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Solution to Some Open Problems on *E*-super Vertex Magic Total Labeling of Graphs

G. Marimuthu[1], M.S. Raja Durga[2], G. Durga Devi[3]
Department of Mathematics
The Madura College
Madurai- 625011
Tamilnadu, India
[1]yellowmuthu@yahoo.com;[2]msrajadurga17@gmail.com;[3]punithadurga1991@gmail.com

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Abstract

Let G be a finite graph with p vertices and q edges. A *vertex magic total labeling* is a bijection f from $V(G)\cup E(G)$ to the consecutive integers 1, 2, ..., $p+q$ with the property that for every $u\in V(G)$, $f(u)+\sum_{v\in N(u)} f(uv)=k$ for some constant k. Such a labeling is *E-super* if $f:E(G)\to\{1, 2, \ldots, q\}$. A graph G is called *E-super vertex magic* if it admits an *E-super vertex magic labeling*. In this paper, we solve two open problems given by Marimuthu, Suganya, Kalaivani and Balakrishnan (Marimuthu et al., 2015).

Keywords: super vertex magic labeling; E-super vertex magic labeling; E-super vertex magic graph

2010 AMS Subject classification: 05C78

1. Introduction

In this paper, we consider only finite simple undirected graphs with order p and size q. For graph theoretic notations we follow (Harary, 1969; Marr and Wallis, 2013). A labeling of a graph G is a mapping that carries a set of graph elements, usually the vertices and edges into a set of numbers, usually integers. Many kinds of labelings have been studied and an excellent survey of graph labeling can be found in (Gallian, 2014).

Sedláček (Sedláček,1963) introduced the concept of magic labeling. Sedláček defined a graph to be magic if it had an edge-labeling, with range the real numbers, such that the sum of the labels around any vertex equalled constant, independent of the choice of vertex. MacDougall et al. (MacDougall et al., 2002) introduced the notion of *vertex magic total labeling*. If G is a finite simple undirected graph with p vertices and q edges, then a *vertex magic total labeling* of G is a bijection f from $V(G)\cup E(G)$ to the integers 1, 2, ..., $p+q$ with the property that for every u in $V(G)$,

$$f(u)+\sum_{v\in N(u)} f(uv)=k,$$

for some constant k, where

$$N(u)=\{v\in V(G): uv\in E(G)\}.$$

They studied the basic properties of vertex magic total graphs and showed some families of graphs having vertex magic total labeling.

MacDougall et al. (MacDougall et al., 2004) further introduced the *super vertex magic total labeling*. They called a vertex magic total labeling f is *super* if $f: V(G)\to\{1,2,\ldots,p\}$. Swaminathan and Jeyanthi **(**Swaminathan and Jeyanthi, 2003) introduced a concept with the name *super vertex magic total labeling*, but with different notion. They called a *vertex magic total labelling* f to be *super* if $f: E(G)\to\{1,2,\ldots,q\}$. To avoid confusion, Marimuthu and Balakrishnan (Marimuthu and Balakrishnan, 2012) called a total labeling f as an *E-super vertex magic total labeling* if $f: E(G)\to\{1,2,\ldots,q\}$. They studied the *E-super vertex magicness* of *even regular graphs*. Most recently Wang and Zhang (Wang and Zhang, 2014) extended the results found in the article (Marimuthu and Balakrishnan, 2012).

Theorem 1.1. (Swaminathan and Jeyanthi, 2003)

A path P_n is E-super vertex magic if and only if n is odd and $n\geq 3$.

Theorem 1.2. (Swaminathan and Jeyanthi, 2003)

If a non-trivial graph G of order p and size q is super vertex magic, then the magic constant k is given by

$$k=q+\frac{p+1}{2}+\frac{q(q+1)}{p}.$$

Theorem 1.3. (Marimuthu and Balakrishnan, 2012)

Every tree T of even order is not E-super vertex magic.

In this article, we partially solve the following open problems given by Marimuthu, Suganya, Kalaivani and Balakrishnan (Marimuthu et al., 2015).

Open problem 1.4. Discuss the E-super vertex magicness of $B_{n,n-t}$, when $t\neq 2$.

Open problem 1.5. Find all E-super vertex magic wounded suns, $C_n^+ - me, m\neq 3$.

