

# An anyon wavefunction for the fractional quantum Hall effect

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## Abstract

An anyon wavefunction (characterized by the statistical factor  $n$ ) projected onto the lowest Landau level is derived for the fractional quantum Hall effect states at filling factor  $\nu = n/(2pn + 1)$  ( $p$  and  $n$  are integers). We study the properties of the anyon wavefunction by using detailed Monte Carlo simulations in disc geometry and show that the anyon ground-state energy is a lower bound to the composite fermion one. Our results suggest that the composite fermions can be viewed as a combination of anyons and a fluid of charge-neutral dipoles.

The fractional quantized Hall effect (FQHE) is one of the most fascinating phenomena in condensed-matter physics [1]. The pioneering work by Laughlin [2] based on the famous trial wavefunction at the filling of  $\nu = 1/(2p + 1)$  revealed that the FQHE arises from the formation of an incompressible quantum fluid that supports quasiparticles and quasiholes carrying fractional charge and statistics. Jain's composite fermion (CF) approach [3] successfully clarified fundamental aspects of the FQHE, which evolved into the description of the FQHE in terms of electron–vortex composites. A CF is the bound state of an electron and an even number of vortices formed in a two-dimensional (2D) system of electrons subject to a strong perpendicular magnetic field. On the basis of the CF theory, the interacting electrons at the Landau level (LL) filling factor  $\nu = n/(2pn + 1)$ ,  $n$  and  $p$  being integers, transform into weakly interacting CFs with an effective filling factor  $\nu^* = n$ , corresponding to  $n$ -filled CF LLs.

The connection between the FQHE and the integer quantum Hall effect (IQHE) has motivated the Chern–Simons (CS) field theoretical approach [4, 5] for the FQHE. Within this field theoretical approach, an even number of magnetic flux quanta ( $\phi_0 = hc/e$  stands for one flux quantum) are attached to the 2D electrons through the introduction of a CS gauge field. In a mean-field approximation where the statistical gauge fluxes are delocalized from the electrons and uniformly spread out in the 2D plane, the average CS gauge field partially cancels the external magnetic field. So far, the fermion CS approach has been very successful for describing the nature of the quantum Hall state at  $\nu = 1/2$  where the CS-flux-generated fictitious magnetic field exactly cancels the external magnetic field at the mean-field level.

An important aspect for developing theories of FQHE is that the Hilbert space is composed solely of states in the lowest Landau level (LLL). This constraint is important conceptually, as it implies the quenching of the kinetic energy and the non-perturbative nature of electron–electron interactions. Because of the drastic reduction of the size of the Hilbert space, numerical studies have been indispensable to the development of our understanding of the FQHE. Although the FQHE can be considered as an IQHE of CFs, it is worth noting that an explicit LLL projection of the latter states is necessary in the construction of Jain’s wavefunction in order to obtain the correct low energy physics [3].

The Chern–Simons field theory is intimately connected to the statistics transmutation of anyons. Halperin [6] demonstrated that an anyon picture inherently leads to an explanation of the quantum Hall hierarchy. Similarly, Ma and Zhang [7] considered ideal anyons subject to a magnetic field. The ideal anyons with statistics  $1/n$  in a strong magnetic field have a ground state that exhibits an IQHE at filling factor  $n$  with quasiparticle excitations of charge  $ne$ . The electron FQHE states are realized with asymmetry in quasiparticle states ( $\nu = n/(2pn + 1)$ ) and quasihole states ( $\nu = n/(2pn - 1)$ ) in the fractional quantum Hall hierarchy. Despite the consensus that anyons play an important role in understanding the FQHE, there remains no explicit, quantitative study of the corresponding ground-state properties.

In this paper we derive an anyon wavefunction for filling factors  $n/(2pn + 1)$ , with  $p$  and  $n$  being integers, that is fully projected onto the LLL. We then study the properties of the anyon wavefunction at these filling factors by carrying out detailed Monte Carlo (MC) simulations. Our calculation provides important information on the connection between anyons and CFs.

