# Hypernetted-chain treatment and the extended shadow wave functions for fractional quantum Hall hierarchical states

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The hypernetted-chain theory is applied to study hierarchical states in the fractional quantum Hall effect. It is noted that a class of wave functions introduced by Girvin [Phys. Rev. B **29**, 6012 (1984)] and MacDonald, Aers, and Dharma-wardana [Phys. Rev. B **31**, 5529 (1985)], based on charge-conjugation procedures, is of the extended shadow wave-function type. The correlation energy, pair distribution function, and static structure function have been calculated in the thermodynamic limit at various filling factors. The results obtained agree with those of previous calculations performed with a finite number of electrons. [S0163-1829(97)08516-0]

## I. INTRODUCTION

The description of strong interparticle correlations in both Bose and Fermi systems with continuous degrees of freedom is a longstanding problem of current interest. More recently, the shadow wave function (SWF) has been proposed[1,2] as a new variational ansatz to compute the properties of solid and liquid $^4$He at $T=0$ K.

The SWF in the bosonic case is given by

$$\Psi^B_{\rm SWF}(\vec{r}_1,\ldots,\vec{r}_N)=\prod_{i<j}^{N} f_{\rm pp}(r_{ij})\int \prod_{i=1}^{N} \chi(|\vec{r}_i-\vec{s}_i|) \times\prod_{i<j}^{N} f_{\rm ss}(s_{ij})d\vec{S}. \quad (1)$$

The quantity $\vec{S}$ denotes the set of coordinates $[\vec{s}_i]$, so-called ''shadow'' variables, associated with the particles. The respective correlation factors may be written as

$$f_{\rm pp}(r_{ij})=e^{-U_{\rm pp}(r_{ij})/2} \quad (2)$$

and

$$f_{\rm ss}(s_{ij})=e^{-U_{\rm ss}(s_{ij})}, \quad (3)$$

where $U_{\rm pp}(r_{ij})$ and $U_{\rm ss}(s_{ij})$ represent, respectively, the particle-particle and shadow-shadow (pseudo)potential. The structure of $f_{\rm ss}(s_{ij})$ is the same as that of $f_{\rm pp}(r_{ij})$, namely, it heals out to unity at large intershadow distances, whereas the ''correlation'' $\chi(x)$ between a particle and its associated shadow heals out to zero.[1] Physically, the shadow variables $\vec{s}_i$ can be thought of as mimicking the quantum correlation ''holes'' which the particles carry around themselves in the dense system.

The physical interpretation of a SWF, as well as the request of more variational freedom and of full symmetry under exchange of any particle with any hole, suggests further extended forms for the SWF, so that a type of so-called extended shadow wave function (ESWF) was proposed.[3]

The ESWF is of the form

$$\Psi^B_{\rm ES}(\vec{r}_1,\ldots,\vec{r}_N)=\prod_{i<j}^{N} f_{\rm pp}(r_{ij})\int \prod_{i,j}^{N,M} f_{\rm ps}(|\vec{r}_i-\vec{s}_j|) \times\prod_{i<j}^{M} f_{\rm ss}(s_{ij})d|\vec{S}, \quad (4)$$

where $N$ and $M$ are, respectively, the number of particles and the number of shadows. The extension which Eq. (4) represents over the standard SWF of Eq. (1) concerns two aspects.

First, in the ESWF all shadows are correlated with all real particles rather than being in a one-to-one correspondence as in Eq. (1), allowing the possibility that the number and location of ''holes'' becomes different from those of the real particles. This form also allows for a description of lattice vacancies. The second aspect, which is related to the first, is that all three ''correlation'' functions $f_{\rm pp}(x)$, $f_{\rm ps}(x)$, and $f_{\rm ss}(x)$ must heal out to unity at large values of $x$.