2. Solution to the Problems

Definition 2.1.

A broom $B_{n,d}$ is defined by attaching *n-d* pendant edges with any one of the pendant vertices of the path P_d.

Theorem 2.2.

The broom $B_{n,n-1}$ is *E*-super vertex magic if and only if *n* is odd and $n \geq 3$.

Proof:

As the broom $B_{n,n-1}$ is isomorphic to a path P_n the results follow immediately from Theorem 1.1. □

Theorem 2.3.

The broom $B_{n,n-t}$ is not *E*-super vertex magic for $n-t \geq 2, t \geq 3.$

Proof:

Let

$$V(B_{n,n-t}) = \{v_1, v_2, ..., v_{n-t}, u_1, u_2, ..., u_t\}$$

and let

$$E(B_{n,n-t}) = \{v_i v_{i+1}, 1 \leq i \leq n-t-1\} \cup \{v_{n-t} u_j, 1 \leq j \leq t\}.$$

Then, $B_{n,n-t}$ has $p = n$ vertices and $q = n-1$ edges.

As $B_{n,n-t}$ is a tree, according to Theorem 1.3, *n* can't be even.

Assume that *n* is odd.

Suppose $B_{n,n-t}$ is *E*-super vertex magic. Then, there exists a *E*-super vertex magic total labeling of $B_{n,n-t}$, say, *f*. According to Theorem 1.2 the magic constant *k* is given by

$$\begin{aligned} k &= q + \frac{p+1}{2} + \frac{q(q+1)}{p} \\ &= n-1+\frac{n+1}{2}+\frac{(n-1)n}{n} \\ &= n-1+\frac{n+1}{2}+n-1 \\ &= 2n-2+\frac{n+1}{2} \\ &= \frac{5n-3}{2}. \end{aligned}$$

Now, we determine the smallest possible value of the sum

$$f(v_{n-t})+\sum_{u\in N(v_{n-t})} f(v_{n-t}u)= f(v_{n-t})+\sum_{j=1}^{t} f(v_{n-t}u_j)+ f(v_{n-t-1}v_{n-t}).$$

The smallest possible value of the set $\{f(v_{n-t}u_j):1\le j\le t\}$ is

$$\{k-(p+q)+j-1 = k-2n+j:1\le j\le t\}.$$

Also, the smallest possible values of $f(v_{n-t-1}v_{n-t})$ and $f(v_{n-t})$ are 1 and *n*, respectively. Therefore, the smallest possible value of the above sum is:

$$\begin{aligned} n+\sum_{j=1}^{t}(k-2n+j)+1 &= n+1+tk-2nt+(1+2+...+t)\\ &= n+1+tk-2nt+\frac{t(t+1)}{2}\\ &= n+1+t\left(\frac{5n-3}{2}\right)-2nt+\frac{t^2+t}{2}\\ &= \frac{(t+2)n+(t-1)^2+1}{2}, t\ge 3\\ &> \frac{5n-3}{2}\\ &= k. \end{aligned}$$

Therefore, the value of the sum at v_{n-t} is at least $n+\sum_{j=1}^{t}(k-2n+j)+1$ exceeds *k*, a contradiction. Hence, the result follows. □

Note that

$$n+\sum_{j=1}^{t}(k-2n+j)+1$$

exceeds *k* by

$$\frac{(t-3)n+(t-1)^2+4}{2}.$$

Definition 2.4.

The sun graph C_n^+ is defined as follows:

$$V(C_n^+)=\{v_1,v_2,\ldots,v_{2n}\}.$$
$$E(C_n^+)=\{v_iv_{i+1},\ 1\le i\le n-1\}\cup\{v_nv_1\}\cup\{v_iv_{n+i},1\le i\le n\}.$$

The graph $C_n^+-me, m\le n$ is obtained by removing $\{v_{2n},v_{2n-1},\ldots,v_{2n-m+1}\}$ and the edges adjacent to them from C_n^+. This is referred as a *wounded sun*.

Theorem 2.5.
A wounded sun $C_n^+ - me$, $3 \neq m \leq n$, is *E*-super vertex magic if and only if *m* is odd and $m \neq 1$.

