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Peristaltic Pumping of a Non-Newtonian Fluid

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Abstract

The flow induced by sinusoidal peristaltic motion of the tube wall of a non-Newtonian fluid obeying Herschel-Bulkley equation (a general rheological equation that represents a power-law, Bingham and Newtonian fluid for particular choice of parameters) under long wavelength and low Reynolds number approximation is investigated. The results obtained for flow rate, pressure drop and friction force are discussed both qualitatively and quantitatively and compared with other related studies. It is found that the pressure drop increases with the flow rate and yield stress but decreases with the increasing amplitude ratio. The flow behaviour index shows significant impact on the magnitude of the pressure drop. The pressure-flow rate relationships in Bingham and Newtonian fluid models are found to be linear whereas the same are non-linear in power- law and Herschel-Bulkley models. The friction force possesses the character similar to the pressure drop (an opposite character to the pressure rise) with respect to any parameter.

Keyword: Herschel-Bulkley, Peristaltic Wave, Pressure Drop, Flow Rate, Friction Force, Reynolds Number

AMS 2000 Subject Classification Numbers: 76Z05

1. Introduction

Pumping of fluids through flexible tubes by means of the peristaltic wave motion of the tube wall has been the subject of engineering and scientific research for over four decades. Engineers and physiologists term the phenomenon of such flow as peristalsis. It is a form of fluid transport induced by a progressive wave of area contraction or expansion along the walls of a distensible duct containing a liquid or mixture. Besides its various engineering applications (e.g., heart-lung machines, finger and roller pumps, etc.), it is known to be significant mechanism responsible for fluid transport in many biological organs including in swallowing food through esophagus, urine transport from kidney to bladder through the ureter, movement of chyme in gastrointestinal tract, transport of spermatozoa in the ductus efferentes of the male reproductive tracts, and, in cervical canal, in movement of ovum in the female fallopian tubes, transport of lymph in lymphatic vessels, and in the vasomotion of small blood vessels such as arterioles, venules and capillaries.

Latham (1966) was probably the first to study the mechanism of peristaltic pumping in his M.S. Thesis. Shapiro et al. (1969) and Jaffrin and Shapiro (1971) explained the basic principles and brought out clearly the significance of the various parameters governing the flow. The literature on the topic is quite extensive by now and a review of much of the literature up to the year 1983, arranged according to the geometry, the fluid, the Reynolds number, the wave number, the amplitude ratio and the wave shape was presented in an excellent article by Srivastava and Srivastava (1984).

The significant contributions to the subject between the years 1984 and 1994 were well referenced by Srivastava and Saxena (1995). The important studies of the recent years include the investigations of Srivastava and Srivastava (1997), Mekheimer et al. (1998), El-Shehawey and Husseny (2000), El-Hakeem and El-Misery (2001), Muthu et al. (2001), Srivastava (2002), Misra and Pandey (2002), Hayat et al. (2002,2003), Mekheimer (2003), Hayat et al. (2004), Siddiqui et al. (2004), Mishra and Rao (2004), Rao and Mishra (2004), Hayat et al. (2005), Hayat and Ali (2006a,b), Srivastava (2007), Medhavi (2008), Medhavi and Singh (2008) and a few others.

In both the mechanical and physiological situations, most of the studies conducted in the literature considered the transported fluid to be a Newtonian fluid. It is well accepted that a large number of fluids of practical importance behave like a non-Newtonian fluid. A survey of the topic indicates that a few studies (Raju and Devenathan, 1972, 1974; Becker, 1980; Shukla and Gupta, 1982; Bohme and Friedrich, 1983; Srivastava and coworkers, 1984, 1985, 1995; Misra and Pandey, 2002, Vijravelua et al., 2005, etc.) have used non-Newtonian fluids in their works. It may be mentioned that Srivastava and coworkers (1984, 1995) and Misra and Pandey (2002) made their studies a particular reference to peristaltic pumping of blood in small vessels.