In the case of an ESWF, the cluster diagrams of the pair distribution function are characterized by only three different types of points: $p, s^R$, and $s^L$, where $p$ denotes particle and $s^{R,L}$ denotes, respectively, right or left shadow coordinates. In fact, the normalization of an ESWF is given by

 

$$\langle \Psi^B_{\rm ES}|\Psi^B_{\rm ES}\rangle = \int \left[ \prod_{i<j}^{N} f^2_{\rm pp}(r_{ij}) \prod_{i,j}^{N,M} f_{\rm ps}(|\vec{r}_i - \vec{s}^L_j|) \times \prod_{i<j}^{M} f_{\rm ss}(s^L_{ij}) \prod_{i,j}^{N,M} f_{\rm ps}(|\vec{r}_i - \vec{s}^R_j|) \times \prod_{i<j}^{M} f_{\rm ss}(s^R_{ij}) \right] d\vec{S}^L d\vec{S}^R d\vec{R}, \quad (5)$$

and coincides with the partition function of a classical three-component system $(p, s^R, s^L)$ interacting via the following (pseudo)potentials:

$$U_{\rm pp} = -\ln f^2_{\rm pp}, \quad (6)$$

$$U_{{\rm ps}^R} = U_{{\rm ps}^L} = -\ln f_{\rm ps}, \quad (7)$$

$$U_{s^R s^R} = U_{s^L s^L} = -\ln f_{\rm ss}, \quad (8)$$

and

$$U_{s^R s^L} = 0. \quad (9)$$

The normalization integral (5) contains as integrand the square $\Psi^B_{\rm ES}(\vec{r}_1, \ldots, \vec{r}_N) \times \Psi^B_{\rm ES}(\vec{r}_1, \ldots, \vec{r}_N)$ of the extended shadow wave function $\Psi^B_{\rm ES}$ as defined in expression (4). The real particle coordinates $\vec{r}_1, \ldots, \vec{r}_N$ must, of course, be taken to be identical in both factors. However, two independent sets of shadow coordinates, over both of which must be integrated in formula (5), are needed for explicitly representing the square of wave function $\Psi^B_{\rm ES}$. In Eq. (5) these two independent sets of shadow particles are distinguished by the superscripts $L$ and $R$, referring to ''left'' and ''right'' shadows, respectively. Every real particle is correlated with every other real particle as well as with every ''left'' and ''right'' shadow. Every ''left'' shadow is correlated in turn with every other ''left'' shadow. The same applies to the ''right'' shadows. Correlations between ''left'' and ''right'' shadows are absent.

## II. METHOD

As a first application of this formalism to the fractional quantum Hall effects (FQHE), we find that hierarchical states[4,5] may be described microscopically by electronic wave functions which can be expressed as an ESWF. A possible way of constructing wave functions for the second level of the hierarchy is to first particle-hole conjugate[6,7] the Laughlin parent state $\Psi_{\nu_0}(z_1, \ldots, z_N)$ with filling factor $\nu_0 = 1/m$, where $m = 3$ and 5:

$$\Psi_{1-1/m}(z_1, \ldots, z_N) = \hat{C}[\Psi_{1/m}(z_1, \ldots, z_N)] = \int d\vec{S}_M \Psi^*_{1/m}(\vec{S}_M) \times \Phi_{\nu=1}(z_1, \ldots, z_N; \vec{S}_M). \quad (10)$$

The short-hand notation $\vec{S}_M$ represents the coordinates $\vec{s}_1, \ldots, \vec{s}_M$ of the $M$ holes, where $N$ and $M$ satisfy[7] the relation $N + M = mM$. The second hierarchy state $\Psi_\nu$ is then obtained by multiplying $\Psi_{1-1/m}(z_1, \ldots, z_N)$ with $\Pi^N_{j<k}(z_j - z_k)^p$, where $p$ must be an even integer such that

$$\frac{1}{\nu} = \frac{1}{1-\nu_0} + p. \quad (11)$$

It follows that

$$\Psi_\nu = \prod_{j<k}^{N} (z_j - z_k)^{p+1} e^{-\sum_{j=1}^{N} (|z_j|^2/4l_0^2)} \int \prod_{j,k}^{N,M} (z_j - s_k) \times \prod_{j<k}^{M} (s^*_j - s^*_k)^m (s_j - s_k) e^{-\sum_{j=1}^{M} (|s_j|^2/2l_0^2)} d\vec{S}, \quad (12)$$

where $l_0 = \sqrt{\hbar/eB}$ is the magnetic length, and $z_k$ and $s_k$ are given in complex notation $x_k + iy_k$.

For instance, the Laughlin parent state $\nu_0 = 1/3$ with $m = 3$ generates for $p = 2$ the daughter state $\nu = 2/7$, whereas for $p = 0$ it produces the state $\nu = 2/3$. Similarly, the Laughlin state $\nu_0 = 1/5$ generates for $p = 0$ the daughter state $\nu = 4/5$, and so on.