## 1. The Chern–Simons transformation

The Hamiltonian for 2D electrons subject to the perpendicular magnetic field  $\mathbf{B} = -B\hat{\mathbf{z}}$  is given by

$$\hat{H} = \hat{H}_0(B) + \hat{V} = \frac{1}{2m} \sum_{j=1}^N (\mathbf{p}_j + e\mathbf{A}_j)^2 + \hat{V}, \quad (1)$$

where  $m$  and  $-e$  are the electron’s mass and charge,  $N$  is the total number of electrons and  $\hat{V}$  is the Coulomb interaction energy. We adopt the symmetric gauge in which the vector potential is given by  $\mathbf{A}_j = (B/2)(y_j, -x_j, 0)$ . With the use of complex coordinates  $z_j = x_j + iy_j$ , and the notation  $\chi = \prod_{i<j}^N (z_i - z_j)$ , the CS transformation amounts to multiplying the many-body wavefunction of the CS transformed Hamiltonian by  $(\chi/|\chi|)^\alpha$  and  $e\mathbf{A}_j \rightarrow e\mathbf{A}_j + \alpha\mathbf{A}_j \equiv e\mathbf{A}_j + \alpha \sum_{k \neq j}^N (-(y_j - y_k), x_j - x_k, 0)/|\mathbf{r}_j - \mathbf{r}_k|^2$ .

After the CS transformation, the kinetic energy operator  $\hat{H}_0(\alpha)$  can be expressed in terms of destruction and creation operators,  $a(\alpha)$  and  $a^\dagger(\alpha)$ , as (all lengths are measured in units of magnetic length  $l \equiv \sqrt{\hbar/e|B|}$ )

$$\hat{H}_0(\alpha) = \hbar\omega_c \sum_{j=1}^N a_j^\dagger(\alpha) a_j(\alpha) + \frac{1}{2} N \hbar\omega_c, \quad (2)$$

where  $\omega_c = e|B|/m$  is the cyclotron frequency and

$$a_j(\alpha) = \frac{1}{\sqrt{2}} \left( 2 \frac{\partial}{\partial z_j^*} + \frac{z_j}{2} + \alpha \frac{\partial \ln \chi^*}{\partial z_j^*} \right), \quad (3)$$

$$a_j^\dagger(\alpha) = \frac{1}{\sqrt{2}} \left( -2 \frac{\partial}{\partial z_j} + \frac{z_j^*}{2} + \alpha \frac{\partial \ln \chi}{\partial z_j} \right). \quad (4)$$

The operators  $a_j^\dagger(\alpha)$  and  $a_j(\alpha)$  satisfy Bose commutation relations,  $[a_j(\alpha), a_k^\dagger(\alpha)] = \delta_{jk}$ .

The angular momentum operator  $\hat{L}(\alpha)$  after the transformation reads

$$\hat{L} \rightarrow \hat{L}(\alpha) = \sum_{j=1}^N (b_j^\dagger(\alpha)b_j(\alpha) - a_j^\dagger(\alpha) a_j(\alpha)), \tag{5}$$

where the operators  $b(\alpha)$  and  $b^\dagger(\alpha)$  describe the LLL degeneracy,

$$b_j(\alpha) = \frac{1}{\sqrt{2}} \left( 2 \frac{\partial}{\partial z_j} + \frac{z_j^*}{2} - \alpha \frac{\partial \ln \chi}{\partial z_j} \right), \tag{6}$$

$$b_j^\dagger(\alpha) = \frac{1}{\sqrt{2}} \left( -2 \frac{\partial}{\partial z_j^*} + \frac{z_j}{2} - \alpha \frac{\partial \ln \chi^*}{\partial z_j^*} \right), \tag{7}$$

satisfying  $[b_j(\alpha), b_k^\dagger(\alpha)] = \delta_{jk}$ .

So far, two types of CS transformation have been discussed in the literature. One considers  $\alpha$  to be an integer ( $\alpha = 2p$  has been used in the fermion CS theory to describe the reduction of the effective magnetic field felt by the CFs [4, 8, 9]; and  $\alpha$ , an odd integer, has been used to describe the formation of the Bose condensate in the FQHE [10]). The other choice concerns  $\alpha = 1/n$  and has been used for the study of anyon superconductivity in the absence of a magnetic field [11].

We recall that for  $\hat{H}_0(B)$ , the LLL wavefunction has the form  $\Psi_{\text{LLL}}(B) = W(N) f(\{z_j\}, \{\partial/\partial z_j\})$ , where  $W(N) = \exp(-\sum_{i=1}^N |z_i|^2/4l^2)$  [3]. Due to the symmetry properties of parity ( $\mathcal{P}$ ) and time reversal ( $\mathcal{T}$ ) of the Hamiltonian (under  $\mathcal{PT}$ :  $z \leftrightarrow -z^*$ ,  $B \leftrightarrow -B$ ,  $\alpha \leftrightarrow -\alpha$ ), the corresponding wavefunction for  $\hat{H}_0(-B)$  is  $\Psi_{\text{LLL}}(-B) = W(N) f(\{z_j^*\}, \{\partial/\partial z_j^*\})$ .  $\Psi_{\text{LLL}}(B)$  and  $\Psi_{\text{LLL}}(-B)$  are the eigenfunctions for  $\hat{L}(\alpha = 0)$  as well, with the eigenvalues  $N(N - 1)/2\nu$  and  $-N(N - 1)/2\nu$  ( $\nu$  is the filling factor), respectively. Furthermore,  $f$  should be an antisymmetric function for electrons.