The present article deals with the flow of a special type of a non-Newtonian fluid obeying the Herschel-Bulkley equation. It is to note that Herschel-Bulkley equation can be reduced to mathematical models which describe the behavior of Bingham, power-law and Newtonian fluids for particular choice of parameters involved. Since the Herschel-Bulkley fluid is a general fluid representing power-law, Bingham and Newtonian fluids, it is strongly believed that the study covers up a wide range of applications in engineering as well as in physiology.

2. Formulation of the Problem and Analysis

Consider the asymmetric flow of a non-Newtonian (Herschel-Bulkley) fluid in a uniform circular tube with a sinusoidal wave traveling down its wall. The geometry of the wall surface is described (Fig. 1) as

$$H\ (x,\ t) = a + b\ Sin\ \frac{2\pi}{\lambda}\ (x - ct), \tag{1}$$

where a is the mean radius of the tube, b is the amplitude of the wave, λ is the wavelength, c is the wave propagation speed and t is the time.

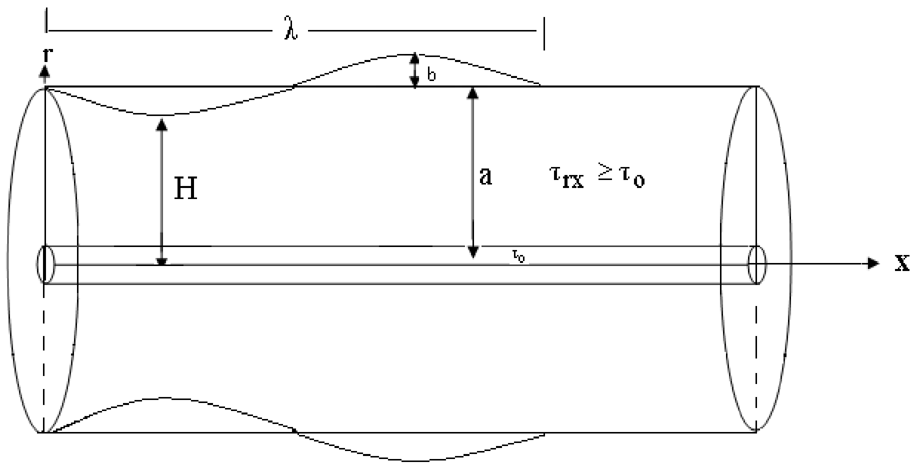

Fig.1 Geometrical representation of peristaltic waves in a circular tube.

The appropriate equation in the wave frame of reference (moving with speed c) under long wavelength approximation and neglecting the inertia terms (Shapiro et al., 1969), is written as

$$\frac{1}{r}\ \frac{\partial}{\partial r}\ (r\,\tau_{rx}) \quad = -\frac{dp}{dx}\ , \tag{2}$$

where τ_{rx} for Herschel-Bulkley fluid is given (Chaturani and Narasimman, 1988)

$$-\frac{\partial u}{\partial r} = f(\tau) = \frac{1}{k}(\tau_{rx} - \tau_0)^n,\ \ \tau_{rx} \geq \tau_0\ , \tag{3}$$

$$\frac{\partial u}{\partial r} = f(\tau) = 0, \quad \tau_{rx} \leq \tau_0\ , \tag{4}$$

where p (dyne cm^{-2}) is the pressure, k $[(cp)^n s^{1-n}]$ and n ($\geq$1) are consistency and flow behavior indexes, respectively and representing the non-Newtonian effects, τ_0 (dyne cm^{-2}), is the yield stress, (x, r) are (axial, radial) coordinates, and u ($cm\ s^{-1}$) is the axial velocity of the fluid. Relation (4) corresponds to the vanishing of the velocity gradient in the region in which $\tau_{rx} \leq \tau_0$ and implies a plug flow. When shear stress in the fluid is very high (i.e., $\tau_{rx} \geq \tau_0$), the power-law behavior is indicated. It is worth mentioning that above Herschel-Bulkley fluid model reduces to Bingham fluid when n =1, k= μ (Newtonian viscosity); to power-law fluid when τ_0=0 and to Newtonian fluid when n =1, $k = \mu$ and τ_0= 0. It is important to note that the plug core radius increases with yield stress, τ_0 and also with the flow behavior index, n (Chaturani and Narasimman, 1988).

