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E-super vertex magic labelling of graphs and some open problems

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Abstract

Let G be a finite graph with p vertices and q edges. A vertex magic total labelling is a bijection from the union of the vertex set and the edge set to the consecutive integers 1, 2, 3, . . . , $p + q$ with the property that for every u in the vertex set, the sum of the label of u and the label of the edges incident with u *is* equal to k for some constant k. Such a labelling is E-super, if the labels of the edge set is the set $\{1, 2, 3, . . . , q\}$. A graph G is called E-super vertex magic, if it admits an E-super vertex magic labelling. In this paper, we establish an E-super vertex magic labelling of some classes of graphs and provide some open problems related to it.

Keywords: Super vertex magic labelling; *E*-super vertex magic labelling

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1. Introduction

In this paper, we consider only finite simple undirected graphs. The set of vertices and edges of a graph G will be denoted by $V(G)$ and $E(G)$ respectively, and we let $p = |V(G)|$ and $q = |E(G)|$. For graph theoretic notations we follow Harary (1969), and Marr and Wallis (2013). A labelling of graph G is a mapping that carries a set of graph elements, usually the vertices and edges into a set of numbers, usually integers. Many kinds of labellings have been studied and an excellent survey of graph labelling can be found in Gallian (2013).

Sedlacek (1963) introduced the concept of magic labelling. Suppose that G is a graph with q edges. We shall say that G is magic if the edges of G can be labeled by the numbers 1, 2, 3, . . . , q so that the sum of labels of all the edges incident with any vertex is the same. MacDougall et al. (2002) introduced the notion of vertex magic total labelling. If G is a finite simple undirected graph with p vertices and q edges, then a vertex magic total labelling is a bijection f from $V(G) \cup E(G)$ to the integers 1, 2, . . . , $p + q$ with the property that for every u in $V(G)$,

$$f(u) + \sum_{v \in N(u)} f(uv) = k,$$

for some constant k. They studied the basic properties of vertex magic graphs and showed some families of graphs having vertex magic total labelling.

MacDougall et al. (2004) further introduced the super vertex magic total labelling. They called a vertex magic total labelling is super, if $f(V(G)) = \{1, 2, . . . , p\}$. Swaminathan and Jeyanthi (2003) introduced a concept with the name super vertex magic labelling, but with different notion. They call a vertex magic total labelling is super, if $f(E(G)) = \{1, 2, . . . , q\}$. To avoid confusion, Marimuthu and Balakrishnan (2012) called a total labelling as an E-super vertex magic total labelling, if $f(E) = \{1, 2, . . . , q\}$. They studied some basic properties of such a labelling. There are number of papers dealing with E-super vertex magic labelling(also called super vertex magic total labelling or strong vertex magic total labelling), see for example, Gray (2006), Gray (2007), Gray and MacDougall (2009), Gray and MacDougall (2010), Gray and MacDougall (2012), Jeremy Holden, Dan McQuillan and James McQuillan (2009), Tao-Ming Wang and Guang-Hui Zhang (2014).

Theorem 1.1. (Swaminathan and Jeyanthi, 2003)

A path P_n is E-super vertex magic if and only if n is odd and $n \geq 3$.

Theorem 1.2. (Swaminathan and Jeyanthi, 2003)

If a non- trivial graph G is super vertex magic, then the magic constant k is given by $k = q + \frac{p+1}{2} + \frac{q(q+1)}{p}$.

Theorem 1.3. (Marimuthu and Balakrishnan, 2012)

If p is even, then every tree T is not E-super vertex magic.

In this paper, we find some families of E-super vertex magic graphs such as stars, spiders and brooms. Also we provide a labelling scheme for the graphs wounded suns $C_n^+ - 3e$ for all $n \geq 3$.

2. Some family of *E*-super vertex magic graphs

Definition 2.1.

A broom $B_{n,d}$ is defined by attaching $n - d$ pendent edges with any one of the pendent vertices of the path P_d.

Theorem 2.2.

The broom $B_{n,n-2}$ is *E*-super vertex magic if and only if *n* is odd.

Proof:

Let

$$V(B_{n,n-2}) = \{v_1, v_2, \ldots, v_{n-2}, u_1, u_2\}$$
$$E(B_{n,n-2}) = \{v_i v_{i+1} : 1 \le i \le n-3\} \cup \{v_{n-2}u_1, v_{n-2}u_2\}.$$

Therefore, $B_{n,n-2}$ has n vertices and *n*- 1 edges.

