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Mehler-Fock Transformation of Ultradistribution

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Abstract

This paper deals with the testing function space Z and its dual Z', which is known as ultradistrbution. Some theorems and properties are investigated for the Mehler-Fock transformation and its inverse for the ultradistribution.

Keywords: Mehler-Fock Transform; Legendre Function; Testing Function Space; Distribution Space; Ultra Distribution

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1. Introduction

One of the distribution space, known as ultradistribution, has been defined on many transformations by researchers Pathak (1997) and Zemanian (1965). In an earlier work Loonker and Banerji (2008), authors have studied the Mehler-Fock transform to the tempered distribution.

The Mehler – Fock transformation is defined as [cf. Yakubovich and Luchko (1994, p. 149)]

$$M[f(x)] = F(r) = \int_1^\infty P_{-\frac{1}{2}+ir}(x) f(x) dx, \qquad r \geq 0, \tag{1}$$

and its inversion is given by

$$f(x) = \int_0^\infty r \tanh(\pi r) P_{-\frac{1}{2}+ir}(x) F(r) dr, \quad x > 1. \tag{2}$$

The *generalization* of the *Mehler-Fock transformation* is given by [cf. Pathak (1997, p. 343)]

$$F(r)=\int_0^\infty f(x)P^{m,n}_{-\frac{1}{2}+ir}(\cosh x)\sinh x dx \quad , \tag{3}$$

where $P^{m,n}_{-\frac{1}{2}+ir}(\cosh x)$ is the generalized Legendre function, defined for complex values of the parameters k, m and n by

$$P^{m,n}_{k}(z)=\frac{(z+1)^{n/2}}{\Gamma(1-m)(z-1)^{m/2}}\ {}_2F_1\left[k+\frac{n-m}{2}+1;-k+\frac{n-m}{2};1-m;\frac{1-z}{2}\right], \tag{4}$$

for complex z not lying on the cross-cut along the real x-axis from 1 to $-\infty$.

The **inversion formula** of (3) is

$$f(x)=\int_0^\infty \chi(r)P^{m,n}_{-\frac{1}{2}+ir}(\cosh x)F(r)dr, \tag{5}$$

where

$$\chi(r)=\Gamma\left(\frac{1-m+n}{2}+ir\right)\Gamma\left(\frac{1-m+n}{2}-ir\right)\Gamma\left(\frac{1-m-n}{2}+ir\right)\Gamma\left(\frac{1-m-n}{2}-ir\right)$$

$$\times\left[\Gamma(2ir)\Gamma(-2ir)\pi 2^{n-m-2}\right]^{-1}, \tag{6}$$

when $m=n$, (3) and (5) can be written as

$$F(r)=\int_0^\infty P_{-\frac{1}{2}+ir}(\cosh\alpha)\sinh\alpha\ f(\alpha)d\alpha, \tag{7}$$

and

$$f(\alpha)=\int_0^\infty r\tanh(\pi r)P_{-\frac{1}{2}+ir}(\cosh\alpha)F(r)dr, \tag{8}$$

whereas for $m=n=0$, (3) and (5) reduce to (1) and (2), respectively.

The **Parseval relation** for the Mehler-Fock transformation is defined as [Sneddon (1974, pp. 393-94)]

$$\int_0^\infty r\tanh(\pi r)F(r)G(r)dr=\int_1^\infty f(x)g(x)dx, \tag{9}$$

whose convolution is

$$M[f*g]=M[f]\cdot M[g]. \tag{10}$$

The asymptotic behaviour for (4) is defined [Pathak (1997, p.345)] as

$$P^{m,n}_{-1/2+ir}(\cosh x) = \begin{cases} O(x^{-\operatorname{Re} m}), & x \to 0+ \\ O(e^{-(1/2)x}), & x \to \infty, \end{cases} \tag{11}$$

and

$$P^{m,n}_{-1/2+ir}(\cosh x) = \begin{cases} O(1), & r \to 0+ \\ 2^{1/2(n-m-1)} \pi^{-1/2} (\sinh x)^{-1/2} (ir)^{m-1/2} \left[e^{irx} + ie^{-i(m\pi + xt)} + O(r^{-1}) \right], & r \to \infty. \end{cases} \tag{12}$$