An introduction of the following non-dimensional variables

$$r' = \frac{r}{a} \quad , x' = \frac{x}{\lambda} \quad , t' = \frac{ct}{\lambda} \,, u' = \frac{u}{c} \,,$$
$$\tau_0' = (a/ck)^{1/n}\, \tau_0\,, \quad p' = (a^{n+1}/ck\lambda^n)^{1/n} p, \tag{5}$$

into equations (2) – (4), after dropping primes, yields

$$\frac{1}{r}\frac{\partial}{\partial r} r\left[\left(-\frac{\partial u}{\partial r}\right)^{1/n} + \tau_0\right] = -\frac{dp}{dx}, \quad \tau_{rx} \geq \tau_0, \tag{6}$$

$$\frac{\partial u}{\partial r} = 0, \quad \tau_{rx} \leq \tau_0\,, \tag{7}$$

where τ_{rx} and τ_0 are dimensionless shearing and yield stresses, respectively.

The non-dimensional boundary conditions are

$$u = -1 \text{ at } \quad r = H/a = h = 1 + \phi \sin 2\pi x,$$
$$\tau_{rx} \text{ is finite at } r = 0, \tag{8}$$

where

$$\phi = b/a.$$

The expression for velocity, u obtained as the solution of equation (6) under the boundary conditions (8), is given as

$$u = -1 + \frac{(-1)^{n+1}}{n+1}\left(\frac{2}{dp/dx}\right)\left(\frac{r}{2}\frac{dp}{dx} + \tau_0\right)^{n+1} - \left(\frac{h}{2}\frac{dp}{dx} + \tau_0\right)^{n+1}. \tag{9}$$

One derives the expression for the fluid velocity in the plug flow region by substituting the radius of the plug flow region for the radial variable r in equation (9) which obviously satisfies equation (7) in the plug flow region. Now considering that the radius of the plug flow area to be small as compared to the non-plug flow area (Srivastava and Saxena, 1995), the instantaneous volumetric flow rate, q $(= q'/\pi a^2 c, q'$ being the flux in moving system which is same as in stationary system) is calculated as

$$q = 2\int_0^h rudr$$
$$= -h^2 + \frac{h^3\tau_h^n}{(n+3)}\left\{\left(1-\frac{\tau_0}{\tau_h}\right)^{n+1}\left[1+\frac{2}{(n+2)}\frac{\tau_0}{\tau_h}+\frac{2}{(n+1)(n+2)}\left(\frac{\tau_0}{\tau_h}\right)^2\right]\right.$$
$$\left.-\frac{2}{(n+1)(n+2)}\left(\frac{\tau_0}{\tau_h}\right)^{n+3}\right\}, \tag{10}$$

where $\tau_h = (-h/2)\, dp/dx$ is the shear stress at the wall. Under the condition, $\tau_0/\tau_h \ll 1$, equation (10) yields

$$-\frac{dp}{dx} = \frac{2}{h^{3/n+1}}[(q+h^2)(n+3)]^{1/n} + \frac{2(n+3)}{(n+2)}\frac{\tau_0}{\tau_h}. \tag{11}$$

Following the analysis of Shapiro et al. (1969), one obtains the mean volume flow rate, Q over a period as

$$Q = q + 1 + \frac{\phi^2}{2}. \tag{12}$$

Since the pressure drop, $\Delta p = p(0) - p(1)$, across one wavelength is same whether measured in moving or stationary coordinate system, it is therefore calculated using equation (12) into (11) as

$$\Delta p = -\int_0^1 \frac{dp}{dx}dx$$
$$= \frac{2(n+3)}{(n+2)}\frac{\tau_0}{\sqrt{1-\phi^2}} + 2(n+3)^{1/n}\int_0^1 \frac{(Q-1-\phi^2/2+h^2)^{1/n}}{h^{3/n+1}}dx. \tag{13}$$