Suppose *n* is odd. Define a total labelling $f : V \cup E \to \{1, 2, \ldots, 2n-1\}$ as follows:

$$f(v_i) = 2n - i - 2,\ 1 \le i \le n-2$$
$$f(u_1) = 2n - 1\ ;\ f(u_2) = 2n - 2.$$

For 1 ≤ *i* ≤ *n*- 3, define the edge label as follows:

$$f(v_i v_{i+1}) = \begin{cases} \dfrac{i}{2}, & \text{if } i \text{ is even} \\ \dfrac{n+i+2}{2}, & \text{if } i \text{ is odd} \end{cases}$$

$$f(v_{n-2}u_1) = \frac{n-1}{2},\ f(v_{n-2}u_2) = \frac{n+1}{2}.$$

Clearly *f* is an *E*-super vertex magic total labelling. Conversely, suppose that $B_{n,n-2}$ is an *E*-super vertex magic graph. The magic constant *k* is given by

$$
\begin{aligned}
k &= q + \frac{p+1}{2} + \frac{q(q+1)}{p} \\
&= n - 1 + \frac{n+1}{2} + \frac{n-1(n-1+1)}{n} \\
&= n - 1 + \frac{n+1}{2} + n - 1 \\
&= 2n - 2 + \frac{n+1}{2},
\end{aligned}
$$

which is an integer only when n is odd.

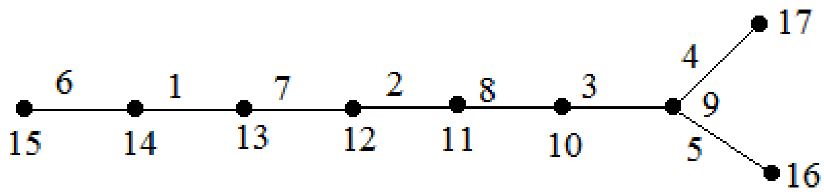

Figure 1. An E-super vertex labelling of $B_{9,7}$

Definition 2.3.

A subdivision of an edge uv in a graph is obtained by removing the edge uv, adding a new vertex w, and adding edges uw and vw. A spider S^n is the graph formed by subdividing all the edges of a star $K_{1,n}$ for $n \geq 1$ at once.

Theorem 2.4.

A spider S^n is E- super vertex magic if and only if $n \leq 4$.

Proof:

Let S^n be an E-super vertex magic graph. It is clear that S^n has $2n+1$ vertices and $2n$ edges. Then for any E-super vertex magic labelling of S^n, the magic constant k is given by

$$
\begin{aligned}
k &= q + \frac{p+1}{2} + \frac{q(q+1)}{p} \\
&= 2n + \frac{2n+2}{2} + \frac{2n(2n+1)}{2n+1} \\
&= 2n + n + 1 + 2n \\
&= 5n + 1.
\end{aligned}
$$

If $n > 5$, let v be the vertex of maximum degree d in S^n. If we assign the first d smallest labels to the edges incident with v, then the sum of label of v and the labels on the edges incident with v exceeds the magic constant $5n + 1$. Note that if we assign the labels to the edges other than $1, 2, \ldots, d$, then

$$f(u) + \sum_{v \in N(u)} f(uv) = k \text{ exceeds } 5n + 1.$$

Suppose that $n = 5$ and assume that S^n is E-super vertex magic. Then, there exists an E-super vertex magic labelling f for S^n. The magic constant k is $5(5) + 1 = 26$. Let u be the vertex of degree 5, let v_i $(1 \le i \le 5)$ be the vertices of degree 2 and let w_i $(1 \le i \le 5)$ be the vertices of degree 1 in S^n. $f(u) = 11$, $f(uv_i) = i, 1 \le i \le 5$. It can be easily verified that

$$f(u) + \sum_{i=1}^{5} f(uv_i) = 26.$$

It is clear that it is impossible to have another labelling to obtain the magic constant 26. From the definition, it is evident that 21 must be used for a vertex. If we assign 21 to any one of the vertices v_i $(1 \le i \le 5)$, k will exceed 26. If we assign 21 to any one of the vertices w_i $(1 \le i \le 5)$, then we must assign 5 to the edge incident with that vertex. But 5 has been used earlier. Thus this argument forces us to conclude that S^5 is not E-super vertex magic. Hence, S^n is not E-super vertex magic if $n \ge 5$.