Similarly, the function $\chi(r)$, defined by (6), possesses the following asymptotic behaviour [cf. Pathak (1997, p.345)]

$$\chi(r) = \begin{cases} O(r^2), & r \to 0+ \quad |\operatorname{Re} n| < 1 - \operatorname{Re} m \\ \dfrac{(ir)^{1-2m}}{\pi 2^{n-m-+2}} [1 + O(r^{-1})], & r \to \infty. \end{cases} . \tag{13}$$

The distributional generalized Mehler-Fock transform $f \in \mathbb{M}'^{\alpha}_{\beta}(I)$, where $\alpha \geq \operatorname{Re}(m)$ and $\beta \leq 1/2$, is defined as [Pathak (1997, p. 346)]

$$F(r) := \left\langle f(x), P^{m,n}_{-\frac{1}{2}+ir}(\cosh x) \right\rangle, \quad r \geq 0, \tag{14}$$

where the space $\mathbb{M}'^{\alpha}_{\beta}(I)$) is the dual of the space $\mathbb{M}^{\alpha}_{\beta}(I)$, which is the collection of all infinitely differentiable complex valued function φ defined on I such that for every non-negative integer *q*,

$$\gamma_q(\varphi) = \sup_{0<t<\infty} \left| \varsigma(x) \nabla^q_x \varphi(x) \right| < \infty, \tag{15}$$

Where

$$\nabla_x = \left(D^2_x + (\coth x) D_x + \frac{m^2}{2(1-\cosh x)} + \frac{n^2}{2(1+\cosh x)} \right), \tag{16}$$

And

$$\varsigma(x) = \varsigma_{\alpha,\beta}(x) = \begin{cases} O(x^{\alpha}), & x \to 0, \\ O(x^{\beta}), & x \to \infty. \end{cases} . \tag{17}$$

The topology over $\mathbb{M}^{\alpha}_{\beta}(I)$ is generated by separating the collection of seminorms $\{\gamma_q\}^{\infty}_{q=0}$ and is a sequentially complete locally convex topological vector space. $\mathcal{D}(I)$, the space of infinitely differentiable functions of compact support with the usual topology, is a linear subspace of $\mathbb{M}_{\beta}^{\alpha}(I)$.

From the properties of the hypergeometric functions, the generalized Legendre function [Pathak (1997, p.346)], satisfies the following differential equation

$$D^2 y + (\coth x) Dy + \left[\frac{m^2}{2(1-\cosh x)} + \frac{n^2}{2(1+\cosh x)} + \left(r^2 + \frac{1}{4}\right)\right] y = 0 . \tag{18}$$

Therefore,

$$\nabla_x P_{-1/2+ir}^{m,n}(\cosh x) = -\left(r^2 + \frac{1}{4}\right) P_{-1/2+ir}^{m,n}(\cosh x) . \tag{19}$$

Relations (11) and (12) prove the boundedness for the Legendre function [Pathak (1997), pp. 346-347, Lemma 11.3.1, Eqn. (11.3.2)]

$$\left| \varsigma(x) \left(\frac{\partial}{\partial r}\right)^q P_{-1/2+ir}^{m,n}(\cosh x) \right| \le C \varsigma(x) x^q \left(\frac{\pi}{2}\right)^{1/2} \Gamma\left(\frac{1}{2} - m_r\right) P_{-1/2}^{m_r,0}(\cosh x) , \tag{20}$$

where C is a constant independent of x and r and, m_r is $\operatorname{Re}(m)$.

The **differentiability** of the Mehler-Fock transform is defined by [Pathak (1997, p.347)]

$$F'(r) := \left\langle f(x), \left(\frac{\partial}{\partial r}\right) P_{-\frac{1}{2}+ir}^{m,n}(\cosh x) \right\rangle, \tag{21}$$

where $f \in \mathbb{M}'^{\alpha}_{\beta}(I)$, $\alpha \ge \operatorname{Re}(m), \operatorname{Re}(m) < 1/2, \quad \beta \le 1/2, r \ge 0$.