One can get useful information on the explicit form of  $f$  by employing the CS transformation and the  $\mathcal{PT}$  symmetry property of the LLL wavefunctions. For our purpose, we are interested in deriving a general equation for the LLL projection by requiring

$$a_j(\alpha)\Psi_{\text{LLL}} = 0. \tag{8}$$

From equations (3) and (8), we have

$$2 \frac{\partial \ln f}{\partial z_j^*} + \alpha \frac{\partial \ln \chi^*}{\partial z_j^*} = 0. \tag{9}$$

Note that for  $f(\{z_j\}, \{\partial/\partial z_j\})$ , equation (9) yields a trivial solution  $\alpha = 0$ . For  $f(\{z_j^*\}, \{\partial/\partial z_j^*\})$ , equation (9) is an operator equation and the solution is a functional of  $\chi^*$ . If we approximate  $f(\{z_i^*\}, \{\partial/\partial z_i^*\})$  by a function form  $f(\{z_i^*\})$ , there exists a simple solution:

$$f(\{z_j^*\}) = (\chi^*)^{-\alpha/2}. \tag{10}$$

Furthermore, using the constraint that  $\Psi_{\text{LLL}}(-B)$  is the eigenfunction of  $\hat{L}(\alpha)$  with the eigenvalue  $N(N - 1)/2\nu$ , one determines  $\alpha$  in terms of the filling factor  $\alpha = -2/\nu$ .

By putting everything together, we arrive at the LLL wavefunction for fillings of  $\nu = n/(2pn + 1)$ :

$$\Psi_{\text{LLL}}(B) = \prod_{i < j}^N (z_i - z_j)^{2p + \frac{1}{n}} \exp\left(-\sum_{i=1}^N |z_i|^2/4l^2\right). \tag{11}$$

A few remarks are immediately in order. (i) For  $n = 1$ , equation (11) coincides with Laughlin's wavefunction for  $\nu = 1/(2p + 1)$ . For  $n > 1$ , the wavefunction describes an incompressible quantum liquid state with filling factor  $\nu = n/(2pn + 1)$  ( $p$  and  $n$  are integers)

and is not antisymmetric (as required for a fermion state). It has anyon symmetry (characterized by the statistical parameter  $n$ ). The LLL anyon wavefunction is expected to serve as a lower bound for the energy of the electronic system. Equation (11) describes composite anyons in that the anyons are attached by  $2p$  vortices. (ii) The transformation employed is a unitary one, in contrast to that used by Ma and Zhang [7]. As a result, our approach conserves the density of the system upon transformation. Nevertheless it should be noted that the resulting wavefunctions are of the same form. The quasiparticle and quasiholes obey fractional statistics and have fractional charge [7]. Therefore, there exists an asymmetry for the quasiparticle and quasihole states. The quasiparticle states ( $\nu = n/(2pn + 1)$ ) correspond to anyon screening (i.e., the anyon-flux-generated fictitious magnetic field cancels the residual external fields), while the quasihole states ( $\nu = n/(2pn - 1)$ ) correspond to anyon anti-screening. (iii) The average CS-generated magnetic field reverses the external magnetic field. This can be seen to be in agreement with the CS transformations for CFs ( $\alpha = 2p$ ) for the reduction of the effective magnetic fields for the CFs. For the part associated with the anyons, the coefficient  $2/n$  in the CS transform can be understood as follows: the  $1/n$ -CS field generates a fictitious magnetic field that cancels the external magnetic field, while the additional  $1/n$ -CS field reverses the direction of the external magnetic field with the same magnitude which corresponds to  $n$ -filled Landau bands. It becomes clear that after expansion of the anyon's wavefunction in the fermion representation where the average field corresponds to  $n$ -filled Landau bands, the resulting wavefunction after the LLL projection becomes Jain's CF wavefunction. It is worth pointing out that the cancellation or the effective reduction of the magnetic field is a dynamical feature associated with the CS gauge field. (iv) According to the corresponding state theory of the global phase diagram [12], the incompressible quantum fluid states described by the anyon wavefunctions correspond to superfluid states. For the FQHE, the transition between different integer quantum Hall states is predicted to follow a floating-up picture, i.e., only transitions between adjacent  $n$  are allowed. For the floating-up scenario, one of us [13] argued that the integer quantum Hall transitions correspond to the transition from a superfluid state for anyons characterized by  $n$  ( $n$ -anyons) to insulating states of  $(n \pm 1)$ -anyons. This is in agreement with the hierarchy picture for equation (11).