If $n \le 4$, E-super vertex magic labelling of spider S^n is given in Figure 2. □

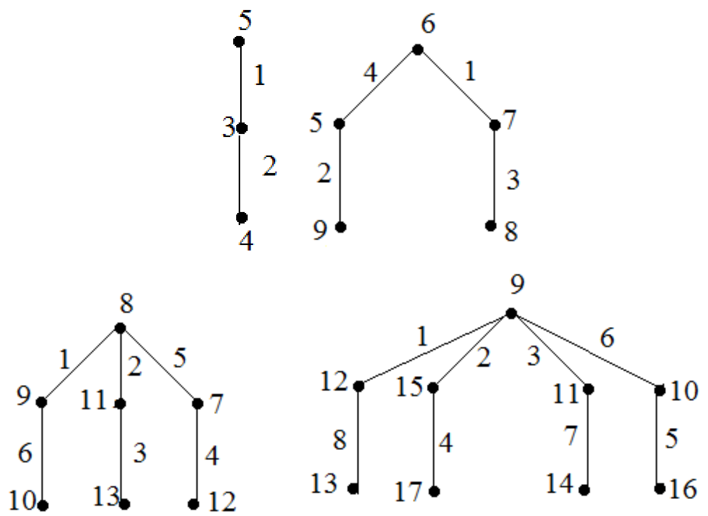

Figure 2. *E*-super vertex magic spiders

Theorem 2.5.

A graph $K_{1,t}$ is E- super vertex magic if and only if $t = 2$.

Proof:

By Theorem 1.1, $K_{1,2}$ is an E- super vertex magic graph. Conversely, assume that $K_{1,t}$ is E-super vertex magic. Suppose that either $t = 1$ or $t \ge 3$. When $t = 1$, $K_{1,t} = P_2$, which is not E-

super vertex magic, by Theorem 1. 2. Suppose that $t \geq 3$. Marimuthu and Balakrishnan (Marimuthu and Balakrishnan, 2012) proved that every tree of even order is not an *E*- super vertex magic. Therefore, $K_{1,t}$ is not an *E*- super vertex magic, when *t* is odd. Now we consider $K_{1,t}$ when *t* is even. Since $K_{1,t}$ is *E*- super vertex magic, the magic constant *k* is given by

$$\begin{aligned} k &= t + \frac{t+2}{2} + \frac{t(t+1)}{t+1} \\ &= t + \frac{t+2}{2} + t \\ &= 2t + \frac{t+2}{2}, \end{aligned}$$

which is an integer, since *t* is even.

Let *v* be a vertex of maximum degree *t*. If we assign the first *t* smallest integers to the edges incident with *v*, then the sum of the label of *v* and the edge labels exceeds the magic constant $2t + (t+2)/2$. Hence $K_{1,t}$ is not *E*- super vertex magic for $t \geq 3$ and therefore $t = 2$.

Definition 2.6.

A sun graph C_n^+ is defined as follows:

$$V(C_n^+) = \{v_1, v_2, ..., v_{2n}\}.$$
$$E(C_n^+) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n v_1\} \cup \{v_i v_{n+i} : 1 \leq i \leq n\}$$

The graph $C_n^+ - 3e$ is obtained by removing $\{v_{2n}, v_{2n-1}, v_{2n-2}\}$ and the edges adjacent to them from C_n^+. This is referred as a wounded sun.

Theorem 2.7.

A wounded sun is *E*-super vertex magic for all $n \geq 3$.

Proof:

Let $V(C_n^+ - 3e) = \{v_1, v_2, ..., v_n\} \cup \{v_{n+1}, v_{n+2}, ..., v_{2n-3}\}$ where v_{n+i}, $1 \leq i \leq n-3$ are pendent vertices. Let $E(C_n^+ - 3e) = \{v_i v_{i+1} : 1 \leq i \leq n\} \cup \{v_i v_{p-(i-1)} : 1 \leq i \leq n-3\}$. Therefore, $C_n^+ - 3e$ has $2n$-3 vertices and $2n$-3 edges.

Define a total labelling $f : V \cup E \rightarrow \{1, 2, ..., 4n-6\}$ as follows:

$$f(v_1) = q+1,$$
$$f(v_n) = 3n-3,$$
$$f(v_{n-i}) = 2n-2+i,\ 1 \leq i \leq n-2,$$

$f(v_{p-(i-1)}) = p+n+i,\ 1 \le i \le n\text{-}3,$

$f(v_{n-2}v_{n-1}) = q,$

$f(v_i v_{i+1}) = i,\ 1 \le i \le n\text{-}3,$

$f(v_{n-1}v_n) = n\text{-}2,$

$f(v_n v_1) = n\text{-}1,$

$f(v_i v_{p-(i-1)}) = p\text{-}i,\ 1 \le i \le n\text{-}3.$

Clearly f is an E-super vertex magic labelling with magic constant $k = 5n\text{-}\ 6$, from Theorem 1.2.

