Exact results for a composite-fermion wave function

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It is noted that the radial distribution function and the interaction energy per particle can be exactly computed for a class of composite fermion wave functions obtained at the mean-field level of the Chern-Simons theory for the fractional quantum Hall effect. These results can be quite instructive and useful to test different numerical methods and approximations used to study these phenomena. [S0163-1829(99)50308-1]

The fractional quantum Hall effect¹ (FQHE) results from a strongly correlated incompressible liquid state^{2,3} formed at special uniform densities $\rho(\nu)$ of a two-dimensional (2D) electronic system that is subjected to a very strong transverse magnetic field B. The dominant sequence of fractional Hall states occurs when filling of the lowest Landau level is ν = p/(2mp+1), where p = 1, 2, ... and m = 0, 1, ... are integers. Much of the theoretical work on FQHE is based on the study of the properties of a 2D fully spin-polarized (spinless) system of N interacting electrons embedded in a uniform positive background. The electrons with charge -e(e>0) and mass m_{e} are considered confined in the x-y plane of area Ω and subjected to a perpendicular magnetic field $\vec{B} = (0,0,B)$, which is generated from the symmetric gauge vector potential $\vec{A}(\vec{r}) = [-(B/2)y, (B/2)x, 0]$. We will consider the thermodynamic limit of an infinite system defined as the limit of N electrons in a sample of area Ω , where N and Ω go to infinity with the density kept constant.

The many-electron system is described by the Hamiltonian $\hat{H} = \hat{K} + \hat{V}$, where \hat{K} is the kinetic energy operator

$$\hat{K} = \frac{1}{2m_e} \sum_{j=1}^{N} \left[-i\hbar \vec{\nabla}_j + e\vec{A}(\vec{r}_j) \right]^2,$$
(1)

and

$$\hat{V} = \sum_{j < k}^{N} v(|\vec{r}_{j} - \vec{r}_{k}|) - \rho(\nu) \sum_{j=1}^{N} \int d^{2}r v(|\vec{r}_{j} - \vec{r}|) + \frac{\rho(\nu)^{2}}{2} \int d^{2}r_{1} \int d^{2}r_{2}v(|\vec{r}_{1} - \vec{r}_{2}|)$$
(2)

is the total electron-electron, electron-background, and background-background interaction potential, where $v(|\vec{r}_j - \vec{r}_k|) = (1/4\pi\epsilon_0)(e^{2/\epsilon}|z_j - z_k|)$ is the interaction potential, $z_j = x_j + iy_j$ is the location of the *j*th electron in complex coordinates, and ϵ is the dielectric constant of the background.

It has become clear in recent years that many essential features of FQHE can be understood straightforwardly in terms of a new kind of particle, called a composite fermion (CF), which is a bound state of an electron and an even number of vortices of the many-body quantum wave function^{4,5} formed at the electronic densities $\rho(\nu) = \nu/[2\pi l_0(B)^2]$ where $l_0(B) = \sqrt{\hbar/(eB)}$ is the electronic

magnetic length. The fundamental property of the CF's is that they experience a reduced effective field $B^*=B(1 - 2m\nu)$ so that the liquid of strongly correlated electrons at *B* is equivalent to a liquid of weakly interacting CF's at B^* . Since the degeneracy of each Landau level is proportional to the magnetic field, the degeneracy N_s^* of each CF Landau level will be smaller than the corresponding N_s for the electrons and will be given by $N_s^* = N_s(1-2m\nu)$. As a result, the effective filling factor of CF's will be an integer number $\nu^* = p$ and will correspond to stable electronic filling factors $\nu = \nu^*/(2m\nu^* + 1)$, where $\nu^* = p = 1, 2, ...$ is the number of filled CF Landau levels.

There are two calculational schemes based on the intuitive physics above. One constructs explicit wave functions⁴ while the second scheme employs a Chern-Simons (CS) field theory⁶ approach to investigate the CF state. Although the two schemes are based on the same physics, a precise quantitative relationship between them is not clear. The wave function Ψ_{ν}^{CS} given by

$$\Psi_{\nu}^{CS} = \prod_{j < k}^{N} \frac{(z_j - z_k)^{2m}}{|z_j - z_k|^{2m}} \Phi_p(B^*),$$
(3)

where $\Phi_p(B^*)$ is the Slater determinant wave function of *p* filled Landau levels, evaluated at the magnetic field shown in the argument, is obtained at the mean-field level of the CS field theory and describes electronic fillings $\nu = p/(2mp + 1)$.

