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Asymptotically Double Lacunary Equivalent Sequences in Topological Groups

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Abstract

In this paper we study the concept of asymptotically double lacunary statistical convergent sequences in topological groups and prove some inclusion theorems.

Keywords: Pringsheim Limit Point; Asymptotically equivalence; P-convergent; Double Lacunary Sequence; Double Statistical Convergence; Topological Groups

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1. Introduction

The idea of statistical convergence of double sequences was first introduced by (Mursaleen and Edely, 2003) while the idea of statistical convergence of single sequences was first studied by (Fast, 1951) and the rapid developments were started after the papers of (Salat, 1980) and (Fridy, 1985). Nowadays it has become one of the most active areas of research in the field of summability theory. (Schoenberg, 1959) gave some basic properties of statistical convergence and also studied the concept as a summability method. Later on statistical convergence was studied by (Connor, 1992), (Freedman, A.R., Sember, J.J. and Raphael, M., 1978), (Esi and Esi, 2010), (Esi and Tripathy, 2007), (Tripathy, 2003), (Esi, 2012) and many others.

(Marouf, 1993) presented definitions for asymptotically equivalent sequences and asymptotically regular matrices. In (Patterson, 2003), Patterson extended those concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for non-negative summability matrices. In (Patterson and Savas, 2006), Patterson and Savas extended the definitions presented in (Patterson, 2003) to lacunary sequences.

This paper extend the definitions presented in (Esi, 2009) to double lacunary sequences in topological groups.

In this paper we study the concept of asymptotically double lacunar statistical sequences on topological groups. Since the study of convergence in topological groups is important, we feel that the concept of asymptotically double lacunary statistical convergent sequences in topological groups would provide a more general framework for the subject.

Definition 1. (Fast, 1951; Fridy, 1985) A single sequence $x = (x_k)$ is said to be statistically convergent to the number L if for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \left|\left\{ k \leq n : \; |x_k - L| \geq \varepsilon \right\}\right| = 0.$$

In this case we write $st - \lim x = L$ or $x_k \to L\,(st)$.

Definition 2. (Freedman, A.R., Sember, J.J. and Raphael, M., 1978) By a lacunary sequence $\theta = (k_r)$, $r = 0, 1, 2, \ldots$, where $k_0 = 0$, we mean an increasing sequence of non-negative integers with $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. The space of lacunary strongly convergent sequence N_θ was defined in (Freedman, A.R., Sember, J.J. and Raphael, M., 1978) as follows:

$$N_\theta = \left\{ x = (x_i) : \; \lim_{r \to \infty} \frac{1}{h_r} \sum_{i \in I_r} |x_i - L| = 0 \;\text{ for some } L \right\}.$$

The following definition is due to (Fridy and Orhan, 1993).

Definition 3. (Fridy and Orhan, 1993) Let $\theta = (k_r)$ be a lacunary sequence; the single sequence $x = (x_k)$ is $st^\theta-$convergent to L provided that for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r} \left|\left\{ k \in I_r : \; |x_k - L| \geq \varepsilon \right\}\right| = 0.$$

In this case we write $st^\theta - \lim x = L$ or $x_k \to L\left(st^\theta\right)$.

In 1900 Pringsheim presented the following definition for the convergence of double sequences.

Definition 4. (Pringsheim, 1900) A double sequence $x = (x_{k,l})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\varepsilon > 0$ there exists $N \in \mathbf{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > N$. We shall describe such an $x = (x_{k,l})$ more briefly as"$P-$convergent".

We shall denote the space of all P-convergent sequences by c^2. By a bounded double sequence we shall mean there exists a positive number K such that $|x_{k,l}| < K$ for all (k,l) and denote such bounded sequences by $\|x\|_{(\infty,2)} = \sup_{k,l} |x_{k,l}| < \infty$. We shall also denote the set of all bounded double sequences by l^2_∞. We also note that in contrast to the case for single sequence a $P-$convergent double sequence need not be bounded.

Definition 5. (Mursaleen and Edely, 2003) A real double sequence $x = (x_{k,l})$ is said to be statistically convergent to L, provided that for each $\varepsilon > 0$

$$P - \lim_{m,n} \frac{1}{mn} \left|\{(k,l): \ k \leq m \text{ and } l \leq n, |x_{k,l} - L| \geq \varepsilon\}\right| = 0.$$

In this case we write $st_2 - \lim x = L$ or $x_{k,l} \to L\,(st_2)$.