Seen from the perspective of the ESWF approach, the hole variable $\vec{s}_i$ may be interpreted as a shadow variable. The analytic form of the shadow-shadow and shadow-particle correlations is, in the present case, fixed by the lowest Landau level constraint. In this sense, this is a variational wave function with no adjustable parameters.

A comparison between $\Psi_\nu$ and $\Psi^B_{\rm ES}$ shows that $\Psi_\nu$ can be identified (after removal of the exponential factor which is relevant only in the trivial long-wavelength limit) with a $\Psi^B_{\rm ES}$, where

$$f_{pp}(jk) = (z_j - z_k)^{p+1}, \quad (13)$$

$$f_{ps}(jk) = (z_j - s_k), \quad (14)$$

and

$$f_{ss}(jk) = (s^*_j - s^*_k)^m (s_j - s_k). \quad (15)$$

Normalization of $\Psi_\nu$ yields

$$\langle \Psi_\nu|\Psi_\nu\rangle = \int \left[ \prod_{j<k}^{N} |z_j - z_k|^{2(p+1)} \prod_{j,k}^{N,M} (z^*_j - s^{L*}_k) \times \prod_{j<k}^{M} (s^L_j - s^L_k)^m (s^{L*}_j - s^{L*}_k) \prod_{j,k}^{N,M} (z_j - s^R_k) \times \prod_{j<k}^{M} (s^{R*}_j - s^{R*}_k)^m (s^R_j - s^R_k) \right] d\vec{R}\, d\vec{S}^L d\vec{S}^R. \quad (16)$$

Writing

$$|z_j - z_k|^{2(p+1)} = e^{-U_{\rm pp}}, \quad (17)$$

$$(z_j - s^R_k) = e^{-U_{\rm pR}}, \quad (18)$$

and

$$(s^{R*}_j - s^{R*}_k)^m (s^R_j - s^R_k) = e^{-U_{\rm RR}}, \quad (19)$$

we see that the normalization condition for $\Psi_\nu$ coincides with the partition function of a three-component system $(p, R, L)$ interacting via the (pseudo)potentials

$$U_{\rm pp}=-2(p+1)\ln|z_j-z_k|, \tag{20}$$

$$U_{\rm pR}=U_{\rm pL}=-\ln|z_j-s_k^R|-i\,\theta_{jR,kR}, \tag{21}$$

$$U_{\rm RR}=U_{\rm LL}=-(m+1)\ln|s_j^R-s_k^R|+i(m-1)\,\theta_{jR,kR}, \tag{22}$$

and

$$U_{\rm RL}=0, \tag{23}$$

where $\tan(\theta_{jk})=y_{jk}/x_{jk}$.

For our $p$, $R$, $L$ system we can apply a three-component hypernetted-chain (HNC) treatment. This will improve upon the one-component HNC calculation of MacDonald, Aers, and Dharma-wardana,[6] based on the introduction of an *ad hoc* effective potential mimicking the effect of the hole variables. While this is an approximation[6] (which we will presently overcome), it did allow inclusion of elementary diagrams in a simple way.

In the present work, for the sake of simplicity, we neglect the elementary diagrams, so we use the so-called HNC/0 approximation. There is a major difference from the previously known HNC/0 treatments, since here the correlations $f_{\rm ps}(jk)$ and $f_{\rm ss}(jk)$, and, consequently, several other HNC/0 quantities are complex functions and depend on both the $x_{jk}$ and $y_{jk}$ components of the interparticle coordinate $z_{jk}$.

The long-range parts $U^l_{\rm ps}(jk)$ and $U^l_{\rm ss}(jk)$ bring an angular dependence into all the HNC quantities. However, it turns out that the nodal functions $N_{\alpha\beta}(jk)$ and non-nodal functions $X_{\alpha\beta}(jk)=g_{\alpha\beta}(jk)-1-N_{\alpha\beta}(jk)$ have a long-range behavior exactly given by $-U^l_{\alpha\beta}(jk)$ and $U^l_{\alpha\beta}(jk)$, so that all pair distribution functions $g_{\alpha\beta}(jk)$ are short ranged.[3]