When q is non-negative integer depending on f, the **asymptotic behavior** of the Mehler-Fock transform is

$$F(r) = \begin{cases} O(1), & r \to 0, \\ O(r^{2q}), & r \to \infty, \end{cases} \tag{22}$$

where

$$|F(r)| \le C' \max_q \left(r^2 + \frac{1}{4}\right)^q . \tag{23}$$

For the operator $\nabla_x^* : \mathbb{M}'^{\alpha}_{\beta}(I) \to \mathbb{M}'^{\alpha}_{\beta}(I)$ under already stated symbols and for $f \in \mathbb{M}'^{\alpha}_{\beta}(I)$, $\varphi \in \mathbb{M}_{\beta}^{\alpha}(I)$, we define the operator transformation formula by

$$\left\langle (\nabla_x^*)^q f(x), \phi(x) \right\rangle = \left\langle f(x), \nabla_x^q \phi(x) \right\rangle, \tag{24}$$

and for *f* being the generalized Mehler-Fock transformation,

$$M\left[(\nabla_t^*)^q f(x)\right] = (-1)^q \left(\frac{1}{4} + r^2\right)^q M\left[f(x)\right] \quad . \tag{25}$$

2. The Space *Z* and its Dual Z'

Let *Z* be the space of the entire function $\varphi(z)$ on l – complex variables such that there exists constants $C_{s,t}$ and $a_1, a_2, \ldots, a_l$ such that

$$\left|z^t\right|\left|D^s \varphi(z)\right| \le C_{s,t} \exp\left(a_1|y_1| + \cdots + a_l|y_l|\right), \tag{26}$$

for all $t = (t_1, t_2, \ldots t_l), t_j \ge 0$ and $s = (s_1, s_2, \ldots s_l), s_j \ge 0$. If the sequence $\{\varphi_\upsilon\}$ satisfies (26), it is said to converge in the sense of *Z*, where *s* and *t* are non-negative integers and $y = \mathrm{Im}(z)$. A sequence $\{\varphi_\upsilon\}$ of functions in *Z* is said to converge in the sense of *Z* if and only if

(i) The sequence $\{\varphi_\upsilon\}$ converges to a limit function uniformly on every compact set in the z – plane.

(ii) For every $s \ge 0, \{D^s \varphi_\upsilon\}$ converges to $\{D^s \varphi\}$ uniformly on every compact set in the *z*-plane.

(iii) There exists constants $C_{s,t}$ and a, independent of υ such that

$$|z|^t \left|D^s \varphi_\upsilon(z)\right| \le C_{s,t} e^{a|y|}, \quad \forall z \text{ and } \upsilon.$$

The dual of the space *Z* is defined as Z' and the function *f* is said to be in Z' if and only if it satisfies the following:

(i) Linearity: If φ_1 and φ_2 are testing functions and α and β are two complex numbers, then

(ii)

$$(f, \alpha\varphi_1 + \beta\varphi_2) = \alpha(f, \varphi_1) + \beta(f, \varphi_2).$$

(iii) Continuity: If $\{\varphi_\upsilon\}_{\upsilon=1}^\infty$ converges in *Z* to zero, then the sequence of numbers $\{(f, \varphi_\upsilon)\}_{\upsilon=1}^\infty$ also converges to zero.

If $\{\varphi_\upsilon\}_{\upsilon=1}^\infty$ converges in *Z* to a limit function φ that is not identically zero, then $\{(f, \varphi_\upsilon)\}_{\upsilon=1}^\infty$ converges to (f, φ). The generalized Mehler-Fock transform of distributions in ☟ ′ (dual of Δ) are generalized function in Z'.

The results proved in the section that follows, satisfy the testing function space *Z* whose Mehler-Fock transformation are in ☟ ′, and satisfy (26) also.