So few many-body problems are solvable that a certain interest may be attached to any model that is exactly solvable. Exact solutions, when available, enable one to gauge the efficiency of different numerical methods and to carefully test many approximations.

It seems not to have been previously observed that an exact solution for the radial distribution function and interaction energy per particle can be obtained for the above CS wave function. As we show in this article, the mathematics is simple, but the results are, however, quite instructive and the comparison with other numerical estimates is interesting. A possible way to find the interaction energy per particle,

$$u_{\nu} = \frac{1}{N} \frac{\langle \Psi_{\nu}^{CS} | \hat{V} | \Psi_{\nu}^{CS} \rangle}{\langle \Psi_{\nu}^{CS} | \Psi_{\nu}^{CS} \rangle} = \frac{\rho(\nu)}{2} \int d^{2}r_{12} [g_{\nu}(r_{12}) - 1] v(r_{12}),$$
(4)

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8132
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depends on the ability to compute exactly the radial distribution function $g_{\nu}(r_{12})$ that in terms of Ψ_{ν}^{CS} is defined as

$$g_{\nu}(r_{12}) = \frac{N(N-1)}{\rho(\nu)^2} \frac{\int d^2 r_3 \cdots d^2 r_N |\Psi_{\nu}^{CS}|^2}{\int d^2 r_1 \cdots d^2 r_N |\Psi_{\nu}^{CS}|^2},$$
(5)

and for the system under consideration will depend only on the interparticle spatial distance $r_{12} = |\vec{r}_1 - \vec{r}_2|$. For a general many-body wave function, the radial distribution function can be computed only approximately, but in the present case one observes that $|\Psi_{\nu}^{CS}|^2 = |\Phi_p(B^*)|^2$, and this corresponds to a squared Slater determinant of p = 1, 2, ... filled CF Landau levels formed at the reduced magnetic field B^* . At this stage the calculation of $g_{\nu}(r_{12})$ is simpler and can be performed exactly.

For a magnetic field B^* applied in the *z* direction which, as usual, is generated by a symmetric gauge vector potential $\vec{A}(\vec{r})$, the eigenstates of the ideal Hamiltonian

$$\hat{H}_{0} = \frac{1}{2m_{e}} [-i\hbar\vec{\nabla} + e\vec{A}(\vec{r})]^{2}$$
(6)

for the various CF Landau levels n = 0, 1, 2, ... are given by

$$\varphi_{n,l}(z) = \frac{1}{\sqrt{2^{n}n!}} \exp\left[\frac{|z|^{2}}{4l_{0}(B^{*})^{2}}\right] \left[2l_{0}(B^{*})\frac{\partial}{\partial z}\right]^{n} \\ \times \left\{\varphi_{0,l}(z) \exp\left[-\frac{|z|^{2}}{4l_{0}(B^{*})^{2}}\right]\right\},$$
(7)

where $l_0(B^*)$ is the CF's magnetic length and

$$\varphi_{0,l}(z) = \frac{1}{\sqrt{2^{l}l!}} \left[\frac{z}{l_0(B^*)} \right]^{l} \varphi_{0,0}(z), \tag{8}$$

$$\varphi_{0,0}(z) = \frac{1}{\sqrt{2\pi l_0(B^*)^2}} \exp\left[-\frac{|z|^2}{4l_0(B^*)^2}\right],\tag{9}$$

and $l=0,1,\ldots,(N_s^*-1)$ is the angular momentum quantum number for the CF's. The radial distribution function $g_{\nu}(r_{12})$ will be given by

$$g_{\nu}(r_{12}) = 1 - \frac{1}{g_s} |l_p(r_{12})|^2,$$
 (10)