The multicomponent HNC/0 method is a generalization of the one-component one and an iteration scheme similar to that can be adopted here. In the case of a multicomponent system the corresponding HNC/0 quantities are given by the following equations:

$$g_{\alpha\beta}(r_{12})=e^{-U_{\alpha\beta}(r_{12})+N_{\alpha\beta}(r_{12})}, \tag{24}$$

$$N_{\alpha\beta}(r_{12})=\sum_\gamma \rho_\gamma\int X_{\alpha\gamma}(r_{13})[X_{\gamma\beta}(r_{32})+N_{\gamma\beta}(r_{32})]d\vec{r}_3, \tag{25}$$

and

$$X_{\alpha\beta}(r_{12})=g_{\alpha\beta}(r_{12})-1-N_{\alpha\beta}(r_{12}), \tag{26}$$

where $\rho_\gamma$ represents the densities of the different types of particles of the multicomponent system.

We recall that the different ''particles'' identified as $p$ (particle), $R$ (right shadow), and $L$ (left shadow) have densities $\rho_p$, $\rho_R$, and $\rho_L$. Since $\Psi^B_{\rm ES}$ is symmetric under the exchange of shadow variables $\vec{s}_i$, irrespective of $\vec{r}_i$, there are only four independent HNC/0 quantities, i.e, $N_{\rm pp}$, $N_{\rm pR}=N_{\rm pL}$, $N_{\rm RR}=N_{\rm LL}$, and $N_{\rm RL}$.

Among the three components $p$, $R$, and $L$, there are only four independent pairings: (pp), (pR), (RR), and (RL).

Introducing the two-dimensional Fourier transforms

$$\widetilde{F}_{pp}(q)=\rho_p\int e^{i\vec{q}\cdot\vec{r}}F_{\rm pp}(r)d\vec{r}, \tag{27}$$

$$\widetilde{F}_{ps}(q)=\sqrt{\rho_p\rho_s}\int e^{i\vec{q}\cdot\vec{r}}F_{\rm pr}(r)d\vec{r}, \tag{28}$$

and

$$\widetilde{F}_{\rm ss}(q)=\rho_s\int e^{i\vec{q}\cdot\vec{r}}F_{\rm ss}(r)d\vec{r}, \tag{29}$$

with $\rho_s=\rho_R=\rho_L$, the general formula of Eq. (25) becomes

$$\widetilde{N}_{\alpha\beta}(q)=\sum_\gamma \widetilde{X}_{\alpha\gamma}(q)[\widetilde{X}_{\gamma\beta}(q)+\widetilde{N}_{\gamma\beta}(q)]. \tag{30}$$

As $R$ and $L$ are Hermitian conjugated, we have $\widetilde{F}_{\rm pp}(q)=\widetilde{F}^*_{\rm pp}(q)$, $\widetilde{F}_{\rm pR}(q)=\widetilde{F}^*_{\rm pL}(q)$, $\widetilde{F}_{\rm RR}(q)=\widetilde{F}^*_{\rm LL}(q)$, $\widetilde{F}_{\rm RL}(q)=\widetilde{F}^*_{\rm LR}(q)$, whereas $\widetilde{F}_{\rm pR}(q)=\widetilde{F}_{\rm Rp}(q)$, $\widetilde{F}_{\rm pL}(q)=\widetilde{F}_{\rm Lp}(q)$.

Keeping this in mind, after some algebra we determine all relations between HNC/0 quantities in the Fourier space for the four possible independent pairings: (pp), (pR), (RR), and (RL).

$$\widetilde{X}_{\rm pR}(q)+\widetilde{N}_{\rm pR}(q)=\frac{\widetilde{X}_{\rm pR}(q)[1-\widetilde{X}^*_{\rm RR}(q)]+\widetilde{X}^*_{\rm pR}(q)\widetilde{X}^*_{\rm RL}(q)}{D(q)}, \tag{31}$$

$$\widetilde{X}_{\rm RR}(q)+\widetilde{N}_{\rm RR}(q)=\frac{[1-\widetilde{X}_{\rm pp}(q)][1-\widetilde{X}^*_{\rm RR}(q)]-\widetilde{X}^*_{\rm pR}(q)^2}{D(q)}-1, \tag{32}$$

$$\widetilde{X}^*_{\rm RL}(q)+\widetilde{N}^*_{\rm RL}(q)=\frac{\widetilde{X}^*_{\rm RL}(q)[1-\widetilde{X}_{\rm pp}(q)]+|\widetilde{X}_{\rm pR}(q)|^2}{D(q)}, \tag{33}$$