The non-dimensional friction force, F (= $(a^{-n+1}/ck\pi^n\lambda^n)^{1/n}$ F', F' is the friction force at the wall in the stationary coordinate system which is same as in moving system) is thus obtained as

$$F = \int_0^1 h^2\left(-\frac{dp}{dx}\right)dx$$
$$= \frac{2(n+3)}{(n+2)}\tau_0 + 2(n+3)^{1/n}\int_0^1 \frac{(Q-1-\phi^2/2+h^2)^{1/n}}{h^{3/n-1}}dx. \tag{14}$$

The closed from analytical evaluation of the complicated integrals involved in equations (13) and (14) seems to be a formidable task and therefore they will be evaluated numerically. Further, to compare the results of the analysis with other theoretical studies analytically and to observe the qualitative effects of non-Newtonian behavior, we return to equations (13) and

(14). When n =1 the results obtained in equations (13) and (14) reduce to the results for Bingham fluid analysis as

$$\Delta p = \frac{8}{(1-\phi^2)^{7/2}}\left\{\frac{1}{3}\,(1-\phi^2)^3\,\tau_0 + Q(1+\frac{3}{2}\,\phi^2) + \frac{\phi^2}{4}(\phi^2-16)\right\}, \tag{15}$$

$$F = \frac{8}{(1-\phi^2)^{3/2}}\quad\left\{Q-1-\frac{\phi^2}{2} + (1+\frac{\tau_0}{3})\;(1-\phi^2)^{3/2}\right\} \tag{16}$$

When τ_0 = 0, equations (13) and (14) yield the results for a power-law fluid analysis as

$$\Delta p = 2\,(n+3)^{1/n}\int_0^1 \frac{(Q-1-\phi^2/2+h^2)^{1/n}}{h^{3/n+1}}\,dx, \tag{17}$$

$$F = 2\,(n+3)^{1/n}\int_0^1 \frac{(Q-1-\phi^2/2+h^2)^{1/n}}{h^{3/n-1}}\,dx, \tag{18}$$

which correspond to the results obtained in Srivastava and Srivastava (1985) for uniform diameter tube. Also, when n =1 and τ_0 = 0, equations (13) and (14) yield the results for a Newtonian fluid model as

$$\Delta p = \frac{8}{(1-\varphi^2)^{7/2}}Q(1+\frac{3}{2}\;\;\varphi^2)+\frac{1}{4}\varphi^2(\varphi^2-16), \tag{19}$$

$$F = \frac{8}{(1-\varphi^2)^{3/2}}Q-1-\frac{\varphi^2}{2}+(1-\varphi^2)^{3/2}, \tag{20}$$

which are the same results as obtained in Shapiro et al. (1969). The results of Shapiro et al. (1969) are also derived either by setting τ_0 = 0 in equations (15) and (16) or n =1 in equations (17) and (18).

Finally, in view of the complicated form of the expression for Δp given in equation (13), it is difficult to obtain the analytical expression for the flow rate, Q for zero pressure drop. However, the pressure drop for zero flow rate, $(\Delta p)_{Q=0}$, which is of particular interest is derived as

$$(\Delta p)_{Q=0} = \frac{2\,(n+3)}{(n+2)}\;\frac{\tau_0}{\sqrt{1-\phi^2}} + 2\,(n+3)^{1/n}\int_0^1 \frac{(h^2-1-\phi^2/2)^{1/n}}{h^{3/n+1}}\,dx. \tag{21}$$

With n =1 and τ_0 = 0, equation (21) reduces to

$$\left(\Delta p\right)_{Q=0} = \frac{2\,\phi^2\,(\phi^2 - 16)}{(1-\phi^2)^{7/2}}, \tag{22}$$

which is the same expression as obtained in Shapiro et al. (1969) for pressure drop at zero flow rate.