Definition 6. (Savas and Patterson, 2006) The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary sequence if there exist two increasing sequences of integers such that

$$k_o = 0, \ h_r = k_r - k_{r-1} \to \infty \ \text{ as } r \to \infty$$

and

$$l_o = 0, \ \bar{h}_s = l_s - l_{s-1} \to \infty \ \text{ as } s \to \infty.$$

Notation 1: $k_{r,s} = k_r k_s$, $h_{r,s} = h_r \overline{h_s}$, $\theta_{r,s}$ is determined by

$$I_{r,s} = \{(k,l): \ k_{r-1} < k \leq k_r \text{ and } l_{s-1} < l \leq l_s\},$$

$$q_r = \frac{k_r}{k_{r-1}}, \ \overline{q_s} = \frac{l_s}{l_{s-1}} \text{ and } q_{r,s} = q_r \overline{q_s}.$$

Definition 7. ((Savas and Patterson, 2006)) Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence; the double number sequence $x = (x_{k,l})$ is $st_2^\theta - convergent$ to L provided that for every $\varepsilon > 0$,

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \left|\{ \ (k,l) \in I_{r,s}: \ \ |x_{k,l} - L| \geq \varepsilon\}\right| = 0.$$

In this case, we write $st_2^\theta - \lim_{k,l} x_{k,l} = L$ or $x_{k,l} \to L\left(st_2^\theta\right)$.

2. Definitions and Notations

By X, we will denote an abelian topological Hausdorff group, written additively, which satisfies the first axiom of countability. In (Cakalli, 1996), a single sequence $x = (x_k)$ in X is said to be statistically convergent to an element $L \in X$ if for each neighbourhood U of 0,

$$\lim_n \frac{1}{n} \left|\{k \leq n: \ x_k - L \notin U\}\right| = 0.$$

In this case we write $st\,(X) - \lim_k x_k = L$ or $x_k \to L\,(st\,(X))$.

In (Cakalli, 1995), the concept of lacunary statistical convergence was defined by Cakalli as follows: Let $\theta = (k_r)$ be a lacunary sequence; the single sequence $x = (x_k)$ in X is said to be

$st^{\theta}-convergent$ to L (or lacunary statistically convergent to L in X) if for each neighbourhood U of 0,

$$\lim_{r}\frac{1}{h_r}\left|\{\ k\in I_r:\ x_k-L\notin U\}\right|=0.$$

In this case we write $st^{\theta}(X)-\lim_k x_k=L$ or $x_k\to L\left(st^{\theta}(X)\right)$.

In 2010, Cakalli and Savas defined double statistical convergence in topological group X as follows:

Definition 8. ((Cakalli and Savas, 2010)) A double sequence $x=(x_{k,l})$ is said to be statistically convergent to a point $L\in X$ if for each neighbourhood U of 0,

$$P-\lim_{n,m}\frac{1}{nm}\left|\{(k,l):k\leq n,l\leq m;\ x_{k,l}-L\notin U\}\right|=0.$$

In this case we write $st_2(X)-\lim_{k,l}x_{k,l}=L$ or $x_{k,l}\to L\left(st_2(X)\right)$ and denote the set of all statistically convergent double sequences in X by $st_2(X)$.

Now we give the two new definitions related to double lacunary asymptotically equivalent sequences in topological groups as follows:

Definition 9. (Patterson and Savas, 2006) Let $\theta_{r,s}=\{(k_r,l_s)\}$ be a double lacunary sequence. A sequence $x=(x_{k,l})$ is said to be st_2^{θ}-convergent to L in X (or double lacunary statistical convergent to L in X) if for each neighbourhood U of 0

$$P-\lim_{r,s}\frac{1}{h_{r,s}}\left|\{\ (k,l)\in I_{r,s}:\ x_{k,l}-L\notin U\}\right|=0.$$