where the "statistical exchange" factor is computed from $l_p(r_{12}) = \hat{\rho}_p(\vec{r}_1, \vec{r}_2) / \rho(\nu)$ and the spin degeneracy g_s of each Landau state is $g_s = 1$, because we are considering fully spin-polarized (spinless) electrons (CF's). The electronic density $\rho(\nu)$ is the same as the CF's density and the (reduced) one-body density matrix $\hat{\rho}_p(\vec{r}_1, \vec{r}_2)$ that corresponds to the dynamically uncorrelated state $\Phi_p(B^*)$ is given by

$$\hat{\rho}_p(z_1, z_2) = g_s \sum_{n=0}^{p-1} \sum_{l=0}^{N_s^* - 1} \varphi_{n,l}^*(z_1) \varphi_{n,l}(z_2), \qquad (11)$$

where we are using the complex coordinates z_i instead of the two-dimensional vectors \vec{r}_i . In the above equation, the sum is extended over all occupied CF states, where $\varphi_{n,l}(z)$ and $\varphi_{n,l}^*(z)$ are, respectively, the single particle wave function

and its conjugate. Using a standard algebra, one can easily prove that the contribution to $\hat{\rho}_p(z_1, z_2)$ coming from the *n*th CF Landau level is

$$\sum_{l=0}^{N_s^*-1} \varphi_{n,l}^*(z_1)\varphi_{n,l}(z_2)$$
$$= L_n \left(\frac{|z_1 - z_2|^2}{2l_0(B^*)^2}\right) \sum_{l=0}^{N_s^*-1} \varphi_{0,l}^*(z_1)\varphi_{0,l}(z_2), \qquad (12)$$

where $L_n(x) = (e^x/n!)(d^n/dx^n)(x^ne^{-x})$ are the Laguerre polynomials of order n = 0, 1, ... By using the formulas given in Eqs. (8) and (9) one can express the second term of Eq. (12) as

$$\sum_{l=0}^{N_s^*-1} \varphi_{0,l}^*(z_1) \varphi_{0,l}(z_2) = \frac{1}{2 \pi l_0 (B^*)^2} \exp\left[-\frac{|z_1|^2 + |z_2|^2}{4 l_0 (B^*)^2}\right] \sum_{l=0}^{N_s^*-1} \frac{1}{l!} \left(\frac{z_1^* z_2}{2 l_0 (B^*)^2}\right)^l.$$
(13)

In the thermodynamic limit both the density and the filling factor $\nu = N/N_s$ are kept constant as the number of electrons N and the LL degeneracy N_s go to infinity. Since the degeneracy N_s^* of each CF Landau level is directly proportional to N_s then also $(N_s^* - 1)$ goes to infinity in the thermodynamic limit. At this point the summation over l in Eq. (13) is extended from 0 to ∞ and one obtains

$$\sum_{l=0}^{\mathsf{V}_{s}^{*}-1} \varphi_{0,l}^{*}(z_{1})\varphi_{0,l}(z_{2}) = \frac{1}{2\pi l_{0}(B^{*})^{2}} \exp\left[-\frac{|z_{1}|^{2}+|z_{2}|^{2}}{4l_{0}(B^{*})^{2}}\right] \exp\left[\frac{z_{1}^{*}z_{2}}{2l_{0}(B^{*})^{2}}\right].$$
 (14)

By substituting Eq. (14) in Eq. (12), one notes that, in the thermodynamic limit, the one-body density matrix $\hat{\rho}_p(z_1, z_2)$ given from Eq. (11) depends solely on p and not N_s^* . After some algebra, the statistical exchange term is found to be complex

$$l_{p}(z_{1}, z_{2}) = \left\{ \frac{1}{p} \sum_{n=0}^{p-1} L_{n} \left[\frac{|z_{1} - z_{2}|^{2}}{2l_{0}(B^{*})^{2}} \right] \right\} \\ \times \exp\left[-\frac{|z_{1} - z_{2}|^{2}}{4l_{0}(B^{*})^{2}} \right] \exp\left[\frac{i}{2l_{0}(B^{*})^{2}} \left| (\vec{r}_{1} \times \vec{r}_{2})_{z} \right| \right],$$
(15)

but since the radial distribution function is found by applying Eq. (10), the phase factor of $l_p(z_1,z_2)$ disappears and we obtain

$$g_{\nu}(r_{12}) = 1 - \exp\left[-\frac{r_{12}^2}{2l_0(B^*)^2}\right] \left\{\frac{1}{p} \sum_{n=0}^{p-1} L_n\left(\frac{r_{12}^2}{2l_0(B^*)^2}\right)\right\}^2,$$
(16)

where $r_{12} = |z_1 - z_2|$ is the interparticle distance. From this result one may erroneously think that the radial distribution

