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Modified Variational Iteration Method for Second Order Initial Value Problems

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Abstract

In this paper, we introduce a modified variational iteration method for second order initial value problems by transforming the integral of iteration process. The main advantages of this modification are that it can overcome the restriction of the form of nonlinearity term in differential equations and improve the iterative speed of conventional variational iteration method. The method is applied to some nonlinear second order initial value problems and the numerical results reveal that the modified method is accurate and efficient for second order initial value problems.

Keywords: Variational iteration method; Initial value problems; Reproducing kernel method

MSC (2000) No.: 65L05; 65D15

1. Introduction

In this paper, we consider the following second order initial value problem:

$$\begin{cases} u''(t) + f(t,u,u') = 0, \quad 0 \le t \le 1, \\ u(0) = \alpha, \ u'(0) = \beta, \end{cases} \tag{1}$$

where f is continuous.

Ordinary differential equations are important tools in solving real-world problems. A wide variety of natural phenomena are modeled by second order ordinary differential equations. They have been applied to many problems, in physics, engineering, biology and so on. Many nonlinear problems are difficult to be solved analytically.

The VIM was proposed originally by He (1999, 2000, 2006, 2007, 2008). This method is based on the use of Lagrange multipliers for identification of optimal values of parameters in a functional. This method gives rapidly convergent successive approximations of the exact solution if such a solution exists. Furthermore, VIM does not require discretization of the problem. Thus the variational iteration method is suitable for finding the approximation of the solution without discretization of the problem. It was successfully applied to various linear and nonlinear problems Mohyud-Din (2009), Noor (2007, 2008).

In this paper, we modify VIM for second order initial value problems by transforming the integral of iteration process. The main advantage of this method is that the method does not share the drawbacks of the conventional VIM, namely, the restriction of the form of the nonlinearity term. Furthermore, the modified VIM can improve the iteration speed of conventional VIM.

The rest of the paper is organized as follows. In the next Section, the VIM is introduced. The modified VIM is introduced in Section 3. The numerical examples are presented in Section 4. Section 5 ends this paper with a brief conclusion.

2. Analysis of He's variational iteration method

Consider the differential equation

$$Lu + Nu = g(x), \tag{2}$$

where L and N are linear and nonlinear operators, respectively, and $g(x)$ is the source inhomogeneous term. The VIM was introduced by He where a correct functional for (2) can be written as

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda[Lu_n(t) + N\tilde{u}_n(t) - g(t)]dt, \tag{3}$$

where λ is a general Lagrangian multiplier He (1999, 2000), which can be identified optimally via variational theory, and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$. By this method, it is firstly required to determine the Lagrangian multiplier λ that will be identified optimally. The successive approximations $u_{n+1}(x)$, $n \geq 0$, of the solution $u(x)$ will be readily obtained upon using the determined Lagrangian multiplier and any selective function $u_0(x)$. Consequently, the solution is given by

$$u(x) = \lim_{n\to\infty} u_n(x).$$

For variational iteration method, the key is the identification of the Lagrangian multiplier. For linear problems, their exact solutions can be obtained by only one iteration step due to the fact that the Lagrangian multiplier can be identified exactly. For nonlinear problems, the Lagrangian multiplier is difficult to be identified exactly. To overcome the difficulty, we apply restricted variations to nonlinear terms. Due to the approximate identification of the Lagrangian multiplier, the approximate solutions converge to their exact solutions relatively slowly. It should be specially pointed out that the more accurate the identification of the multiplier, the faster the approximations converge to their exact solutions.

3. Modified VIM for (1)

For (1), according to VIM, He (2007), we have the following iteration formula:

$$u_{n+1}(t) = u_0(t) + \int_0^t (s-t)\, f(s, u(s), u'(s))ds. \tag{4}$$

However, it may be difficult to perform the integral that appear in (4) analytically, except for simple $f(s, u(s), u'(s))$. Therefore, it is difficult to perform iteration many times and obtain higher-order approximation. To overcome the difficulty, we shall modify iteration formula (4) by performing integral in (4) using reproducing kernel method. Put

$$w_n(t) = \int_0^t (s-t)\, f(s, u(s), u'(s))ds. \tag{5}$$

Differentiation of both sides of (5) yields

$$w_n'(t) = -\int_0^t f(s, u(s), u'(s))ds. \tag{6}$$

And it is easy to see that $w_n(0) = 0$.

Differentiation of both sides of (6) gives

$$w_n''(t) = -f(s, u(s), u'(s)). \tag{7}$$

Clearly, $w_n'(0) = 0$.