In this case, we write $st_2^{\theta}(X)-\lim_{k,l}x_{k,l}=L$ or $x_{k,l}\to L\left(st_2^{\theta}(X)\right)$ and

$$st_2^{\theta}(X)=\left\{x=(x_{k,l}):\ for\ some\ L,\ st_2^{\theta}-\lim_{k,l}x_{k,l}=L\right\}$$

and in particular

$$st_2^{\theta}(X)_o=\left\{x=(x_{k,l}):\ st_2^{\theta}-\lim_{k,l}x_{k,l}=0\right\}.$$

Definition 10. (Esi, 2009) Let $\theta_{r,s}=\{(k_r,l_s)\}$ be a double lacunary sequence. A double number sequence $x=(x_{k,l})$ is $N_{\theta}(X)$ $-$convergent to L in X provided that

$$P-\lim_{r,s}\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}\ \&\ x_{k,l}-L\notin U}\left|x_{k,l}-L\right|=0.$$

Definition 11. ((Esi, 2009)) Let $\theta_{r,s}=\{(k_r,l_s)\}$ be a double lacunary sequence. The double non-negative sequences $x=(x_{k,l})$ and $0\notin y=(y_{k,l})$ are said to be asymptotically double lacunary statistical equivalent to L in X

$$P-\lim_{r,s}\frac{1}{h_{r,s}}\left|\left\{\ (k,l)\in I_{r,s}:\ \left|\frac{x_{k,l}}{y_{k,l}}-L\right|\geq\varepsilon\right\}\right|=0.$$

In this case, we write $x \overset{S^L_{\theta_{r,s}}}{\sim} y$ and simply asymptotically double lacunary statistical equivalent if $L = 1$.

Definition 12. The double non-negative sequences $x = (x_{k,l})$ and $0 \notin y = (y_{k,l})$ are said to be asymptotically double equivalent to L in X if for each neighbourhood U of 0

$$P - \lim_{n,m} \frac{1}{mn} \left| \left\{ (k,l) : k \leq n, l \leq m; : \ \frac{x_{k,l}}{y_{k,l}} - L \notin U \right\} \right| = 0.$$

In this case, we write $x \overset{S^L(X)}{\sim} y$ and simply asymptotically double statistical equivalent if $L = 1$ in X. Furthermore, let $S^L(X)$ denote the set of all double sequences $x = (x_{k,l})$ and $y = (y_{k,l})$ such that $x \overset{S^L(X)}{\sim} y$ in X.

Definition 13. Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence. The double non-negative sequences $x = (x_{k,l})$ and $0 \notin y = (y_{k,l})$ are said to be asymptotically double lacunary equivalent to L in X if for each neighbourhood U of 0

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \left| \left\{ \ (k,l) \in I_{r,s} : \ \frac{x_{k,l}}{y_{k,l}} - L \notin U \right\} \right| = 0.$$

In this case, we write $x \overset{S^L_{\theta_{r,s}}(X)}{\sim} y$ and simply asymptotically double lacunary statistical equivalent if $L = 1$ in X. Furthermore, let $S^L_{\theta_{r,s}}(X)$ denote the set of all double sequences $x = (x_{k,l})$ and $0 \notin y = (y_{k,l})$ such that $x \overset{S^L_{\theta_{r,s}}(X)}{\sim} y$ in X.

Definition 14. Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence. The double non-negative sequences $x = (x_{k,l})$ and $0 \notin y = (y_{k,l})$ are said to be asymptotically double lacunary $N_{\theta_{r,s}}(X)_L$ −equivalent to L in X if for each neighbourhood U of 0

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s} \ \& \ x_{k,l} - L \notin U} \left| \frac{x_{k,l}}{y_{k,l}} - L \right| = 0.$$

In this case, we write $x \overset{N_{\theta_{r,s}}(X)_L}{\sim} y$ and simply asymptotically double lacunary $N_{\theta_{r,s}}(X)-$ equivalent if $L = 1$ in X.

3. Main Results

Theorem 1.

Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence. Then,

(i). (a) If $x \overset{N_{\theta_{r,s}}(X)_L}{\sim} y$, then $x \overset{S^L_{\theta_{r,s}}(X)}{\sim} y$.

(i). (b) $N_{\theta_{r,s}}(X)_L$ is a proper subset of $x \overset{S^L_{\theta_{r,s}}(X)}{\sim} y$.

(ii). If $\left(\frac{x_{k,l}}{y_{k,l}}\right) \in l^2_\infty$ and $x \overset{S^L_{\theta_{r,s}}(X)}{\sim} y$ then $x \overset{N_{\theta_{r,s}}(X)_L}{\sim} y$.