ORION CIFTJA

TABLE I. Exact results for the interaction energy per particle u_{ν} corresponding to the Chern-Simons wave function, Ψ_{ν}^{CS} at fillings $\nu = p/(2p+1)$. The interaction energy is in units of $(1/4\pi\epsilon_0)[e^{2}/\epsilon l_0(B)]$, where $l_0(B)$ is the magnetic length of the electrons. In the fifth column we report the available variational Monte Carlo results of Kamilla and Jain (Ref. 7) obtained in the spherical geometry.

| т | р | $\nu = \frac{p}{2mp+1}$ | Exact | Ref. 7 |
|---|----------|-------------------------|--|-------------|
| 1 | 1 | 1/3 | $-\frac{1}{2}\sqrt{\frac{\pi}{6}} = -0.361800$ | -0.3619(90) |
| 1 | 2 | 2/5 | $-\frac{11}{16}\sqrt{\frac{\pi}{10}} = -0.385343$ | -0.3848(16) |
| 1 | 3 | 3/7 | $-\frac{107}{128}\sqrt{\frac{\pi}{14}} = -0.395990$ | -0.3947(15) |
| 1 | 4 | 4/9 | $-\frac{1971}{2048}\sqrt{\frac{\pi}{18}} = -0.402\ 064$ | -0.4007(16) |
| 1 | 5 | 5/11 | $-\frac{35205}{32768}\sqrt{\frac{\pi}{22}} = -0.405\ 992$ | |
| 1 | 6 | 6/13 | $-\frac{924747}{786432}\sqrt{\frac{\pi}{26}}=-0.408742$ | |
| 1 | 7 | 7/15 | $-\frac{2662079}{2097152}\sqrt{\frac{\pi}{30}} = -0.410776$ | |
| 1 | 8 | 8/17 | $-\frac{91033507}{67108864}\sqrt{\frac{\pi}{34}} = -0.412341$ | |
| 1 | 9 | 9/19 | $-\frac{27800473833}{19327352832}\sqrt{\frac{\pi}{38}}=-0.413583$ | |
| 1 | 10 | 10/21 | $-\frac{130215126675}{85899345920}\sqrt{\frac{\pi}{42}} = -0.414592$ | |
| 1 | ∞ | 1/2 | $-\frac{4}{3\pi} = -0.424413$ | |

function at fillings $\nu = p/(2mp+1)$ will depend only on p and not in m, but a deeper check tells that the dependence on m is "hidden" in the reduced field $B^* = B/(2mp+1)$ to which the CF's are subjected. We can further simplify this



FIG. 1. Exact values for the interaction energy per particle u_{ν} corresponding to Ψ_{ν}^{CS} for fillings $\nu = p/(2p+1)$ (square) and $\nu = p/(4p+1)$ (circle). In the $p \to \infty$ limit we obtain the respective results for fillings $\nu = 1/2$ (dotted line) and $\nu = 1/4$ (dashed line). The energies are expressed in the standard units of $(1/4\pi\epsilon_0)[e^2/\epsilon l_0(B)]$.

expression by noting that $\sum_{n=0}^{p-1} L_n(x) = L_{p-1}^1(x)$, where $L_n^k(x) = (-1)^k (d^k/dx^k) [L_{n+k}(x)]$ are the associated Laguerre polynomials of order n = 0, 1, ... and degree k = 0, 1, ...

Since both electrons and CF's have the same density, we can relate the CF magnetic length $l_0(B^*)$ to the true electronic magnetic length $l_0(B)$ by the simple relation



FIG. 2. Radial distribution function $g_{\nu}(r)$ for the first two states of the series $\nu = p/(2p+1)$, p = 1,2,... and for the filling $\nu = 1/2$ obtained as the $p \rightarrow \infty$ limit of such a series.