## 2. Monte Carlo calculations

It is readily observable that the systems described by the anyon wavefunction can be mapped to a one-component plasma with inverse temperature  $(4p + 2/n)$  [2]. The system under consideration consists of  $N$  particles moving in a 2D space subject to a strong perpendicular magnetic field and embedded in a uniform neutralizing background of positive charge. Our goal is to calculate the thermodynamic limit of the expectation value of the potential energy operator and other quantities. To do so we perform detailed MC simulations in disc geometry [14] and extract the thermodynamic estimate of various quantities by extrapolating the finite  $N$  results.

In our simulations we adopt the well-known Metropolis algorithm [15]. The expectation value of any operator is then estimated by averaging its value over numerous configurations. For each  $N$  we routinely perform MC simulations employing several million configurations. All the results that we report here were obtained after discarding 100 000 'equilibration' MC steps and using  $2 \times 10^6$  MC steps for averaging purposes. The thermodynamic estimate of the correlation energy per particle is obtained by fitting  $E = \langle \hat{V} \rangle / N$  with a second-order polynomial in  $1/\sqrt{N}$  by using systems with  $N = 4, 16, 36, 64, 100, 144$  and 196 particles.

In table 1 we show the thermodynamic estimates of the correlation energy per particle for a number of FQHE states described by the anyon wavefunction. The results are rounded in the last digit. We compare the energies obtained from the anyon wavefunction with the

**Table 1.** Thermodynamic estimates for the correlation energy per particle of various FQHE states described by the anyon and Jain CF wavefunctions [3]. Energies are in units of  $e^2/l$ .

$p$	$n$	$\nu$	$E$	$E_{CF}$
1	1	1/3	-0.4095	-0.4098
	2	2/5	-0.4422	-0.4328
	3	3/7	-0.4550	-0.4423
	4	4/9	-0.4618	-0.4474
	5	5/11	-0.4661	-0.4508
	$\infty$	1/2	-0.4844	-0.4653
2	1	1/5	-0.3274	-0.3275
	2	2/9	-0.3432	-0.3428
	3	3/13	-0.3490	-0.3483
	4	4/17	-0.3520	-0.3512
	5	5/21	-0.3538	—
	$\infty$	1/4	-0.3615	—

corresponding results of the LLL projected Jain's CF wavefunction as obtained after MC simulations in spherical geometry [16].

The energy difference between Jain's CF and anyon states can be interpreted as an exchange correction in that the anyon wavefunction does not satisfy the Pauli principle required for fermions (except for  $n = 1$ ). As can be seen from table 1, the energy of the anyon state,  $E$ , serves as a lower bound for the CF's energy  $E_{CF}$ . It is interesting to note that such an exchange correction,  $E_X$ , satisfies the relation

$$E_X = E_{CF} - E = (1 - 1/n)\Delta_p, \quad (12)$$

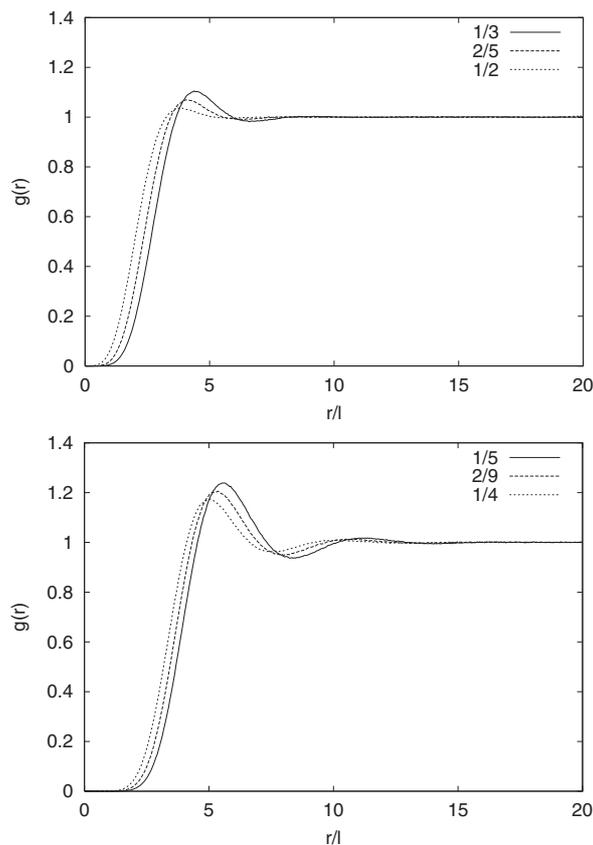
where  $\Delta_p$  can be readily extracted from table 1. Our calculations show that  $\Delta_1 = 0.019e^2/l$  and  $\Delta_2 \simeq 0.001e^2/l$ .