Naturally, one obtains

$$\begin{cases} w_n''(t) = -f(t, u_n, u_n'), \quad 0 \le t \le 1, \\ w_n(0) = 0, \ \ w_n'(0) = 0. \end{cases} \tag{8}$$

Now we solve (8) using reproducing kernel method (RKM). In order to solve (8) using RKM, we first construct a reproducing kernel space $W_2^3[0,1]$ in which every function satisfies the initial conditions of (8).

Reproducing kernel Hilbert space $W_2^3[0,1]$ is defined as $W_2^3[0,1] = \{u(x) | u(x), u'(x), u''(x)$ are absolutely continuous real value functions, $u'''(x) \in L^2[0,1], u(0) = u'(0) = 0\}$. The inner product and norm in $W_2^3[0,1]$ are given, respectively, by

$$(u(y), v(y))_{W_2^3} = u(0)v(0) + u'(0)v'(0) + u(1)v(1) + \int_0^1 u'''v'''dy$$

and

$$\|u\|_{W_2^3} = \sqrt{(u,v)_{W_2^3}}, \quad u, v \in W_2^3[0,1].$$

By Geng and Cui (2007), it is easy to obtain its reproducing kernel (RK)

$$k(x,y)=\begin{cases}k_1(x,y),\ y\le x,\\ k_1(y,x),\ y\ge x,\end{cases}$$

where

$$k_1(x,y)=\frac{y^2(-(x^2(-126+10x-5x^2+x^3))+5(x-1)xy^2-(x^2-1)y^3)}{120}.$$

In (8), put $Lw_n(t)=w_n''(t)$, it is clear that $L:W_2^3[0,1]\to W_2^1[0,1]$ is a bounded linear operator. Put $\varphi_i(t)=\overline{k}(t_i,t)$ and $\psi_i(t)=L^*\varphi_i(t)$ where $\overline{k}(s,t)$ is the RK of $W_2^1[0,1]$, L^* is the adjoint operator of L. The orthonormal system $\{\overline{\psi}_i(t)\}_{i=1}^{\infty}$ of $W_2^3[0,1]$ can be derived from Gram-Schmidt orthogonalization process of $\{\psi_i(t)\}_{i=1}^{\infty}$,

$$\overline{\psi}_i(t)=\sum_{k=1}^{i}\beta_{ik}\psi_k(t),\ (\beta_{ii}>0, i=1,2,\cdots).$$

By RKM presented in Geng and Cui (2007), we have the following theorem.

Theorem 1. For (8), if $\{t_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then $\{\psi_i(t)\}_{i=1}^{\infty}$ is the complete system of $W_2^3[0,1]$ and $\psi_i(t)=L_s k(t,s)|_{s=t_i}$.

Theorem 2. If $\{t_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, and the solution of (8) is unique, then the solution of (8) is

$$w_n(t)=-\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}f(t_k,u_n(t_k),u_n'(t_k))\overline{\psi}_i(t). \tag{9}$$

Now, the approximate solution $w_n^N(t)$ can be obtained by taking N terms in the series representation of $w_n(t)$ and

$$w_n^N(t)=-\sum_{i=1}^{N}\sum_{k=1}^{i}\beta_{ik}f(t_k,u_n(t_k),u_n'(t_k))\overline{\psi}_i(t). \tag{10}$$

Therefore, we obtain the following modified iteration formula of (4):

$$u_{n+1}(t) = u_0(t) + w_n(t) = u_0(t) - \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik} f(t_k, u_n(t_k), u_n'(t_k))\overline{\psi}_i(t) \tag{11}$$

and

$$u_{n+1}(t) = u_0(t) + w_n^N(t) = u_0(t) - \sum_{i=1}^{N}\sum_{k=1}^{i}\beta_{ik} f(t_k, u_n(t_k), u_n'(t_k))\overline{\psi}_i(t) \tag{12}$$

Beginning with $u_0(t) = \alpha + \beta t$, according to (12), one can obtain the n th iteration approximation $u_n^N(t)$.

Remark. Since $u_0(0) = \alpha,\ u_0'(0) = \beta$, also, $u_n^N(0) = 0,\ (u_n^N)'(0) = 0$, hence, in the process of iteration, we can guarantee that the nth iteration approximation $u_n^N(t)$ obtained from (11) satisfies the initial conditions of (1).