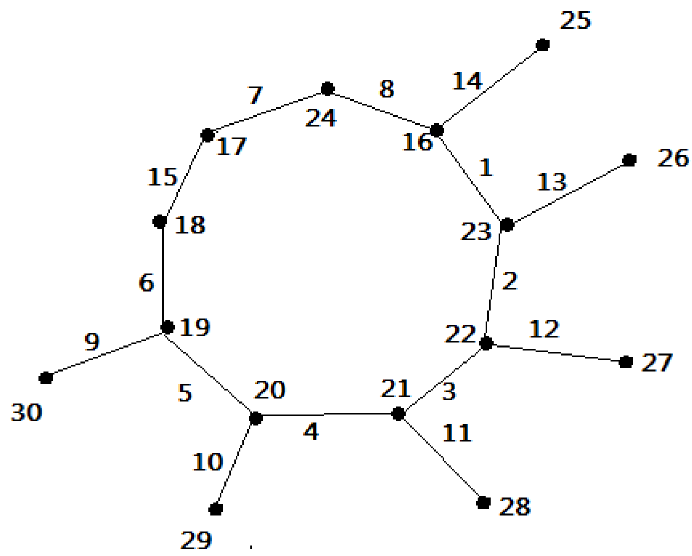

Figure 3. An E-super vertex magic labelling of $C_9^+ - 3e$

3. Conclusion and Scope

In this paper, we have found some E-super vertex magic trees such as stars, spiders and brooms. It is possible to find an E-super vertex magic labelling for trees of odd order. Thus we have the following problem.

Open problem 1.

Characterize all E-super vertex magic trees of odd order.

In Theorem 2.2, we have proved that the brooms $B_{n,n-d}$ are E-super vertex magic, where d = 2. It is possible to discuss the remaining cases.

Open problem 2.

Discuss the E-super vertex magicness of $B_{n,n-d}$ when $d \neq 2$.

Marimuthu and Balakrishnan (2012) showed that every sun graph C_n^+ is not *E*-super vertex magic for all $n \geq 3$. But the removal of edges from C_n^+ results in an *E*-super vertex magic graph from Theorem 2.7.

Open problem 3.

Find all *E*-super vertex magic wounded sun $C_n^+ - me$, $m \neq 3$.

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REFERENCES

Gallian, J.A. (2013). A dynamic survey of graph labelling, *Electron. J. Combin.* 16: #DS6.
Gray, I.D. (2006). New construction methods for vertex-magic total labelling of graphs, Ph.D.Thesis, University of Newcastle, Newcastle, Australia.
Gray, I.D. (2007). Vertex-Magic Total Labelling's of Regular Graphs, *SIAM J. Discrete Math.,* 21(1): 170–177.
Gray,I.D and MacDougall, J.A. (2010). Vertex-magic labelling of non-regular graphs, *Australas. J. Combin.,* 46 :173–183.
Gray, I.D and MacDougall, J.A. (2012). Vertex-magic labelling of regular graphs: Disjoint unions and assemblages, *Discrete Appl.Math.*, 160:1114– 1125.
Gray, I.D and MacDougall, J.A. (2009). Vertex-magic total labelling: mutations, *Australas. J. Combin.,* 45 :189–206.
Harary, F. (1969). *Graph Theory*, Addison- Wesley.
Jeremy Holden, Dan McQuillan and James McQuillan (2009). A conjecture on strong magic labelling of 2-regular graphs, *Discrete Math.,* 309 :4130–4136.
MacDougall, J.A, Miller, M, Slamin, and Wallis, W.D. (2002). Vertex magic total labelling of graphs, *Util. Math.,* 61: 3–21.
MacDougall, J.A, Miller, M, Sugeng, K.A. (2004). Super vertex magic total labelling of graphs, Proc. of the 16th Australian Workshop on Combinatorial Algorithms, pp. 222– 229.
Marimuthu, G and Balakrishnan, M. (2012). E-super vertex magic labelling of graphs, *Discrete Appl. Math.,* 160 :1766–1774.
Marr, M and Wallis, W.D. (2013). *Magic Graphs*, Birkhauser- Springer, Second edition, Boston.
Sedlacek, J., Problem 27. (1963).*Theory of Graphs and its Applications*, Proc. Symposium, pp. 163–167.
Swaminathan, Vand Jeyanthi, P. (2003). Super vertex magic labelling, *Indian J. Pure Appl. Math., 34 (6): 935–939.*
Tao-Ming Wang and Guang-Hui Zhang, (2014). Note on E-super vertex magic graphs, *Discrete Appl. Math.,* 178: 160–162.