omega(n + 1/2)$ where $n = 0, 1, 2, \ldots$ The normalized eigenfunctions are:

$$\Phi_n(x) = N_n \ \exp\left(-\frac{\alpha^2 x^2}{2}\right) H_n(\alpha\, x) \quad ; \quad N_n = \sqrt{\frac{\alpha}{\sqrt{\pi}\, 2^n\, n!}} \ , \tag{2}$$

where N_n is the normalization constant, $\alpha = \sqrt{m\omega/\hbar}$ is a parameter with the dimensionality of an inverse length and $H_n(\alpha\, x)$ are Hermite polynomials. Calculate $(\Delta x)^2 = \langle x^2 \rangle - (\langle x \rangle)^2$ and $(\Delta p_x)^2 = \langle \hat{p}_x^2 \rangle - (\langle \hat{p}_x \rangle)^2$ for an arbitrary quantum state $\Phi_n(x)$. Verify whether the Heisenberg uncertainty principle, $(\Delta x)^2\, (\Delta p_x)^2 \geq (\hbar/2)^2$ is satisfied. **Hint:** Recall that $\langle x \rangle = 0$ and $\langle \hat{p}_x \rangle = 0$, so you need to calculate only $\langle x^2 \rangle$ and $\langle \hat{p}_x^2 \rangle$. The final result should be: $(\Delta x)^2\, (\Delta p_x)^2 = \hbar^2 (n + 1/2)^2$.

Problem 2: Consider the displaced one-dimensional (1D) harmonic oscillator:

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2} m\omega^2 (x - x_0)^2 \ , \tag{3}$$

where m is the mass of the particle, ω is the angular frequency, $\hat{p}_x$ is the linear momentum operator in the x direction and x_0 is the coordinate of the center of the 1D oscillator. Prove that the allowed energy eigenvalues are: $E_n = \hbar\omega(n + 1/2)$ where $n = 0, 1, 2, \dots$ and the normalized eigenfunctions are:

$$\Phi_n(x - x_0) = N_n \exp\left(-\frac{\alpha^2 (x - x_0)^2}{2}\right) H_n[\alpha\,(x - x_0)] \quad ; \quad N_n = \sqrt{\frac{\alpha}{\sqrt{\pi}\, 2^n\, n!}} \ , \tag{4}$$

where N_n is a normalization constant that has been previously defined, $\alpha = \sqrt{m\omega/\hbar}$ is a parameter with the dimensionality of an inverse length and $H_n(z)$ are the Hermite polynomials with z as argument.

Problem 3: Consider a two-dimensional (2D) isotropic harmonic oscillator described by the Hamiltonian:

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{\hat{p}_y^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2) \ , \tag{5}$$

where m is the mass of the particle, ω is the angular frequency, and $\hat{p}_x$, $\hat{p}_y$ are the respective linear momentum operators in the x and y direction. Prove that the allowed energy eigenvalues are of the form: $E_{n_x n_y} = \hbar\omega(n_x + n_y + 1)$ where $n_x = 0, 1, 2, \ldots$ and $n_y = 0, 1, 2, \ldots$ Note $n = n_x + n_y = 0, 1, 2, \ldots$ Find the degeneracy of any given energy eiegenvalue, $E_{n_x n_y n_z}$ in terms of quantum number, n. Verify that the degeneracy of any given energy eigenvalue $E_{n_x n_y}$ in terms of number n is: $D_n = (n + 1)$. **Note:** If degeneracy is one, that means that the energy eigenvalue is **non degenerate**.

Problem 4: Consider a three-dimensional (3D) isotropic harmonic oscillator described by the Hamiltonian:

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{\hat{p}_y^2}{2m} + \frac{\hat{p}_z^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2 + z^2) \ , \tag{6}$$

where m is the mass of the particle, ω is the angular frequency, and $\hat{p}_x$, $\hat{p}_y$, $\hat{p}_z$ are the respective linear momentum operators in the x, y and z direction. Prove that the allowed energy eigenvalues are of the form: $E_{n_x n_y n_z} = \hbar\omega(n_x + n_y + n_z + 3/2)$ where $n_x = 0, 1, 2, \ldots$, $n_y = 0, 1, 2, \ldots$ and $n_z = 0, 1, 2, \ldots$. Note $n = n_x + n_y + n_z = 0, 1, 2, \ldots$ Find the degeneracy of any given energy eiegenvalue, $E_{n_x n_y n_z}$ in terms of quantum number, n. **Note:** If degeneracy is one, that means that the energy eigenvalue is **non degenerate**.

Problem 5: Consider two identical one-dimensional (1D) harmonic oscillators. The Hamiltonian of the two particles in oscillatory motion is:

$$\hat{H} = \frac{\hat{p}_{x_1}^2}{2m} + \frac{1}{2}m\omega^2 x_1^2 + \frac{\hat{p}_{x_2}^2}{2m} + \frac{1}{2}m\omega^2 x_2^2 \,, \tag{7}$$

where the indexes 1 and 2 refer respectively to particle 1 and 2. The energy eigenvalues are: $E_{n_1 n_2} = \hbar\omega(n_1 + n_2 + 1)$ where $n_1 = 0, 1, \ldots$ and $n_2 = 0, 1, \ldots$ The eigenfunctions corresponding to those eigenvalues are: $\Psi_{n_1 n_2}(x_1, x_2) = \Phi_{n_1}(x_1)\,\Phi_{n_2}(x_2)$ where $\Phi_{n_i}(x_i)$ are the normalized eigenfunctions for the 1D oscillator for particle $i = 1, 2$ respectively. The groundstate wave function is: $\Psi_{00}(x_1, x_2)$ and corresponds to the lowest energy $E_{00} = \hbar\omega$. Find the "average relative distance" between particle 1 and 2 in the groundstate:

$$\langle |x_1 - x_2| \rangle = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \, \Psi_{00}(x_1, x_2)^* \, |x_1 - x_2| \, \Psi_{00}(x_1, x_2) \tag{8}$$