3. Mehler-Fock Transformation and the Testing Function Space *Z*

Employing the notions and terminologies of those of Zemanian (1965, p.193), we define the testing function space Z and the ultradistribution space Z' for the Mehler-Fock transformation. Let $\Phi(r)$ be an arbitrary testing function in Δ, whose support is contained in the finite interval $0 < r < \infty$ and the inverse Mehler-Fock transformation of it can be written as an integral over $(0,\infty)$ as

$$\varphi(x) = \int_0^\infty \chi(r) P^{m,n}_{-\frac{1}{2}+ir}(\cosh x)\Phi(r)dr \, . \tag{27}$$

The function $\varphi(x)$ in (27), can be extended to an entire function over the complex z plane $(z = x + iy)$ such that

$$\varphi(z) = \int_0^\infty \chi(r) P^{m,n}_{-\frac{1}{2}+ir}(\cosh z)\Phi(r)dr \, , \tag{28}$$

is analytic for all finite z and converges uniformly over every bounded domain of the z – plane.

The integrand is a continuous function of (z, r) for every complex z and every real r and also is an analytic function of z for every real r. Moreover, if (28) is integrated by parts q times, we obtain

$$\varphi(z) = \int_0^\infty \Phi^{(q)}(r)\chi^{(q)}(r)\int_0^\infty P^{m,n}_{-\frac{1}{2}+ir}(\cosh z)dr \tag{29}$$

$$\varphi(z) = \int_0^\infty \Phi^{(q)}(r)\chi^{(q)}(r)\int_0^\infty [2^{1/2(n-m-1)}\pi^{-1/2}(\sinh x)^{-1/2}(ir)^{m-1/2}$$

$$\times\{e^{irx} + ie^{-i(m\pi+rx)} + O(r^{-1})\} - O(1)]dr \, , \qquad \text{[from (12)]} \tag{30}$$

$$\varphi(z) = \int_0^\infty \Phi^{(q)}(r)\chi^{(q)}(r)\int_0^\infty C_q \;\; \{e^{irx} + ie^{-i(m\pi+rx)} + O(r^{-1})\} - O(1)\}dr \, , \tag{31}$$

i.e.

$$(-iz)^q \varphi(z) = \int_0^\infty \Phi^{(q)}(r)\chi^{(q)}(r) C_q dr \, , \tag{32}$$

such that for all z

$$\left| z^q \phi(z) \right| \le C_q U_q \le N_q, \quad q = 0, 1, 2, \cdots, \tag{33}$$

where

$$C_q = \int_0^\infty \left| \Phi^{(q)}(r)\chi^{(q)}(r) \right| dr \, . \tag{34}$$

Thus, if $\Phi(r)$ is in Δ with its support in $0 \le r \le \infty$, then $\varphi(z)$ can be extended to an entire function with the existence of a set of constants $C_q\,(q = 0,1,2,\ldots)$ such that the inequality (33) is satisfied. The converse holds true. Indeed,

$$\Phi(r) = \int_0^\infty \varphi(z) P^{m,n}_{-\frac{1}{2}+ir}(\cosh z)\sinh z\, dz\,, \tag{35}$$

where $\varphi(z)$ being an entire function, satisfying (33). The path of integration may be shifted in the z – plane onto any line that is parallel to the z-axis, which is justified by the Cauchy's theorem and due to the fact that, for all y in any fixed finite interval, $\varphi(x+iy)$ goes to zero faster than any power of $1/|x|$ as $|x| \to \infty$, according to the relation (33). Thus, such a shifting for every y, yields

$$\Phi(r) = \int_0^\infty \varphi(x+iy) P^{m,n}_{-\frac{1}{2}+ir}(\cosh(x+iy))\sinh(x+iy)dx\,. \tag{36}$$