FIG. 3. Radial distribution function $g_{\nu}(r)$ for the first two states of the series $\nu = p/(4p+1)$, p = 1,2,... and for the filling $\nu = 1/4$ obtained as the $p \rightarrow \infty$ limit of such a series.

 $l_0(B^*)^2 = l_0(B)^2(2mp+1)$, so finally the radial distribution function corresponding to Ψ_{ν}^{CS} for $\nu = p/(2mp+1)$ is written as

$$g_{\nu}(r_{12}) = 1 - \exp\left[-\frac{1}{2(2mp+1)} \left(\frac{r_{12}}{l_0(B)}\right)^2\right] \\ \times \left\{\frac{1}{p} L_{p-1}^1 \left[\frac{1}{2(2mp+1)} \left(\frac{r_{12}}{l_0(B)}\right)^2\right]\right\}^2, \quad (17)$$

in terms of the natural dimensionless distance $r_{12}/l_0(B)$.

The calculation of the interaction energy per particle corresponding to Ψ_{ν}^{CS} for $\nu = p/(2mp+1)$ follows from Eq. (4), where after writing $\rho(\nu) = p/[2\pi l_0(B^*)^2]$ we introduce the dimensionless variable $x = r_{12}/l_0(B^*)$ and obtain the energy in a simple one-dimensional integral form,

$$u_{\nu} = -\frac{1}{2p\sqrt{2mp+1}} \int_{0}^{\infty} dx \exp\left(-\frac{x^{2}}{2}\right)$$
$$\times \left[L_{p-1}^{1}\left(\frac{x^{2}}{2}\right)\right]^{2} \frac{1}{4\pi\epsilon_{0}} \frac{e^{2}}{\epsilon l_{0}(B)}.$$
(18)

In the $p \rightarrow \infty$ limit and for a nonzero value of *m*, the CS wave function Ψ_{ν}^{CS} will correspond to the filling $\nu = 1/(2m)$, where m = 1, 2, ... and, in this limit the radial distribution function is found to be

$$g_{\nu=1/2m}(r_{12}) = 1 - 4 \left[\frac{J_1 \left(\frac{1}{\sqrt{m}} \frac{r_{12}}{l_0(B)} \right)}{\frac{1}{\sqrt{m}} \frac{r_{12}}{l_0(B)}} \right]^2, \quad (19)$$

where $J_1(x)$ is the Bessel function of the first order. It is interesting to note that this radial distribution function corresponds to a 2D system of fully-spin polarized $(g_s = 1)$ free fermions whose Fermi radius is given by $k_F(\nu = 1/(2m))$ $= 1/[\sqrt{m}l_0(B)]$.

The exact interaction energy per particle corresponding to the $\nu = 1/(2m)$ state, where m = 1, 2, ... is computed by applying Eq. (4) and we find that

TABLE II. Exact results for the interaction energy per particle u_{ν} corresponding to the Chern-Simons wave function Ψ_{ν}^{CS} at fillings $\nu = p/(4p+1)$. The interaction energy is in units of $(1/4\pi\epsilon_0)[e^2/\epsilon l_0(B)]$, where $l_0(B)$ is the magnetic length of the electrons.

| | | $v = \frac{p}{p}$ | |
|---|----|-------------------|--|
| т | р | $\sim 2mp+1$ | Exact |
| 2 | 1 | 1/5 | $-\frac{1}{2}\sqrt{\frac{\pi}{10}} = -0.280249$ |
| 2 | 2 | 2/9 | $-\frac{11}{16}\sqrt{\frac{\pi}{18}} = -0.287217$ |
| 2 | 3 | 3/13 | $-\frac{107}{128}\sqrt{\frac{\pi}{26}} = -0.290577$ |
| 2 | 4 | 4/17 | $-\frac{1971}{2048}\sqrt{\frac{\pi}{34}}=-0.292544$ |
| 2 | 5 | 5/21 | $-\frac{35205}{32768}\sqrt{\frac{\pi}{42}} = -0.293835$ |
| 2 | 6 | 6/25 | $-\tfrac{924747}{786432}\sqrt{\frac{\pi}{50}} = -0.294748$ |
| 2 | 7 | 7/29 | $-\frac{2662079}{2097152}\sqrt{\frac{\pi}{58}} = -0.295428$ |
| 2 | 8 | 8/33 | $-\frac{91033507}{67108864}\sqrt{\frac{\pi}{66}} = -0.295954$ |
| 2 | 9 | 9/37 | $-\frac{27800473833}{19327352832}\sqrt{\frac{\pi}{74}} = -0.296373$ |
| 2 | 10 | 10/41 | $-\frac{130215126675}{85899345920}\sqrt{\frac{\pi}{82}} = -0.296715$ |
| 2 | œ | 1/4 | $-\frac{4}{3\pi}\frac{1}{\sqrt{2}} = -0.300105$ |