3. Numerical Results and Discussion

To observe the quantitative effects of various parameters involved in the analysis, computer codes are developed to evaluate the expressions for dimensionless pressure drop, Δp, and friction force, F (equations (13) and (14)) for different values of the parameters (Shapiro et al., 1969; Chaturani and Narasimman, 1988; Srivastava and Saxena, 1995), τ_0, n, ϕ and Q and some critical results are displaced graphically in Figs. 2-7. The results for the parameter values; n =1 and $\tau_0 \neq 0$, $n \neq 1$ and $\tau_0 = 0$, and $n = 1$ and $\tau_0 = 0$ correspond to the results for Bingham, power-law, and Newtonian fluid models, respectively. For ϕ=0 the results of the study reduce to those obtained in no peristalsis case.

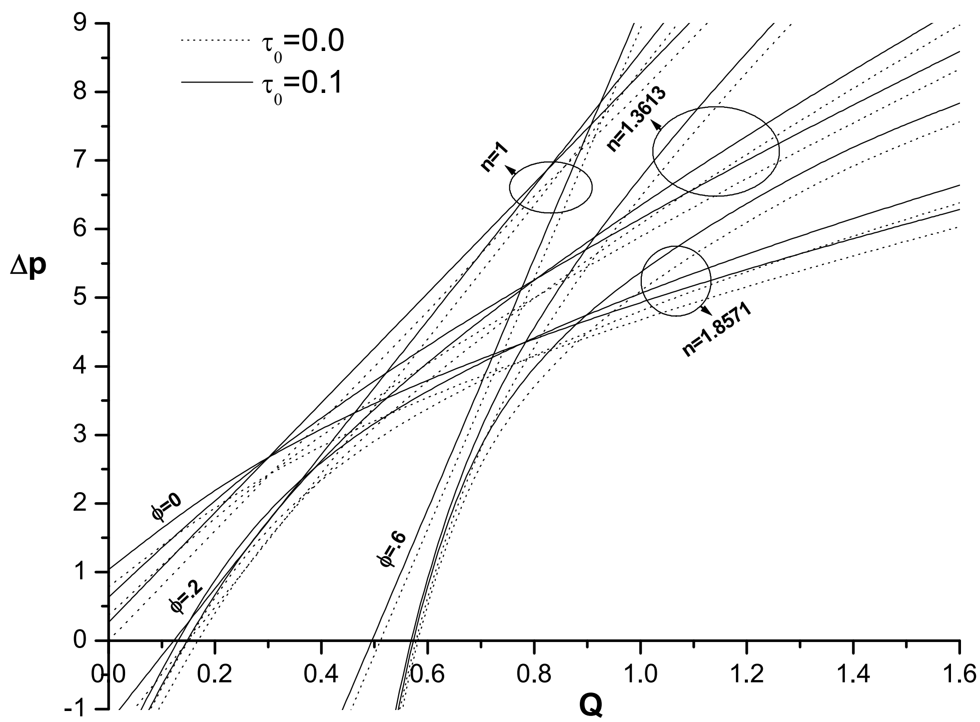

Fig.2 Pressure -flow rate relationship for different ϕ, τ_0 and n.