For the reason, stated above, the last integral (36) converges uniformly for $0 < r < \infty$, which on formal differentiation under the integral sign, yields

$$\Phi^{(1)}(r) = \int_0^\infty \varphi(z) \frac{\partial}{\partial r}\left(P^{m,n}_{-\frac{1}{2}+ir}(\cosh(z))\right)\sinh(z)dz\,,$$

which when continued q times, finally yields

$$\Phi^{(q)}(r) = \int_0^\infty \varphi(z) \frac{\partial^q}{\partial r^q}\left(P^{m,n}_{-\frac{1}{2}+ir}(\cosh(z))\right)\sinh(z)dz$$

$$= \int_0^\infty \varphi(z) C\varsigma(z) z^q \left(\frac{\pi}{2}\right)^{1/2} \Gamma\frac{1}{2} - m_r P^{m_r,0}_{-\frac{1}{2}}(\cosh z)dz \qquad \text{[from (20)]}$$

$$= \int_0^\infty C\varphi(z) z^q (z^\alpha) \left(\frac{\pi}{2}\right)^{1/2} \Gamma\frac{1}{2} - m_r P^{m_r,0}_{-\frac{1}{2}}(\cosh z)dz \qquad \text{[from (17)]}$$

i.e.

$$\Phi^{(q)}(r) = \int_0^\infty \varphi(z) z^q P^{m_r,0}_{-\frac{1}{2}}(\cosh z) A_q dz\,, \tag{37}$$

where A_q is a constant .

Now invoking (33), we obtain from (37)

$$\left|\Phi^{(q)}(r)\right| \le N_q \quad , \quad q = 0,1,2,\ldots. \tag{38}$$

By virtue of the asymptotic behaviour and boundedness property of the function $P^{m,n}_{-1/2+ir}(\cosh z)$, it implies that $\left|\Phi(r)\right| = 0$, for $\left|r\right| > 0$.

Lemma 1:

A necessary and sufficient condition for $\Phi(r)$ to be in Δ, with its support contained in $0 \le r \le \infty$, is that its inverse Mehler-Fock transformation can be extended to an entire function that satisfies the inequality (33).

Proof:

Z is the testing function space, whose Mehler-Fock transformation are elements of Δ. Above Lemma characterizes that *Z* is the space of all entire functions, that satisfy the inequality (33) for some constant N_q, which implies that *Z* is a linear space. Also, the Mehler-Fock transformation and its inverse are linear one-to-one mapping of *Z* onto Δ onto *Z*, respectively. Further, for the convergence in space *Z*, we claim that the sequence $\{\varphi_\upsilon(z)\}_{\upsilon=1}^{\infty}$ converges in *Z* if the following conditions are satisfied

(i) Each $\{\varphi_\upsilon\}$ is in *Z*.

(ii) There exists a constant $N_q\,(q = 0,1,2,\ldots)$, which does not depend upon υ such that for all $z = x + iy, \left|z^q \varphi_\upsilon(z)\right| \le N_q,\ q = 0,1,2,\ldots$.

(iii) $\{\varphi_\upsilon(z)\}_{\upsilon=1}^{\infty}$ converges uniformly on every bounded domain of the *z*- plane.

As a consequence, the limit function φ of $\{\varphi_\upsilon(z)\}_{\upsilon=1}^{\infty}$ is also in *Z*, for which it will satisfy condition (ii), and the uniformity of convergence in condition (iii) ensures that $\varphi(z)$ is analytic for all *z*. Thus, the space *Z* is closed under convergence. The Conditions (ii) and (iii) imply that $\{z^q \varphi_\upsilon(z)\}_{\upsilon=1}^{\infty}$ converges to $z^q \varphi(z)$ uniformly. The lemma is, therefore, completely proved.

Theorem 1:

The sequence $\{\Phi_\upsilon\}_{\upsilon=1}^{\infty}$ converges in Δ to the limit Φ if and only if the inverse Mehler-Fock transformation $\{\varphi_\upsilon(z)\}_{\upsilon=1}^{\infty}$ converges in *Z* to the limit $\varphi = M^{-1}\Phi$, where notations have usual meaning.