The dependence on  $n$  of the exchange correction is of the same form,  $|1 - 1/n|$ , as Ma and Zhang used in the perturbative analysis of ideal anyons in a magnetic field [7]. The size of the exchange correction indicates that the anyon wavefunction captures the essential physics of the FQHE. The largest correction refers to the case of  $\nu = 1/2$ , where  $E$  is merely 4% lower than  $E_{CF}$ . The effect of exchange correction on the energy is greatly reduced with increase of  $p$ . In fact, for  $p > 1$ , the contribution of the exchange correction becomes negligible.

Following the same procedure as described in [14], we have calculated the pair distribution,  $g(r)$ . As seen from figure 1, the calculated  $g(r)$  shows  $p$  ( $=1, 2$ ) 'bumps' for filling factors of  $n/(2pn + 1)$ . This is to be compared with the results for CFs, where more 'bumps' (or 'wiggles') are observable due to the inherent fermion symmetry of Jain's CF wavefunction and the associated Friedel-like oscillations. Moreover, there exist notable differences in the short distance behaviours of  $g(r)$ . The short range behaviour of the pair distribution function obtained from the anyon wavefunction is consistent with  $g(r) \propto r^{4p+2/n}$ . At  $\nu = 1/2$ , the dependence on  $r^4$  is to be compared with the  $g(r) \propto r^2$  dependence obtained from the Rezayi-Read Fermi wavefunction [17].

### 3. Discussion

We are now in the position to discuss the exchange corrections. To this end, it is instructive to focus on the limiting case of  $\nu = 1/2$ . According to the neutral fermion theory [18], at filling of  $\nu = 1/2$ , a liquid of  $\pm 1/2$  charged anyons (neutral fermions) floats on top of a Bose quantum Hall liquid. Since the Bose quantum Hall liquid is described by equation (11) with



**Figure 1.** Pair distribution function for the anyon wavefunction at filling factors  $\nu = 1/3, 2/5, 1/2$  and  $1/5, 2/9, 1/4$  obtained after a MC simulation in disc geometry for  $N = 196$  particles.

$n = \infty$ , the energy for the dipoles (neutral fermions) is nothing but  $\Delta_1$ . The neutral fermions and the Bose quantum Hall fluid are decoupled in that the neutral fermions experience no magnetic field and their contribution to  $\sigma_{xy}$  is 0. The decoupling of the neutral fermions and the Bose quantum fluid has important consequences for the low energy physics. In general, the wavefunctions factorize into products and the corresponding anyon excitations and neutral fermion excitations have different behaviours.

It is tempting to develop a perturbation for the exchange corrections. However, the ‘expansion parameter’,  $|1 - 1/n|$ , is not small. Analogously to the case of  $\nu = 1/2$ , it is reasonable to view CFs as a combination of anyons and dipole interactions. The CFs and anyons experience an effective magnetic field corresponding to  $n$ -filled Landau bands. The ‘gradient correction’ to the anyon wavefunction can be identified as a dipole field that experiences no residual magnetic field. A fluid of dipoles is expected to be compressible [18], and to decouple from the incompressible quantum fluid described by anyons due to the different symmetry properties under  $\mathcal{T}$ . In fact, our results from equation (12) suggest an effect of vortex–charge separation for the CFs. Naturally, the effect of the dipole field is expected to be proportional to  $(1 - 1/n)$ . Further quantitative studies are clearly desirable.

In summary, we introduced and studied the properties of a specific microscopic anyon wavefunction for the FQHE. Our MC results based on this anyon wavefunction provide

a *quantitative* measure of the contributions from the anyon picture of the FQHE, which was discussed *qualitatively* by Ma and Zhang [7]. The comparison with Jain's CF results reveals interesting exchange effects arising from the approximations involved in the anyon wavefunction.

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