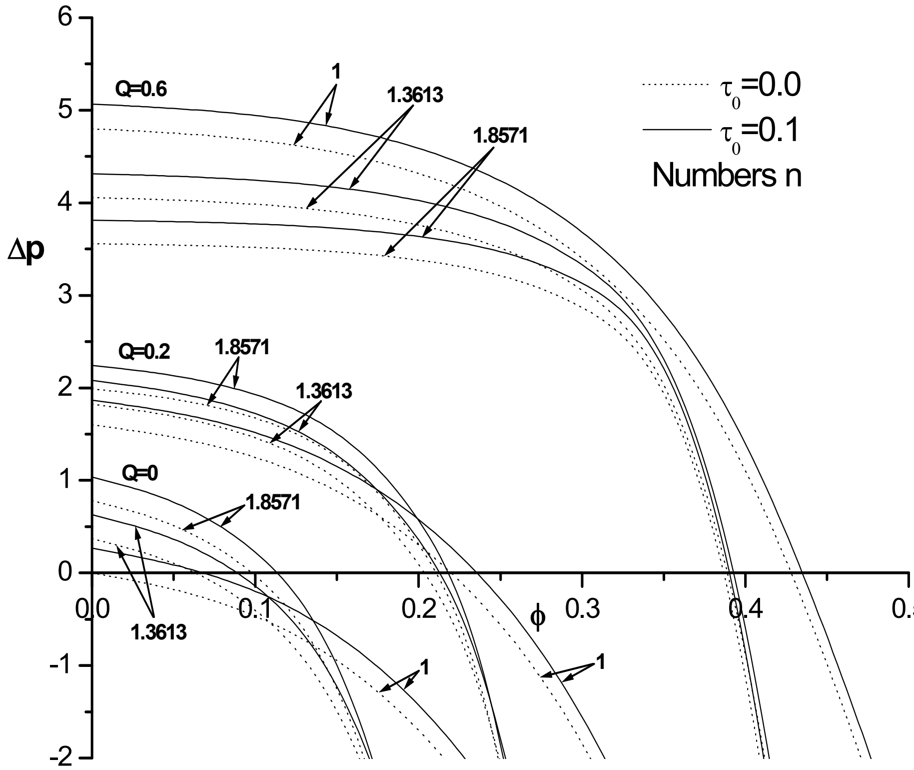

Fig.3 Variation of pressure drop, Δp with ϕ for different τ_0, n and Q.

The pressure drop, Δp increases with the flow rate, Q and its minimum magnitude is achieved at zero flow rate (Fig. 2). A linear relationship exists between the pressure drop and flow rate in Bingham and Newtonian fluids analyses whereas the relationship between these (Δp and Q) are seen non-linear in power-law and Herschel-Bulkey fluid models (Fig. 2).

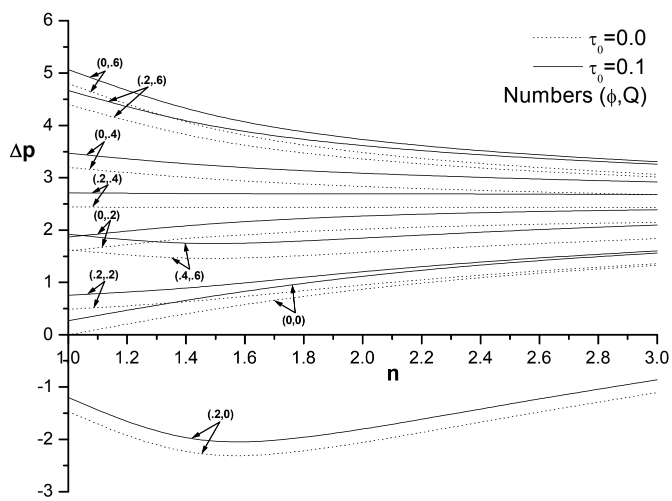

Fig.4 Variation of pressure drop, Δp with n for different τ_0, ϕ and Q.

Pressure drop, Δp is found to be decreasing indefinitely with increasing amplitude ratio, ϕ (Fig. 3). Increasing flow rate significantly affect the magnitudes of pressure drop with the parameter n (Fig. 4) and for any given set of other parameters, pressure drop increases with yield stress (Fig. 5). Although, the radius of the plug flow region is small as compared to the non- plug flow region, the numerical results reveal that even relatively small values of yield stress, τ_0 significantly influence the magnitudes of the flow characteristics.

Friction force, F possesses a character similar to the pressure drop, Δp with respect to any parameter (Fig. 6 and Fig. 7). Fig. 2 - Fig. 7 clearly present a good comparison of the results among different fluid models (Herschel-Bulkley, power-law, Bingham and Newtonian).