and

$$\widetilde{X}_{\rm pp}(q)+\widetilde{N}_{\rm pp}(q)=\frac{|1-\widetilde{X}_{\rm RR}(q)|^2-|\widetilde{X}_{\rm RL}(q)|^2}{D(q)}-1, \tag{34}$$

where the denominator $D(q)$ is given by

$$\begin{aligned}D(q)=&[1-\widetilde{X}_{\rm pp}(q)][|1-\widetilde{X}_{\rm RR}(q)|^2-|\widetilde{X}_{\rm RL}(q)|^2]\\&-|\widetilde{X}_{\rm pR}(q)|^2[\widetilde{X}_{\rm RL}(q)+\widetilde{X}^*_{\rm RL}(q)]\\&-\widetilde{X}_{\rm pR}(q)^2[1-\widetilde{X}^*_{\rm RR}(q)]\\&-\widetilde{X}^*_{\rm pR}(q)^2[1-\widetilde{X}_{\rm RR}(q)].\end{aligned} \tag{35}$$

As in a standard HNC/0 treatment our (pseudo)potentials of the form $U(z)=a\ln(z)+\theta$ are separated into short-range and long-range parts:

$$U^s(z)=-aK_0(Qz) \tag{36}$$

and

$$U^l(z)=a[\ln(z)+K_0(z)]+\theta, \tag{37}$$

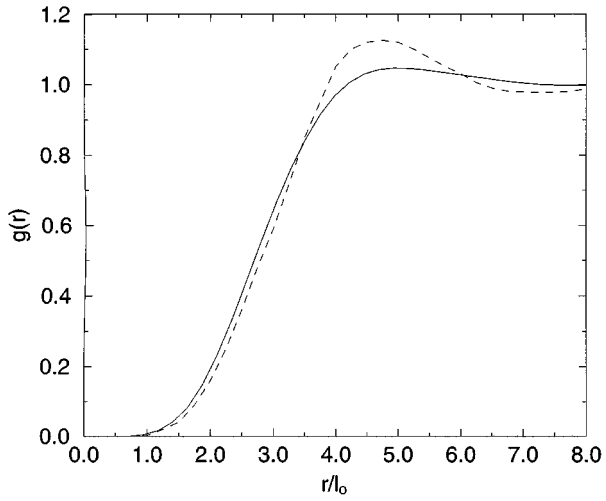


FIG. 1. Pair distribution function obtained from the ESWF (solid line), compared to that of (Ref. 6) (dashed line), for the state $\nu=\frac{2}{7}$.

where $K_0(x)$ is the modified Bessel function and $Q$ the cut-off parameter of the order of unity.

As a consequence, all nodal and non-nodal functions for all independent pairings (pp), (pR), (RR), and (RL) are split into their short- and long-range parts.

## III. RESULTS

A three-component HNC/0 scheme has been implemented using ESWF for the hierarchy states corresponding to $\nu=2/7$, 2/3, and 4/5. For numerical convenience the distances were expressed in dimensionless units $r/l_0$.

In Fig. 1 we plot the pair distribution function $g(r)$ for $\nu=2/7$ as a function of $r/l_0$, and compare it with that of Ref. 6. It clearly shows characteristics of a liquid state. The ground-state energy per particle, computed from

$$u(\nu)=\frac{\rho}{2}\int v(r)[g(r)-1]d^2r, \tag{38}$$

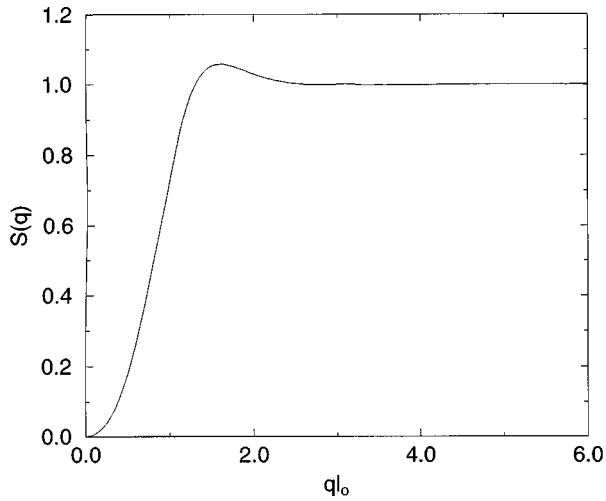


FIG. 2. Structure factor $S(q)$ for $\nu=\frac{2}{7}$ obtained from the ESWF.