Proof:

Let $C_n^+ - me$ be a wounded sun, where *m* is a positive integer such that $3 \neq m \leq n$. Then,

$$V(C_n^+ - me) = \{v_1, v_2, ..., v_n\} \cup \{v_{n+1}, v_{n+2}, ..., v_{2n-m}\}$$

and

$$E\,(C_n^+ - me) = \{v_i v_{i+1},\ 1 \leq i \leq n-1\} \cup \{v_n v_1\} \cup \{v_i v_{n+i},\ 1 \leq i \leq n-m\}.$$

Thus, $C_n^+ - me$ has $p = 2n - m$ vertices and $q = 2n - m$ edges.

Assume that $C_n^+ - me$, $m \neq 3$ is *E*-super vertex magic. Then, by Theorem 1.3, the magic constant *k* is given by

$$\begin{aligned} k &= q + \frac{p+1}{2} + \frac{q(q+1)}{p} \\ &= p + \frac{p+1}{2} + p + 1 \\ &= 2p + 1 + \frac{p+1}{2} \\ &= 2(2n-m) + 1 + \frac{2n-m+1}{2} \\ &= 4n - 2m + 1 + n - \left(\frac{m-1}{2}\right) \\ &= 5n + 1 - \left(\frac{5m-1}{2}\right), \end{aligned}$$

which is an integer only when *m* is odd.

Suppose *m* = 1. Then.

$$V(C_n^+ - e) = \{v_1, v_2, ..., v_n\} \cup \{v_{n+1}, v_{n+2}, ..., v_{2n-1}\}$$

and

$$E\,(C_n^+ - e) = \{v_i v_{i+1},\ 1 \leq i \leq n-1\} \cup \{v_n v_1\} \cup \{v_i v_{n+i},\ 1 \leq i \leq n-1\}.$$

Thus, $C_n^+ - e$ has $p = 2n - 1$ vertices and $q = 2n - 1$ edges. Since $C_n^+ - e$ is *E*-super vertex magic, it has a *E*-super vertex magic labeling. Therefore, for each vertex $v \in V(C_n^+ - e)$, the total weight $wt(v)$, (the sum of the labels of the vertex *v* and that of the edges incident to *v*) is a constant $k = 5n - 1$. Since there are *n*-1 pendent vertices and *n*-1 pendent edges, the only possible set of the labels of the pendent vertices is $\{3n, 3n+1, \ldots, 4n-2 = p+q\}$ and the only possible set of the labels of the pendent edges is $\{n+1, n+2, \ldots, 2n-1 = q\}$. Therefore, at the vertex v_n, the largest possible label of the vertex v_n is 3*n*-1 and the largest possible labels of the two edges incident to this vertex are *n*-1 and *n*. Thus, the largest possible value $wt(v_n)$ is

$$(3n-1)+(n-1)+(n) = 5n-2 < 5n-1 = k.$$

That is, the largest possible value of $wt(v_n)$ is less than *k*. Therefore, $wt(v_n) < k$, a contradiction. Hence, *m* is odd and $m \neq 1$. Conversely, assume that $m \leq n$ is odd, $m \neq 1$ and $m \neq 3$. Then, $m \geq 5$.

If $m = n$, then the wounded sun $C_n^+ - me$ is the cycle C_n. By Theorem 1.2, $C_n^+ - me$ is *E*-super vertex magic.

Now let $5 \leq m \leq n-1$. Define a total labeling

$$f : V \cup E \to \{1, 2, \ldots, 4n-2m\}$$

as follows:

$$f(v_1) = 2n-1,$$
$$f(v_n) = 3n-m,$$
$$f(v_{n-i}) = 2n-m+i, 1 \leq i \leq m-2,$$
$$f(v_{i+1}) = 3n-m-i, 1 \leq i \leq n-m,$$
$$f(v_{n+i}) = 3n-m+i, 1 \leq i \leq n-m,$$
$$f(v_i v_{i+1}) = i, 1 \leq i \leq n-m,$$
$$f(v_1 v_n) = n-m+1,$$
$$f(v_i v_{n+i}) = 2n-i-3\left(\frac{m-1}{2}\right), 1 \leq i \leq n-m,$$
$$f(v_{n-2i} v_{n-2i+1}) = 2n-m+1-i, 1 \leq i \leq \frac{m-1}{2},$$
$$f(v_{n-2i+1} v_{n-2i+2}) = n-\left(\frac{m-3}{2}\right)-i, 1 \leq i \leq \frac{m-1}{2}.$$

