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A Further Result on the Instability of Solutions to a Class of Non-Autonomous Ordinary Differential Equations of Sixth Order

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Abstract

The aim of the present paper is to establish a new result, which guarantees the instability of zero solution to a certain class of non-autonomous ordinary differential equations of sixth order. Our result includes and improves some well-known results in the literature.

Keywords: Non-autonomous differential equation, sixth order, instability.

AMS 2000 Subject Classification Numbers: 34D05, 34D20.

1. Introduction

Consider the non-autonomous vector differential equation of sixth order

$$\begin{aligned} X^{(6)} + AX^{(5)} + BX^{(4)} + C\dddot{X} + d(t)\Phi(X,\dot{X},\ddot{X},\dddot{X},X^{(4)},X^{(5)})\ddot{X} \\ + R(X)\dot{X} + f(t)H(X,\dot{X},\ddot{X},\dddot{X},X^{(4)},X^{(5)})X = 0, \end{aligned} \tag{1}$$

in which $t \in \Re^+$, $\Re^+ = [0,\infty)$ and $X \in \Re^n$; A, B and C are constant $n \times n$-real symmetric matrices; Φ, R and H are continuous $n \times n$-symmetric real matrix functions depending, in each case, on the arguments shown in (1) ; $d : \Re^+ \to \Re^+$, $f : \Re^+ \to \Re^+$ and the functions d and f are also continuous for arguments shown explicitly. Let $J(R(X)X|X)$ denote the linear operator from the matrix $R(X)$ to the matrix

$$J(R(X)X|X) = \left(\frac{\partial}{\partial x_j} \sum_{k=1}^{n} r_{ik} x_k \right) = R(X) + \left(\sum_{k=1}^{n} \frac{\partial r_{ik}}{\partial x_j} x_k \right), \quad (i, j = 1,2,\ldots,n),$$

where $(x_1, x_2, ..., x_n)$ and (r_{ik}) are components of X and R, respectively. It is assumed that the matrix $J(R(X)X|X)$ exists and is symmetric and continuous. From the literature, it can be seen that, so far, many problems about the instability of solutions of various scalar and vector linear and nonlinear differential equations of third-, fourth-, fifth-, sixth-, seventh and eighth order have been investigated by researchers. For some papers carried out on the subject, one can refer to the book of Reissig et al (1974) and the papers of Ezeilo (1978, 1979, 1982), Sadek (2003), Tejumola (2000), Tunç (2004, 2005, 2006, 2007, 2008), C.Tunç and E. Tunç (2005, 2006), and the references listed in these papers. The motivation for the present work has been inspired especially by the papers of Tejumola (2000), Tunç (2004, 2007) and the papers mentioned above. Namely, Tejumola (2000) established some sufficient conditions, which guarantee the instability of the trivial solution $x = 0$ of the following scalar nonlinear differential equation of sixth order:

$$x^{(6)} + a_1 x^{(5)} + a_2 x^{(4)} + a_3 \dddot{x} + \varphi_4(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}, x^{(5)})\ddot{x} \\ + \varphi_5(x)\dot{x} + \varphi_6(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}, x^{(5)}) = 0 .$$

Later, Tunç (2004, 2007, 2008) investigated the same subject for the sixth order nonlinear vector differential equations of the form:

$$X^{(6)} + AX^{(5)} + B(t)\Phi(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)}, X^{(5)})X^{(4)} + C(t)\Psi(\ddot{X})\dddot{X} \\ + D(t)\Omega(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)}, X^{(5)})\ddot{X} + E(t)G(\dot{X}) + H(X) = 0 ,$$

$$X^{(6)} + AX^{(5)} + BX^{(4)} + C\dddot{X} + \Phi(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)}, X^{(5)})\ddot{X} \\ + R(X)\dot{X} + H(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)}, X^{(5)})X = 0,$$

and

$$X^{(6)} + AX^{(5)} + BX^{(4)} + C\dddot{X} + \Phi(t, X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)}, X^{(5)})\ddot{X} \\ + \Psi(X)\dot{X} + H(t, X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)}, X^{(5)})X = 0,$$

respectively. It should be noted that throughout all the above mentioned papers on the subject, it was taken into consideration the Krasovskii's criterion, (see Krasovskii 1955), and the Lyapunov's (1966) second (or direct) method was used as a basic tool to prove the results established there. In this paper, distinctly, we take into consideration a result of Lakshmikantham et al. (1991, Theorem 1.1.9) and also use the Lyapunov's (1966) second (or direct) method as a basic tool to verify our result, which will be given hereafter. The equation considered, (1), and the assumptions will be established here are some different than that in literature (See Tejumola (2000), Tunç (2004, 2007) and the references listed in these papers).