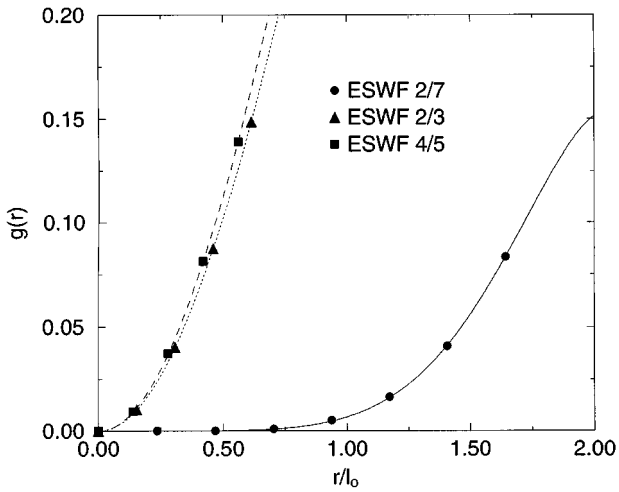


FIG. 3. Small-$r$ behavior for several states.

where $v(r)=e^2/\epsilon r$, is found to be $u(2/7)=-0.374(e^2/\epsilon l_0)$, which is in close agreement with the value $-0.377(3)(e^2/\epsilon l_0)$ of Morf and Halperin.[8] In Fig. 2 we plot the structure factor $S(q)$ for $\nu=2/7$ as a function of $ql_0$.

A further test of the ESWF and a useful source of information for future research is the study of small-$r$ behavior of the pair distribution functions. Diagonalizing numerically the Hamiltonian for a finite system of 4–6 fermions, Yoshioka[9] obtained the coefficients of expansion of $g(r)$ for small $r$ which seem to vary continuously as function of $\nu$. For small $r$ around the origin $r=0$, we can expand our $g(r)$ obtained from the three-component HNC/0 method in the following way:

$$g(r)=\sum_{i=0}^{\infty} c_i\left(\frac{r}{l_0}\right)^{2i}. \tag{39}$$

We least square fitted $g(r)$ in the region $0\leqslant r\leqslant 1.7l_0$ to obtain the coefficients $c_i$.

For $\nu=2/7$ we found $c_0=0$, $c_1=0$, $c_2=0$, $c_3=0.008\,185$, and $c_4=-0.001\,455$, whereas for $i>4$ the coefficients $c_i$ are zero within the limit of our numerical accuracy. The same procedure applied to the state $\nu=4/5$ gives $c_0=0$, $c_1=0.481\,689$, $c_2=-0.133\,997$, $c_3=0.024\,499$, and $c_4=-0.002222$. For the state $\nu=2/3$ we obtained instead $c_0=0$, $c_1=0.435\,377$, $c_2=-0.117\,703$, $c_3=0.021\,723$, and $c_4=-0.001\,994$. These results agree rather closely with the finite-size calculations by Yoshioka.[9]

The small-$r$ behavior of different FQHE pair distribution functions as a function of $r/l_0$ is plotted in Fig. 3. Indeed, there are only small differences between states $\nu=4/5$ and 2/3, as expected.

In Fig. 4 we plot the resulting $g(r)$ for the state $\nu=2/3$, obtained from the ESWF, and compare it with that of Ref. 8. The $g(r)$ for $\nu=2/3$ is in rather good agreement with that obtained by Morf and Halperin[8] using nonantisymmetrized wave functions.

For instance, the ground-state energy per particle was found to be $u(2/3)=-0.510(e^2/\epsilon l_0)$, very close to the result $-0.509(5)(e^2/\epsilon l_0)$ of Morf and Halperin.[8] The ground-state energy per particle of the state $\nu=4/5$ was found to be