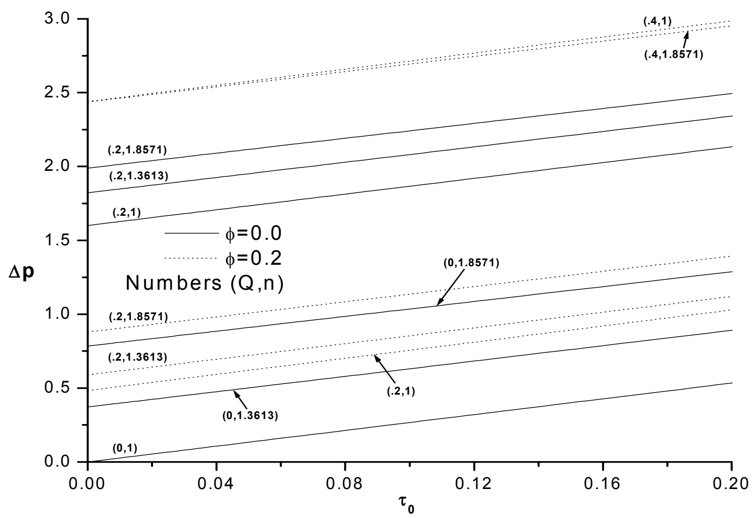

Fig.5 Variation of pressure drop, Δp with τ_0 for different ϕ, Q and n.

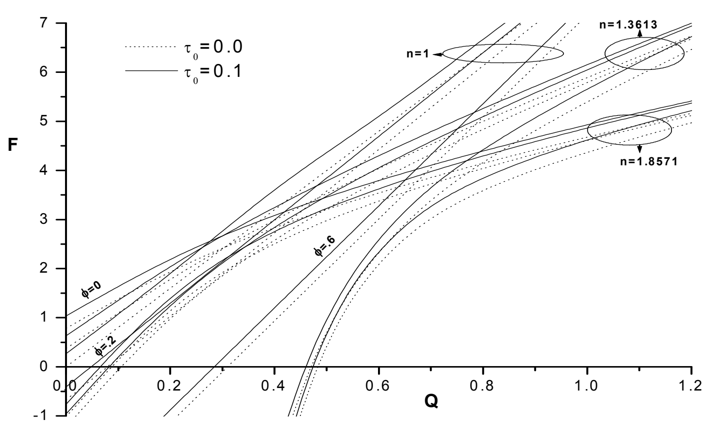

Fig.6 Friction force - flow rate relationship for different ϕ, τ_0 and n.

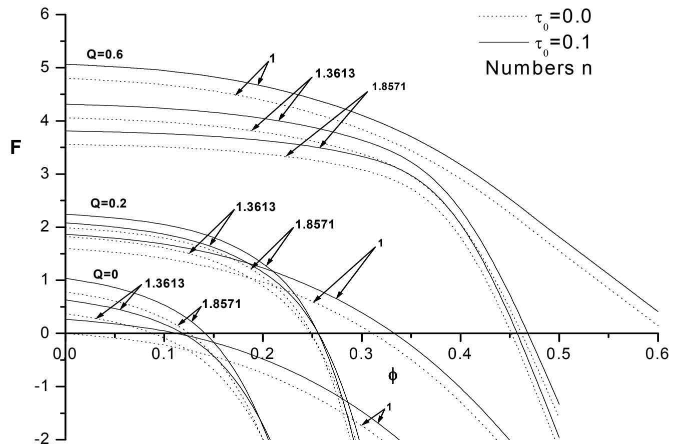

Fig.7 Variation of friction force, F with ϕ for different τ_0, n and Q.

4. Conclusions

A non-Newtonian fluid model obeying Herschel-Bulkley equation has been applied to study the flow induced by means of sinusoidal peristaltic waves using long wavelength approximation. Particular cases (Bingham, power-law and Newtonian fluids), derived from the present non-Newtonian fluid model, have been discussed and compared both qualitatively and quantitatively. Pressure-flow rate relationships in Bingham and Newtonian fluids have been found linear whereas the same are non-linear in power-law and Herschel-Bulkley models. Pressure drop increases with the flow rate and yield stress but decreases with amplitude ratio. Friction force possesses an opposite character to pressure rise (negative of pressure drop). It is strongly believed that the results of the analysis may be applied to discuss the peristaltic induced flow of blood and other physiological fluids (Scott Blair and Spanner, 1974, Whitmore, 1968).

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