Throughout this paper, the symbol $\langle X, Y \rangle$ is used to denote the usual scalar product in $\Re^n$ for given any X, Y in $\Re^n$, that is, $\langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i$, thus $\|X\|^2 = \langle X, X \rangle$. It is also well-known that a real symmetric matrix $A = (a_{ij}), (i, j = 1, 2, ..., n)$ is said to be positive definite if and only if the quadratic form $X^T AX$ is positive definite, where $X \in \Re^n$ and X^T denotes the transpose of X (See Bellman (1997)).

We take into consideration, in place of (1), the equivalent differential system

$$\dot{X} = Y\,,\ \dot{Y} = Z\,,\ \dot{Z} = S\,,\ \dot{S} = T\,,\ \dot{T} = U\,,$$
$$\dot{U} = -AU - BT - CS - d(t)\Phi(X,Y,Z,S,T,U)Z$$
$$- R(X)Y - f(t)H(X,Y,Z,S,T,U)X\,, \tag{2}$$

which was obtained as usual by setting $\dot{X} = Y$, $\ddot{X} = Z$, $\dddot{X} = S$, $X^{(4)} = T$, $X^{(5)} = U$ in (1).

We establish the following theorem:

Theorem 1: In addition to the basic assumptions imposed on A, B, C, d, Φ, R, f and H appearing in system (2), we assume that the following conditions are satisfied: There are constants a_0, a_1, b_0, b_1, c_1 and r_1 such that

A, B, C, Φ, R and H are symmetric such that $\dfrac{\partial r_{ij}}{\partial x_k} = \dfrac{\partial r_{ik}}{\partial r_j}$, $(i, j, k = 1, 2, ..., n)$;

$$0 < a_0 \le \lambda_i(A) \le a_1\,,\ b_0 \le \lambda_i(B) \le b_1 < 0\,,\ 0 \le \lambda_i(C) \le c_1\,,\ 0 \le \lambda_i(R) \le r_1$$

and

$$\lambda_i(f(t)H(X,Y,Z,S,T,U)) < \frac{1}{4b_1}\left[\lambda_i(d(t)\Phi(X,Y,Z,S,T,U))\right]^2,\ (i = 1,2,...,n)\,,$$

for all $t \in \Re^+$ and X, Y, Z, S, T, $U \in \Re^n$. Then the trivial solution of system (2) is unstable.

2. Preliminaries

In order to reach our main result, we will give a basic theorem for the general non-autonomous differential system and two well-known lemmas which play an essential role in the proof of our main result. Consider the differential system

$$\dot{x} = f(t,x)\,,\ x(t_0) = x_0\,,\ t \ge 0\,, \tag{3}$$

where $f \in C\left[R_+ \times S_{(\rho)},\ \Re^n\right]$ and $S_{(\rho)} = \left[x \in \Re^n : |x| < \rho\right]$. Assume, for convenience, that the solutions $x(t) = x(t, t_0, x_0)$ of (3) exist, and are unique for $t \ge t_0$ and $f(t,0) = 0$ so that we have trivial solution $x = 0$. Let us state that the following fundamental instability theorem.

Theorem 2: Assume that there exists a $t_0 \in \Re_+$ and an open set $U \subset S_{(\rho)}$ such that

$V \in C^1\left[[t_0,\infty) \times S_{(\rho)},\ \Re_+\right]$ and for $[t_0,\infty) \times U$,

(i) $0 < V(t,x) \le a(|x|)$, $a \in \kappa$;

(ii) either $V'(t,x) \geq b(|x|)$, $b \in \kappa$, $\kappa = [\sigma \in C[[t_0, \rho), \Re_+]]$ such that $\sigma(t)$ is strictly increasing and $\sigma(0) = 0$ or $V'(t,x) = CV(t,x) + \omega(t,x)$, where $C > 0$ and $\omega \in C[[t_0, \infty) \times U, \Re_+]$;

(iii) $V(t,x) = 0$ on $[t_0, \infty) \times (\partial U \cap S_{(\rho)})$, ∂U denotes boundary of U and $0 \in \partial U$.

Then the trivial solution $x = 0$ of system (3) is unstable.

Proof: See Lakshmikantham et al. (1991, Theorem 1.1.9).

Lemma 1: Let A be a real symmetric $n \times n$ matrix and

$a' \geq \lambda_i(A) \geq a > 0$ $(i = 1,2,...,n)$, where a', a are constants.

Then

$$a'\langle X, X\rangle \geq \langle AX, X\rangle \geq a\langle X, X\rangle$$

and

$$a'^2\langle X, X\rangle \geq \langle AX, AX\rangle \geq a^2\langle X, X\rangle .$$

Proof: See Bellman (1997).