(iii). If $\left(\frac{x_{k,l}}{y_{k,l}}\right) \in l^2_\infty$, then $N_{\theta_{r,s}}(X)_L = x \overset{S^L_{\theta_{r,s}}(X)}{\sim} y$.

Proof:

(i). (a) If $x \overset{N_{\theta_{r,s}}(X)_L}{\sim} y$ then

$$\left|\left\{\ (k,l) \in I_{r,s}:\ \frac{x_{k,l}}{y_{k,l}} - L \notin U\right\}\right| \leq \sum_{(k,l)\in I_{r,s}\ \&\ x_{k,l}-L\notin U} \left|\frac{x_{k,l}}{y_{k,l}} - L\right|$$

and

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}\ \&\ x_{k,l}-L\notin U} \left|\frac{x_{k,l}}{y_{k,l}} - L\right| = 0.$$

This implies that

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \left|\left\{\ (k,l) \in I_{r,s}:\ \frac{x_{k,l}}{y_{k,l}} - L \notin U\right\}\right| = 0.$$

This completes the proof (a).

(i). (b) Let $(k_r) = (2^{r-1})$ and $(l_s) = (2^{s-1})$. Then we have $I_{r,s} = \{(m,n) : 2^{r-1} < m \leq 2^r$ and $2^{s-1} < n \leq 2^s\}$. Define the double sequence $\left(\frac{x_{m,n}}{y_{m,n}}\right)$ as follows: We divide the total double sequences into blocks of $I_{r,s}$ type such that $\frac{x_{2^{r-1}+i,2^{r-1}}}{y_{2^{r-1}+i,2^{r-1}}} = i$ for $1 \leq i \leq \left[\sqrt[3]{h_{r,s}}\right] \left(= \left[\sqrt[3]{2^{r-1}2^{s-1}}\right]\right)$ in each of the first $\left[\sqrt[3]{h_{r,s}}\right]$ number of rows. At other places $\frac{x_{m,n}}{=}0$. The $I_{r,s}$ block will look as follows:

$$\begin{bmatrix}
1 & 2 & 3 & \cdots & \sqrt[3]{h_{r,s}} & 0 & 0 & 0 & \cdots \\
1 & 2 & 3 & \cdots & \sqrt[3]{h_{r,s}} & 0 & 0 & 0 & \cdots \\
1 & 2 & 3 & \cdots & \sqrt[3]{h_{r,s}} & 0 & 0 & 0 & \cdots \\
. & . & . & . & . & . & . & . & . \\
1 & 2 & 3 & \cdots & \sqrt[3]{h_{r,s}} & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & .
\end{bmatrix}$$

We consider the sup-norm for the double sequences, i.e. $\left\|\left(\frac{x_{m,n}}{y_{m,n}}\right)\right\| = \sup_{m,n}\left|\frac{x_{m,n}}{y_{m,n}}\right|$. We have $\left[\sqrt[3]{h_{r,s}}\right] \to \infty$ as $r,s \to \infty$ independent of each other. Thus for a given $\varepsilon > 0$, (very small) we have

$$P - \lim_{r,s\to\infty} \frac{1}{h_{r,s}} \left|\left\{(m,n) \in I_{r,s}, \left|\frac{x_{m,n}}{y_{m,n}}\right| \geq \varepsilon\right\}\right| = P - \lim_{r,s\to\infty} \frac{\left[\sqrt[3]{h_{r,s}}\right]}{h_{r,s}} = 0.$$

This is equivalent to

$$P - \lim_{r,s\to\infty} \frac{1}{h_{r,s}} \left|\left\{(m,n) \in I_{r,s}, \frac{x_{m,n}}{y_{m,n}} \notin U\right\}\right| = P - \lim_{r,s\to\infty} \frac{\left[\sqrt[3]{h_{r,s}}\right]}{h_{r,s}} = 0,$$

where U is an $\varepsilon-$neighborhood of zero, ε being very small. Since we have the first $\left[\sqrt[3]{h_{r,s}}\right]$ number of rows having first $\left[\sqrt[3]{h_{r,s}}\right]$ elements as $1, 2, 3, \ldots, \left[\sqrt[3]{h_{r,s}}\right]$, so we have

$$P - \lim_{r,s\to\infty} \sum_{(m,n)\in I_{r,s}} |x_{m,n}| = P - \lim_{r,s\to\infty} \frac{\left[\sqrt[3]{h_{r,s}}\right]\left(\left[\sqrt[3]{h_{r,s}}\right]\left(\left[\sqrt[3]{h_{r,s}}\right]+1\right)\right)}{2\left[\sqrt[3]{h_{r,s}}\right]} = \frac{1}{2}.$$