Proof:

Let $\{\Phi_\upsilon(r)\}_{\upsilon=1}^{\infty}$ converges in Δ to Φ and that the support of all $\Phi_\upsilon(r)$ be contained in $0 \le r \le \infty$. Then, $\varphi_\upsilon(z)$ and φ are also in *Z*. Also,

$$\left|z^q \varphi_\upsilon(z)\right| = \left|\int_0^\infty \Phi_\upsilon^q(r)\chi^{(q)}(r)C_q\right|dr$$

$$\leq C_q \sup_{0\leq r\leq\infty} \left|\Phi_\upsilon^q(r)\chi^{(q)}(r)\right| \quad . \tag{39}$$

Since $\{\Phi_\upsilon(r)\}_{\upsilon=1}^{\infty}$ converges to $\Phi(r)$ in Δ, $\sup\left|\Phi_\upsilon^q(r)\chi^{(q)}(r)\right|$ is, indeed, uniformly bounded for all values to υ . Thus, φ_υ satisfy the inequality (38).Thereby the Condition (iii) for the convergence in Z is also justified. Indeed, according to (39)

$$\left|\varphi_\upsilon(z)-\varphi(z)\right| \leq C_q \sup_{0\leq r\leq\infty} \left|\Phi_\upsilon(r)-\Phi(r)\right|.$$

As $\upsilon\to\infty$, $\leq \sup\left|\Phi_\upsilon(r)-\Phi(r)\right| \to 0$, on each bounded domain of the z – plane. Thus, $\left|\varphi_\upsilon(z)-\varphi(z)\right| \to 0$ uniformly on each such domain.

Conversely, if $\{\varphi_\upsilon(z)\}_{\upsilon=1}^{\infty}$ converges in Z to φ, then by the Condition (ii) and Lemma 1, all $\Phi_\upsilon(r)$ and $\Phi(r)$ are in Δ and their supports are contained in $0\leq r\leq\infty$. Also, for each non-negative integer q,

$$\begin{aligned}
\left|\Phi_\upsilon^{(q)}(r)-\Phi^{(q)}(r)\right| &= \left|\int_0^\infty [\varphi_\upsilon(z)-\varphi(z)]P_{-1/2}^{m_r,0}(\cosh z)A_q\right| dz \quad , \quad \text{[from (36)]}\\
&\leq \left|\int_0^\infty z^q P_{-1/2}^{m_r,0}(\cosh z)A_q[\varphi_\upsilon(z)-\varphi(z)]\right| dz\\
&\leq N_q \sup_z [\varphi_\upsilon(z)-\varphi(z)]. \qquad \text{[from (37)]}
\end{aligned}$$

By Conditions (ii) and (iii), the above inequality converges to zero. Thus, $\{\Phi_\upsilon(r)\}_{\upsilon=1}^{\infty}$ converges in Δ to $\Phi(r)$. This completes the proof of the theorem.

Theorem 2:

Z is a proper subspace of *S*, where *S* is the testing function space of rapid descent.

Proof:

If φ is in *Z*, then indeed, it is an entire function and, thus, $\varphi(z)$ is infinitely smooth for all *z*. Also, its Mehler-Fock transformation $\Phi(r)$ is in Δ, which implies that $\Phi(r)$ is also in Δ. Therefore, by (38) $\Phi^{(q)}(r)$ is again in Z, so that for each pair of non-negative integer *q*,

$$\left|\Phi^{(q)}(r)\right| \leq N_q \quad , 0\leq r\leq\infty \quad .$$

Hence φ is in *S*. Finally, Δ is subset of *S* and since Δ does not intersect Z, except for the zero function. Therefore, *Z* is truly a proper subspace of *S*. The theorem is completely proved.

Theorem 3:

If the sequence $\{\varphi_\upsilon(z)\}_{\upsilon=1}^{\infty}$ converges in *Z* to φ, then it also converges in *S* to φ.