Lemma 2: Let Q, D be any two real $n \times n$ commuting symmetric matrices. Then,

(i) The eigenvalues $\lambda_i(QD)$,$(i = 1,2,...,n)$, of the product matrix QD are real and satisfy

$$\max_{1\leq j,k\leq n} \lambda_j(Q)\lambda_k(D) \geq \lambda_i(QD) \geq \min_{1\leq j,k\leq n} \lambda_j(Q)\lambda_k(D) .$$

(ii) The eigenvalues $\lambda_i(Q + D)$, $(i = 1,2,...,n)$, of the sum of matrices Q and D are real and satisfy

$$\left\{\max_{1\leq j\leq n} \lambda_j(Q) + \max_{1\leq k\leq n} \lambda_k(D)\right\} \geq \lambda_i(Q + D) \geq \left\{\min_{1\leq j\leq n} \lambda_j(Q) + \min_{1\leq k\leq n} \lambda_k(D)\right\},$$

where $\lambda_j(Q)$ and $\lambda_k(D)$ are, respectively, the eigenvalues of matrices Q and D.

Proof: See Bellman (1997).

3. Proof of Theorem 1

To prove Theorem 1, we construct a scalar differentiable Lyapunov function $V_0 = V_0(t, X, Y, Z, S, T, U)$. This function, V_0, is defined as follows:

$$V_0 = \langle X, U\rangle + \langle X, AT\rangle + \langle X, BS\rangle + \langle X, CZ\rangle$$

$$-\langle Y,T\rangle - \langle Y,AS\rangle - \langle Y,BZ\rangle + \langle Z,S\rangle - \frac{1}{2}\langle Y,CY\rangle$$

$$+\frac{1}{2}\langle Z,AZ\rangle + \int_0^1 \langle \sigma R(\sigma X)X, X\rangle d\sigma .$$

Clearly, $V_0(t,0,0,0,0,0,0) = 0$. Now, subject to the assumptions of Theorem 1, it is a straightforward calculation to see that

$$V_0(0,0,0,\varepsilon,\varepsilon,0,0) = \frac{1}{2}\langle \varepsilon, A\varepsilon\rangle + \langle \varepsilon,\varepsilon\rangle \geq \frac{1}{2}\langle \varepsilon, a_0\varepsilon\rangle + \langle \varepsilon,\varepsilon\rangle$$

$$= (\frac{1}{2}a_0 + 1)\|\varepsilon\|^2 > 0 ,$$

for all arbitrary, $\varepsilon \neq 0$, $\varepsilon \in \Re^n$. In view of the function $V_0 = V_0(t,X,Y,Z,S,T,U)$, the assumptions of Theorem 1, Lemma 1 and Cauchy-Schwarz inequality $|\langle X,Y\rangle| \leq \|X\|\,\|Y\|$, one can easily conclude that there is a positive constant K_1 such that

$$V_0(t,X,Y,Z,S,T,U) \leq K_1\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|S\|^2 + \|T\|^2 + \|U\|^2\right).$$

Now, let $(X,Y,Z,S,T,U) = (X(t),Y(t),Z(t),S(t),T(t),U(t))$ be an arbitrary solution of system (2). By an elementary differentiation along the solution paths of system (2), it can be verified that

$$\dot{V}_0 = \frac{d}{dt}V_0(t,X,Y,Z,S,T,U) = -\langle d(t)\Phi(X,Y,Z,S,T,U)Z, X\rangle$$

$$-\langle f(t)H(X,Y,Z,S,T,U)X, X\rangle$$

$$-\langle BZ,Z\rangle + \langle S,S\rangle - \langle R(X)X,Y\rangle$$

$$+\frac{d}{dt}\int_0^1 \langle \sigma R(\sigma X)X, X\rangle d\sigma . \tag{4}$$

Now, recall that

$$\frac{d}{dt}\int_0^1 \sigma\langle R(\sigma X)X, X\rangle d\sigma = \int_0^1 \langle \sigma R(\sigma X)X,Y\rangle d\sigma + \int_0^1 \langle \sigma J\left(R(\sigma X)X|\sigma X\right)Y, X\rangle d\sigma$$

$$= \int_0^1 \langle \sigma R(\sigma X)X,Y\rangle d\sigma + \int_0^1 \sigma \frac{\partial}{\partial\sigma}\langle \sigma R(\sigma X)Y, X\rangle d\sigma$$

$$= \sigma^2\langle R(\sigma X)X,Y\rangle\Big|_0^1$$

$$= \langle R(X)X,Y\rangle . \tag{5}$$

Combining the estimate (5) with (4), we arrive that

$$\dot{V}_0 = -\langle d(t)\Phi(X,Y,Z,S,T,U)Z,X\rangle - \langle f(t)H(X,Y,Z,S,T,U)X,X\rangle$$