Therefore, we have $x \overset{S^L_{\theta_{r,s}}(X)}{\sim} y$ but not $x \overset{N_{\theta_{r,s}}(X)_L}{\sim} y$.

(ii). Suppose we have the double sequence $\left(\frac{x_{k,l}}{y_{k,l}}\right)$ in l^2_∞ and $x \overset{S^L_{\theta_{r,s}}(X)}{\sim} y$. Then we can assume that

$$\left|\frac{x_{k,l}}{y_{k,l}} - L\right| \leq H, \text{ for all } k \text{ and } l.$$

Given $\varepsilon > 0$

$$\begin{aligned}\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left|\frac{x_{k,l}}{y_{k,l}} - L\right| &= \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s} \ \& \ x_{k,l}-L\notin U} \left|\frac{x_{k,l}}{y_{k,l}} - L\right| \\ &+ \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s} \ \& \ x_{k,l}-L\in U} \left|\frac{x_{k,l}}{y_{k,l}} - L\right| \\ &\leq \frac{H}{h_{r,s}} \left|\left\{ (k,l) \in I_{r,s} : \ \frac{x_{k,l}}{y_{k,l}} - L \notin U\right\}\right| + \varepsilon.\end{aligned}$$

Therefore $x \overset{N_{\theta_{r,s}}(X)_L}{\sim} y$.

(iii). It follows from (i) and (ii). □

Theorem 2.

Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence with $\liminf_r q_r > 1$ and $\liminf_s \overline{q}_s > 1$ then $x \overset{S^L(X)}{\sim} y \Rightarrow x \overset{S^L_{\theta_{r,s}}(X)}{\sim} y$.

Proof:

Suppose that $\liminf_r q_r > 1$ and $\liminf_s \overline{q}_s > 1$ then there exists $\delta > 0$ such that $q_r > 1 + \delta$ and $\overline{q}_s > 1+\delta$. This implies that $\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$ and $\frac{\overline{h}_s}{l_s} \geq \frac{\delta}{1+\delta}$. Since $h_{r,s} = k_r l_s - k_r l_{s-1} - k_{r-1} l_s - k_{r-1} l_{s-1}$, we are granted the following

$$\frac{k_r l_s}{h_{r,s}} \leq \frac{(1+\delta)^2}{\delta^2} \text{ and } \frac{k_{r-1} l_{s-1}}{h_{r,s}} \leq \frac{1}{\delta}.$$

Let $x \overset{S^L(X)}{\sim} y$. We will prove $x \overset{\theta_{r,s}(X)_L}{\sim} y$. Let us take any neighbourhood U of 0. Then for sufficiently large r and s, we have

$$\begin{aligned}&\frac{1}{k_r l_s} \left|\left\{ (k,l) \in I_{r,s} : k \leq k_r \text{ and } l \leq l_s, \ \frac{x_{k,l}}{y_{k,l}} - L \notin U\right\}\right| \\ &\geq \frac{1}{k_r l_s} \left|\left\{ (k,l) \in I_{r,s} : \ \frac{x_{k,l}}{y_{k,l}} - L \notin U\right\}\right| \\ &\geq \frac{\delta^2}{(1+\delta)^2} . \frac{1}{h_{r,s}} \left|\left\{ (k,l) \in I_{r,s} : \ \frac{x_{k,l}}{y_{k,l}} - L \notin U\right\}\right|\end{aligned}$$

This completes the proof. □

Theorem 3.

Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence with $\limsup_r q_r < \infty$ and $\limsup_s \overline{q}_s < \infty$ then $x \overset{S^L_{\theta_{r,s}}(X)}{\sim} y \Rightarrow x \overset{S^L(X)}{\sim} y$.