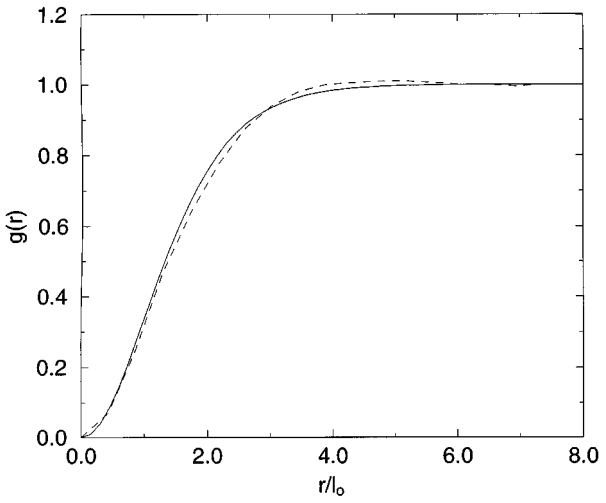


FIG. 4. Pair distribution function $g(r)$ obtained from the ESWF (solid line), compared to that of Morf and Halperin (Ref. 8) (dashed line) for the state $\nu=\frac{2}{3}$.

$u(4/5)=-0.548(e^2/\epsilon l_0)$. Accurate energy values for the states $\nu=2/3$ and $\nu=4/5$ are obtained from those at $\nu=1/3$ and $\nu=1/5$.

Using the Monte Carlo simulation data of Levesque, Weis, and MacDonald,[10] via Eq. (40) for particle-hole symmetry, we compute $u(2/3)$ and $u(4/5)$,

$$\nu u(\nu)=(1-\nu)u(1-\nu)+\left(\frac{\pi}{8}\right)^{1/2}(1-2\nu)\frac{e^2}{\epsilon l_0}, \quad (40)$$

yielding $u(2/3)=-0.518$ $(e^2/\epsilon l_0)$ and $u(4/5)=-0.5519(e^2/\epsilon l_0)$, in reasonable agreement with our approximate values $-0.510(e^2/\epsilon l_0)$ and $-0.548(e^2/\epsilon l_0)$ for functions of the ESWF type.

Finally in Table I we make an overall comparison of the ground-state energy per particle obtained using the ESWF, the results of Dharma-wardana[11] from a parametrization fit, and those of Morf and Halperin.[8]

TABLE I. Ground-state energies per particle $u(\nu)$ in units of $e^2/\epsilon l_0$ for filling factors $\nu=\frac{2}{3}$, $\frac{4}{5}$, and $\frac{2}{7}$. The first row displays our ESWF results. The results of Dharma-wardana (Ref. 11) are reported in the second row; $u(\frac{2}{3})$ and $u(\frac{4}{5})$ are obtained by applying particle-hole conjugation to the corresponding estimates of Ref. 11. The third row represents the results of Morf and Halperin of (Ref. 8).

| $\nu$ | $\frac{2}{3}$ | $\frac{4}{5}$ | $\frac{2}{7}$ |
|---|---|---|---|
| $u(\nu)$ (ESWF) | -0.510 | -0.548 | -0.374 |
| $u(\nu)$ (Ref. 11) | -0.518 | -0.552 | -0.379 |
| $u(\nu)$ (Ref. 8) | -0.509(5) | not available | -0.377(3) |

## IV. CONCLUSION

In the present paper we applied the hypernetted-chain technique to a class of wave functions of the second level hierarchical states of the FQHE, constructed through charge-conjugation procedures.[6,7] These wave functions were found to be of the ESWF type. The model system has been mapped into a three-component mixture of particles, and a multicomponent HNC/0 technique has been applied.

Results for the states at filling factors $\nu=2/7$, 2/3, and 4/5 have been reported. The quantities obtained with this technique are of very good quality, and are fully consistent with several other calculations, essentially confirming the validy of this treatment.

The small-$r$ behavior of the pair function reproduces with very good accuracy the numerical expansion coefficients reported by Yoshioka.[9] The proposed HNC treatment can be used to study several properties of the second-level hierarchy states in the thermodynamic limit.

The main approximation in this work is to neglect the elementary function $E(r)$. It is known that this mainly affects the magnitude of the peak of the pair distribution function. It has also been shown that the scaling approximation applied to the four-point elementary diagrams provides almost exact results[12] for the one-component HNC. The extension of the scaling procedure to the three-component case is not completely straightforward. It is expected that the elementary diagrams which need to be scaled will be the $E_{\rm pp}$ ones. The scaling coefficient may be obtained by requiring consistency on the available sum rules like, for instance, the kinetic-energy sum rule.

The extension of ESWF to higher fractions in the hierarchical scheme requires larger multicomponent systems. The study of ground-state wave functions of the composite fermion type[13] asks for the use of Fermi hyppernetted-chain rather than HNC techniques.[14] Work in this direction is in progress.

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