$$-\langle BZ,Z\rangle + \langle S,S\rangle \ .$$

Hence, the assumptions of Theorem 1 and the fact $\langle S,S\rangle = \|S\|^2$ imply that

$$\dot{V}_0 \geq -\langle d(t)\Phi(X,Y,Z,S,T,U)Z,X\rangle - \langle f(t)H(X,Y,Z,S,T,U)X,X\rangle - b_1\langle Z,Z\rangle$$

$$= -b_1\left\| Z + \frac{d(t)}{2b_1}\Phi(X,Y,Z,S,T,U)X\right\|^2 - \langle f(t)H(X,Y,Z,S,T,U)X,X\rangle$$

$$+\frac{1}{4b_1}\langle d(t)\Phi(X,Y,Z,S,T,U)X, d(t)\Phi(X,Y,Z,S,T,U)X\rangle$$

$$\geq -\langle f(t)H(X,Y,Z,S,T,U)X,X\rangle$$

$$+\frac{1}{4b_1}\langle d(t)\Phi(X,Y,Z,S,T,U)X, d(t)\Phi(X,Y,Z,S,T,U)X\rangle > 0 .$$

Thus, the assumptions of Theorem 1 imply that $\dot{V}_0(t) \geq K_2\|X\|^2$ for all $t \geq 0$, where K_2 is a positive constant, say infinite inferior limit of the function $\dot{V}_0$. Besides, $\dot{V}_0 = 0$ $(t \geq 0)$ necessarily implies that $X = 0$ for all $t \geq 0$, and therefore also that $Y = \dot{X}$, $Z = \dot{Y} = 0$, $S = \ddot{Y} = 0$, $T = \dddot{Y} = 0$, $U = Y^{(4)} = 0$ for all $t \geq 0$. Hence,

$$X = Y = Z = S = T = U = 0 \text{ for all } t \geq 0 .$$

Therefore, subject to the assumptions of the theorem the function V_0 has the entire the criteria of Theorem 2, Lakshmikantham et al. (1991, Theorem 1.1.9). Thus, the basic properties of the function $V_0(t,X,Y,Z,S,T,U)$, which are proved just above verify that the zero solution of system (2) is unstable. The system of equations (2) is equivalent to differential equation (1) and the proof of the theorem is now complete.

Remark:

In order to prove the main result in Tejumola (2000) on the instability of the trivial solution $x = 0$ of scalar autonomous differential equation

$$x^{(6)} + a_1x^{(5)} + a_2x^{(4)} + a_3\dddot{x} + \varphi_4(x,\dot{x},\ddot{x},\dddot{x},x^{(4)},x^{(5)})\ddot{x}$$

$$+\varphi_5(x)\dot{x} + \varphi_6(x,\dot{x},\ddot{x},\dddot{x},x^{(4)},x^{(5)}) = 0 ,$$

the author showed that there exists a continuous Lyapunov function $V = V(x, y, z, w, s, u)$ which has the following properties of Krasovskii's (1955):

(K_1) in every neighborhood of $(0,0,0,0,0,0)$ there exists a point $(\xi, \eta, \zeta, \mu, \gamma, \lambda)$ such that $V(\xi, \eta, \zeta, \mu, \gamma, \lambda) > 0$;

(K_2) the time derivative $\dot{V} = \frac{d}{dt} V(x, y, z, w, s, u)$ along solution paths of the system,

$$x' = y \,,\ y' = z \,,\ z' = w \,,\ w' = s \,,\ s' = u \,,$$

$$u' = - a_1 u - a_2 s - a_3 w - \varphi_4(x, y, z, w, s, u) z - \varphi_5(x) y - \varphi_6(x, y, z, w, s, u),$$

which is equivalent the above equation, is positive semi-definite; and

(K_3) the only solution $(x, y, z, w, s, u) = (x(t), y(t), z(t), w(t), s(t), u(t))$ of the above system, which satisfies $\dot{V} = 0\ (t \geq 0)$ is the trivial solution $(0,0,0,0,0,0)$. It should be noted that we established a Lyapunov function for the non-autonomous vector differential equation (1) and proved the main result, Theorem 1, based on a result of Lakshmikantham et al. (1991, Theorem 1.1.9). For the special case $n = 1$, our assumptions reduce to that of Tejumola (2000) for the case $\varphi_6(x, y, z, w, s, u) = \varphi_6(x, y, z, w, s, u) x$ and Tunç (2007), except some minor differences. These differences are raised because of the non-autonomous case.

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