Proof:

Since $\limsup_r q_r < \infty$ and $\limsup_s \overline{q}_s < \infty$ there exists $H > 0$ such that $q_r < H$ and $\overline{q}_s < H$ for all r and s. Let $x \overset{S^L_{\theta_{r,s}}(X)}{\sim} y$ and $\varepsilon > 0$. Then there exists $r_o > 0$ and $s_o > 0$ such that for every $i \geq r_o$ and $j \geq s_o$

$$B_{i,j} = \frac{1}{h_{i,j}} \left| \left\{ \ (k,l) \in I_{i,j} : \ \frac{x_{k,l}}{y_{k,l}} - L \notin U \right\} \right| < \varepsilon.$$

Let $M = \max\{B_{i,j} : \ 1 \leq i \leq r_o \text{ and } 1 \leq j \leq s_o\}$ and m and n be such that $k_{r-1} < m \leq k_r$ and $l_{s-1} < n \leq l_s$. Thus we obtain the following

$$\begin{aligned}
&\frac{1}{mn} \left| \left\{ (k,l) \in I_{i,j} : \ k \leq m \text{ and } l \leq n, \ \frac{x_{k,l}}{y_{k,l}} - L \notin U \right\} \right| \\
\leq& \frac{1}{k_{r-1} l_{s-1}} \left| \left\{ (k,l) \in I_{i,j} : \ k \leq k_r \text{ and } l \leq l_s, \ \frac{x_{k,l}}{y_{k,l}} - L \notin U \right\} \right| \\
\leq& \frac{1}{k_{r-1} l_{s-1}} \sum_{t,u}^{r_o,s_o} h_{t,u} B_{t,u} + \frac{1}{k_{r-1} l_{s-1}} \sum_{(r_o < t \leq r) \cup (s_o < u \leq s)} h_{t,u} B_{t,u} \\
\leq& \frac{M}{k_{r-1} l_{s-1}} \sum_{t,u}^{r_o,s_o} h_{t,u} + \frac{1}{k_{r-1} l_{s-1}} \sum_{(r_o < t \leq r) \cup (s_o < u \leq s)} h_{t,u} B_{t,u} \\
\leq& \frac{M k_{r_o} l_{s_o} r_o s_o}{k_{r-1} l_{s-1}} + \frac{1}{k_{r-1} l_{s-1}} \sum_{(r_o < t \leq r) \cup (s_o < u \leq s)} h_{t,u} B_{t,u} \\
\leq& \frac{M k_{r_o} l_{s_o} r_o s_o}{k_{r-1} l_{s-1}} + \left(\sup_{t \geq r_o \cup u \geq s_o} B_{t,u} \right) \frac{1}{k_{r-1} l_{s-1}} \sum_{(r_o < t \leq r) \cup (s_o < u \leq s)} h_{t,u} \\
\leq& \frac{M k_{r_o} l_{s_o} r_o s_o}{k_{r-1} l_{s-1}} + \frac{\varepsilon}{k_{r-1} l_{s-1}} \sum_{(r_o < t \leq r) \cup (s_o < u \leq s)} h_{t,u} \\
\leq& \frac{M k_{r_o} l_{s_o} r_o s_o}{k_{r-1} l_{s-1}} + \varepsilon H^2.
\end{aligned}$$

Since k_r and l_s both approach infinity as both m and n approach infinity, it follows that

$$\frac{1}{mn} \left| \left\{ (k,l) \in I_{i,j} : \ k \leq m \text{ and } l \leq n, \ \frac{x_{k,l}}{y_{k,l}} - L \notin U \right\} \right| \to 0.$$

This completes the proof. □

The following theorem is an immediate consequence of Theorem 2 and Theorem 3.

Theorem 4.

Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence with $1 < \liminf_{r,s} q_{rs} \leq \limsup_{r,s} q_{rs} < \infty$. Then $x \overset{S^L(X)}{\sim} y \Leftrightarrow x \overset{S^L_{\theta_{r,s}}(X)}{\sim} y$.

Theorem 5.

If the two double sequences $x = (x_{k,l})$ and $y = (y_{k,l})$ both satisfy $x \overset{S^{L_1}(X)}{\sim} y$ and $x \overset{S^L_{\theta_{r,s}}(X)}{\sim} y$, then $L_1 = L_2$.