Proof:

By Theorem 1, it implies that $\{\Phi_\upsilon(r)\}_{\upsilon=1}^{\infty}$ converges in Δ to $\Phi(r)$. For $q = 0,1,2,\ldots,\{\Phi_\upsilon^{(q)}(r)\}_{\upsilon=1}^{\infty}$ converges in Δ to $\Phi(r)$. Hence $\{z^q \varphi_\upsilon(z)\}_{\upsilon=1}^{\infty}$ converges in *Z* to $z^k \varphi(z)$, which shows that $\Phi_\upsilon^{(q)}(r)$ converges in $\Phi^{(q)}(r)$ uniformly for $0 \le r \le \infty$.

Theorem 4:

For each $\varphi(z)$ in *S* there exists a sequence $\{\varphi_\upsilon(z)\}_{\upsilon=1}^{\infty}$, with the elements exclusively in *Z*, that converges in *S* to φ, that is, *Z* is dense in *S*.

Proof:

Since the Mehler-Fock transformation maps *S* onto itself, Φ is also in *S*. Moreover, Δ is dense in *S*. Therefore, choose a sequence $\{\Phi_\upsilon(r)\}_{\upsilon=1}^{\infty}$ in Δ which converges in *S* to Φ. By the continuity of the inverse Mehler-Fock transformation as a mapping of *S* onto itself, $\{\varphi_\upsilon(z)\}_{\upsilon=1}^{\infty}$, is the sequence we seek.

4. Ultradistribution Spaces Z' for Mehler-Fock Transformation

The ultradistribution for the Mehler-Fock transformation can be defined by the Paresval's relation, as

$$\langle F(r), \Phi(r) \rangle = \langle f(x), \varphi(x) \rangle \tag{40}$$

$$\langle F(r), \varphi(x) \rangle = \langle f(x), \Phi(r) \rangle, \text{ for all } \varphi \text{ in } \Delta, F(r) \in Z'. \tag{41}$$

Theorem 5:

Z' contains S', where S' is the dual of *S*.

Proof:

According to Theorem 2, since *Z* is the subspace of *S*, hence each distribution *f* of slow growth in linear on *Z*. Furthermore, since convergence in *Z* implies convergence in *S* [cf. Theorem 3], *f* is also continuous on *Z*. For regular distribution, the testing function spaces have their identification [Pathak (1997, p. 201)] as

$$Z \subset S \subset S' \subset Z' \quad . \tag{42}$$

From (41), evidently, the Mehler-Fock transformation $F(r)$ of any distribution f is in ☟ $'$. As Φ traverses ☟ , φ traverses Z, so that $F(r)$ is defined as functional, which assigns to each φ in Z the same number that $f(x)$ assigns to $\Phi(r)$. By using Theorem 1, for f in ☟ $'$, $F(r)$ is a continuous linear functional on Z and , thus, is an ultradistribution.

The inverse Mehler-Fock transformation is well served by relations (40) and (41). It follows that Mehler-Fock transformation is a mapping of ☟ $'$ onto Z' and its inverse is the mapping of Z' onto ☟ $'$. This shows the correspondence to be one-to-one. Similarly, through the identification of testing function space with distribution (42), Mehler-Fock transformation for slow growth have the mapping S' onto Z' and its inverse have the mapping Z' onto S', From Theorem 5, similar mapping can easily be proved for (40) and (41).

Commentary:

We state following properties, without detailed proof, of Mehler-Fock transformation for the ultradistribution owing to the basic cause that the linearity and continuity have been defined in the preceding pages.

(i) Addition : $\langle f(z)+g(z),\varphi(z)\rangle = \langle f(z),\varphi(z)\rangle + \langle g(z),\varphi(z)\rangle$

(ii) Multiplication by constant α : $\langle \alpha\ f(z),\varphi(z)\rangle = \langle f(z),\alpha\varphi(z)\rangle$

(iii) Differentiability (Eqn. (21)): $\langle f^{(1)}(z),\varphi(z)\rangle = \langle f(z),\varphi^{(1)}(z)\rangle$

(iv) Operator formula (Eqn. (24)) : $\langle (\nabla_x^*)^q f(x),\phi(x)\rangle = \langle f(x),\nabla_x^q\phi(x)\rangle$,

where $f(z), g(z) \in Z'$ and $\varphi(z) \in Z$.

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