$$u_{\nu=1/2m} = -\frac{4}{3\pi\sqrt{m}} \frac{1}{4\pi\epsilon_0} \frac{e^2}{\epsilon l_0(B)}.$$
 (20)

We calculated exactly the integrals appearing in Eq. (18) for the first 10 values, p = 1, 2, ..., 10 of the series of fillings where m=1 and 2. The interaction energy per particle u_v is shown in Fig. 1 for the series of states v=p/(2p+1)(square) and v=p/(4p+1) (circle). In the $p \rightarrow \infty$ limit the results converge to the values given by Eq. (20), respectively, for the v=1/2 (dotted line) and v=1/4 state (dashed line). In Figs. 2 and 3 we plot the radial distribution function corresponding to the first two values of p for the series of fillings given, respectively, by v=p/(2p+1) and v=p/(4p+1) and we compare them to the radial distribution function of the states v=1/2 and 1/4 obtained, respectively, as the $p\rightarrow\infty$ limit of each of such series and given by Eq. (19).

A better and more instructive idea of these results is given in Tables I and II, where we display the exact analytical values of energy for ten states of the form $\nu = p/(2p+1)$ and $\nu = p/(4p+1)$, respectively. Since different numerical methods and approaches have been used to study these phenomena, we believe that these exact results can be quite useful to gauge their accuracy and their dependence on the number of particles. Extrapolation of finite number calculations in the thermodynamic limit are not always unambiguous and the results we provide can be a good test of accuracy for different approximation schemes. For a long time the quantitative investigations of FQHE have been limited only to systems of a few number of particles, and only recently have we been aware of a method developed by Kamilla and Jain,⁷ which enables a treatment of much bigger systems of 50 particles. By adopting this method to the spherical geometry, they used variational Monte Carlo techniques to compute the expectation values of the interaction and kinetic energy per particle for different wave functions, among them the Ψ_{u}^{CS} wave function. In the spherical geometry, they found that the energies scale approximately linearly as 1/N where N is the number of electrons. In Table I we compare directly our exact results with their $N \rightarrow \infty$ extrapolations of the interaction energy per particle and as can clearly be seen, their results are very accurate for all the relevant fillings they have considered, but eventually the method becomes more difficult to apply and less accurate for states in the $\nu \rightarrow 1/2$ limit.

In summary, we have analytically computed the interaction energy per particle for all states of FQHE that are described by a CS wave function. We provide tables of exact analytical values of the interaction energy per particle for all these states. The data can be used to further test the reliability of different numerical methods employed on the study of FQHE.

This work was carried out at the Ames Laboratory, which is operated for the U.S. Department of Energy by Iowa State University, and was supported by the Director for Energy Research, Office of Basic Energy Sciences of the U.S. Department of Energy.

- ¹For a review, see *The Quantum Hall Effect*, edited by R. E. Prange and S. M. Girvin (Springer-Verlag, New York, 1990); *The Fractional Quantum Hall Effect*, edited by T. Chakraborty and P. Pietiläinen (Springer-Verlag, New York, 1988).
- ²D. C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982).
- ³R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
- ⁴J. K. Jain, Phys. Rev. Lett. **63**, 199 (1989).
- ⁵J. K. Jain, Phys. Rev. B **41**, 7653 (1990).
- ⁶B. I. Halperin, P. A. Lee, and N. Read, Phys. Rev. B **47**, 7312 (1993).
- ⁷R. K. Kamilla and J. K. Jain, Phys. Rev. B 55, 9824 (1997).