Proof:

Let $x \overset{S^{L_1}(X)}{\sim} y$ and $x \overset{S^L_{\theta_{r,s}}(X)}{\sim} y$. Suppose that $L_1 \neq L_2$. Since X is a Hausdorff space, then there exists a symmetric neighbourhood U of 0 such that $L_1 - L_2 \notin U$. Then we may choose a symmetric neighbourhood W of 0 such that $W + W \subset U$. We obtain the following inequality:

$$\begin{aligned}&\frac{1}{k_m l_n}\left|\{(k,l) : k \leq k_m, l \leq l_n;\ z_{k,l} \notin U\}\right| \\ &\quad\leq \frac{1}{k_m l_n}\left|\left\{(k,l) : k \leq k_m, l \leq l_n;\ \frac{x_{k,l}}{y_{k,l}} - L_1 \notin W\right\}\right| \\ &\quad+ \frac{1}{k_m l_n}\left|\left\{(k,l) : k \leq k_m, l \leq l_n;\ L_2 - \frac{x_{k,l}}{y_{k,l}} \notin W\right\}\right|,\end{aligned}$$

where $z_{k,l} = L_1 - L_2$ for all $k, l \in \mathbf{N}$. It follows from this inequality that

$$\begin{aligned}1 \leq &\frac{1}{k_m l_n}\left|\left\{(k,l) : k \leq k_m, l \leq l_n;\ \frac{x_{k,l}}{y_{k,l}} - L_1 \notin W\right\}\right| \\ &+ \frac{1}{k_m l_n}\left|\left\{(k,l) : k \leq k_m, l \leq l_n;\ L_2 - \frac{x_{k,l}}{y_{k,l}} \notin W\right\}\right|.\end{aligned}$$

The second term on the right side of this inequality tends to 0 as $m, n \to \infty$ in Pringsheim sense. To see this, we write

$$\begin{aligned}&\frac{1}{k_m l_n}\left|\left\{(k,l) : k \leq k_m, l \leq l_n;\ L_2 - \frac{x_{k,l}}{y_{k,l}} \notin W\right\}\right| \\ &= \frac{1}{k_m l_n}\left|\left\{(k,l) \in \cup_{r,s=1,1}^{m,n} I_{r,s} :\ L_2 - \frac{x_{k,l}}{y_{k,l}} \notin W\right\}\right| \\ &= \frac{1}{k_m l_n}\sum_{r,s=1,1}^{m,n}\left|\left\{(k,l) \in I_{r,s} :\ L_2 - \frac{x_{k,l}}{y_{k,l}} \notin W\right\}\right| \\ &= \left(\sum_{r,s=1,1}^{m,n} h_{r,s}\right)^{-1}\left(\sum_{r,s=1,1}^{m,n} h_{r,s} t_{r,s}\right),\end{aligned}$$

where $t_{r,s} = \frac{1}{h_{r,s}}\left|\left\{(k,l) \in I_{r,s} :\ L_2 - \frac{x_{k,l}}{y_{k,l}} \notin W\right\}\right|$. Since $x \overset{S^L_{\theta_{r,s}}(X)}{\sim} y$, we know that $P{-}\lim_{r,s} t_{r,s} = 0$. So,

$$P - \lim_{m,n} \frac{1}{k_m l_n}\left|\left\{(k,l) : k \leq k_m, l \leq l_n;\ L_2 - \frac{x_{k,l}}{y_{k,l}} \notin W\right\}\right| = 0. \tag{1}$$

On the other hand, since $x \overset{S^{L_1}(X)}{\sim} y$,

$$P-\lim_{m,n}\frac{1}{k_m l_n}\left|\left\{(k,l): k\le k_m, l\le l_n;\ \frac{x_{k,l}}{y_{k,l}}-L_1\notin W\right\}\right|=0. \tag{2}$$

By (1) and (2) it follows that

$$P-\lim_{m,n}\frac{1}{k_m l_n}\left|\{(k,l): k\le k_m, l\le l_n;\ z_{k,l}\notin U\}\right|=0$$

which is a contradiction. This completes the proof. □

4. Conclusion

The definition of asymptotically equivalent sequences was introduced by Marouf in 1993. Later on it was further investigated from the sequence space point of view and linked with summability theory by several authors. The results obtained in this study are much more general than those obtained earlier.

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