From the above labeling, we have,

$$V(C_n^+ - me) = \{v_1\} \cup \{v_n\} \cup \{v_{n-i}, 1 \leq i \leq m-2\} \cup \{v_{i+1}, 1 \leq i \leq n-m\} \cup \{v_{n+i}, 1 \leq i \leq n-m\}.$$

Now,

$$f(v_1) + \sum_{u \in N(v_1)} f(v_1 u) = f(v_1) + f(v_1 v_2) + f(v_1 v_n) + f(v_1 v_{n+1})$$
$$= (2n-1) + (1) + (n-m+1) + \left(2n-1-3\left(\frac{m-1}{2}\right)\right)$$
$$= 5n - \frac{5m}{2} + \frac{3}{2} = 5n + \frac{3-5m}{2}.$$

$$f(v_n)+\sum_{u\in N(v_n)} f(v_n u)= f(v_n)+ f(v_n v_1)+ f(v_n v_{n-1})$$

$$=(3n-m)+(n-m+1)+\left(n-\left(\frac{m-3}{2}\right)-1\right)$$

$$=5n+\frac{3-5m}{2}.$$

$$f(v_{n-i})+\sum_{u\in N(v_{n-i})} f(v_{n-i} u)= f(v_{n-i})+ f(v_{n-i} v_{n-i-1})+ f(v_{n-i} v_{n-i+1})$$

$$=(2n-m+i)+\left(2n-m+1-\left(\frac{i+1}{2}\right)\right)+\left(n-\left(\frac{m-3}{2}\right)-\left(\frac{i+1}{2}\right)\right)$$

$$=5n+\frac{3-5m}{2}, 1\le i\le m-2.$$

$$f(v_{i+1})+\sum_{u\in N(v_{i+1})} f(v_{i+1} u)= f(v_{i+1})+ f(v_{i+1} v_i)+ f(v_{i+1} v_{i+2})+ f(v_{i+1} v_{n+i+1})$$

$$=(3n-m-i)+(i)+(i+1)+\left(2n-m-(i+1)-\left(\frac{m-3}{2}\right)\right)$$

$$=5n+\frac{3-5m}{2}, 1\le i\le n-m.$$

$$f(v_{n+i})+\sum_{u\in N(v_{n+i})} f(v_{n+i} u)= f(v_{n+i})+ f(v_i v_{n+i})$$

$$=(3n-m+i)+\left(2n-i-3\left(\frac{m-1}{2}\right)\right)$$

$$=5n+\frac{3-5m}{2}, 1\le i\le n-m.$$

Therefore,

for every $v\in V(C_n^+ - me)$, $f(v)+\sum_{u\in N(v)} f(uv)=5n+\frac{3-5m}{2}$.

Hence, f is an *E*-super vertex magic labeling of $V(C_n^+ - me)$ with magic constant

$$k=5n+\frac{3-5m}{2},$$

where *m* is odd, $m\neq 1$ and $m\neq 3$.

An illustration for Theorem 2.5 is given in Figure 1.

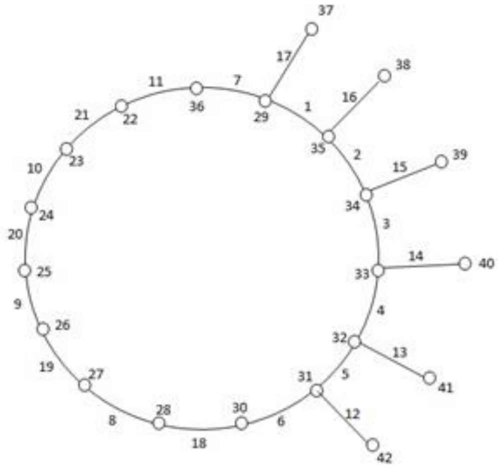

Figure 1. An *E*-super vertex magic labeling of $C_{15}^{+} - 9e$

3. Conclusion and Scope

In this article, we have solved two open problems given by Marimuthu, Suganya, Kalaivani and Balakrishnan (Marimuthu et al., 2015). There is another open problem for further investigation in the same paper which is given as follows:

Open problem 1. Characterize all *E*-super vertex magic trees of odd order.

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