4. Numerical examples

In this section, the present method is applied to some second order initial value problems. Obtained results show that the present method is remarkably effective. In the following examples, we take $t_i = \frac{i-1}{N-1}, i = 1, 2, \cdots, N.$

Example 4.1. Consider the following initial value problem:

$$\begin{cases} u''(t) + e^{-t}\sin tu'(t) + t^3(1-t)u^3(t) = f(t), & 0 < t < 1, \\ u(0) = 1,\ u'(0) = 0, \end{cases}$$

where $f(t)$ is given such that the exact solution is $u(t) = \cos(\pi t)$.

Solution: Beginning with $u_0(t) = 1$, according to (12), one can obtain the approximation $u_n^N(t)$.

When we take $n = 3, 5,\ N = 51$, the numerical results are shown in Figures 1, 2.

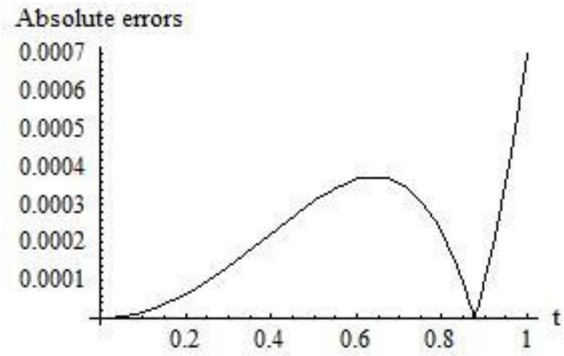

Figure 1: Absolute error $\left|u(t) - u_3^{51}(t)\right|$

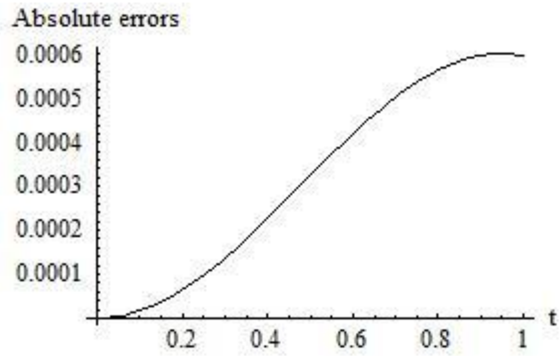

Figure 2: Absolute error $\left|u(t) - u_3^{51}(t)\right|$

Example 4.2. Consider the following initial value problem:

$$\begin{cases} u''(t) + \dfrac{u^3(t)}{5} + \sin\sqrt{u} = f(t), \quad 0 < t < 1, \\ u(0) = 5, \;\; u'(0) = 0, \end{cases}$$

where $f(t)$ is given such that the exact solution is $u(t) = 5\cos(2t)$.

Solution: Beginning with $u_0(t) = 5$, according to (12), one can obtain the approximation $u_n^N(t)$.

Taking $n = 5,\ N = 51, 101,$ the numerical results are shown in Figures 3, 4.

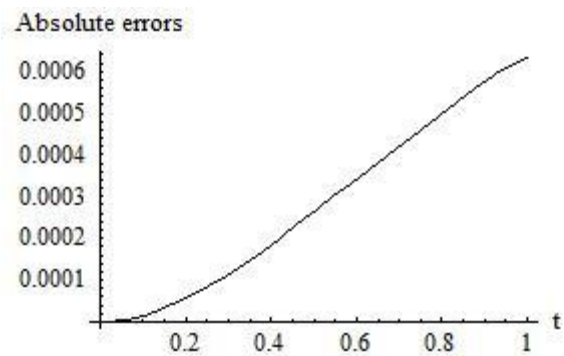

Figure 3: Absolute error $\left|u(t) - u_5^{51}(t)\right|$

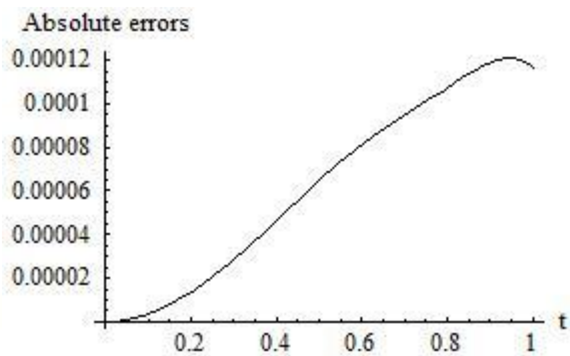

Figure 4: Absolute error $\left|u(t) - u_5^{101}(t)\right|$

5. Conclusion

In this paper, based on RKM, a modification of variational iteration method is presented for second order initial value problems. The modification can overcome the restriction of conventional VIM and improve the iteration speed of conventional VIM. For some second order initial value problems, when the form of nonlinear terms is complex, the higher-order approximation can not be obtained using conventional VIM while the higher-order approximation can be obtained by using the modified VIM.

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