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# Some Results on f-Simultaneous Chebyshev Approximation

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**Abstract**

Let $X$ be Hausdorff topological vector space and $f$ be a real valued continuous function on $X$. In this paper we introduce and study the concept of $f$-simultaneous approximation of a nonempty subset $K$ of $X$ as a generalization to the problem of simultaneous approximation. Further we present some results regarding $f$-simultaneous approximation in the quotient space.

**Keywords:** $f$-best simultaneous approximation; $f$-simultaneous Chebyshev approximation; $f$-simultaneously proximinal sets; $f$-quasi-simultaneously Chebyshev

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## 1. Introduction

Let $K$ be a subset of a Hausdorff topological vector space $X$ and $f$ be a real valued continuous function on $X$. For $x \in X$, set $F_K(x) = \inf\limits_{k \in K} f(x-k)$. A point $k_0 \in K$ is called $f$-best approximation to $x$ in $K$ if $F_K(x) = f(x - k_0)$. The set $P_K^f(x) = \{k_0 \in K : F_K(x) = f(x - \kappa_0)\}$ denotes the set of all $f$-best approximations to $x$ in $K$. Note that this set may be empty. The set $K$ is said to be $f$-proximinal ($f-$Chebyshev) if for each $x \in X$, $P_K^f(x)$ is non-empty (singleton). The notion of $f$-best approximation in a vector space $X$ was given by Breckner and Brosowski, and in a Hausdorff topological vector space $X$ by Narang. For a

Hausdorff locally convex topological vector space and a continuous sublinear functional $f$ on $X$, Breckner, Brosowski and Govindarajulu proved certain results on best approximation relative to the functional $f$. By using the existence of elements of $f$-best approximation some results on fixed point were proved by Pai and Veermani.

As a generalization to the problem of simultaneous approximation see Saidi and Singer, we introduce the concept of best $f$-simultaneous approximation as follows.

**Definition 1.**

Let $f$ be a real valued continuous function on a Hausdorff topological vector space $X$. A subset $A$ of $X$ is called $f$-bounded if there exists $M > 0$ such that $|f(x)| \leq M$ every $x \in A$.

Note that $f$-bounded set need not be bounded in the classical sense, for example if $f(x) = e^{-x}$, the set $[0, \infty)$ is an $f$-bounded subset of real numbers.

**Definition 2.**

Let $X$ be a Hausdorff topological real vector space, $f$ be a real valued continuous function on $X$ and $K$ be a non-empty subset of $X$. A point $k_0 \in K$ is called $f$-best simultaneous approximation in $K$ if there exists an $f$-bounded subset $A$ of $X$ such that

$$F_K(A) = \inf_{k\in K} \sup_{a\in A} |f(a-k)| = \sup_{a\in A} |f(a- k_0)| .$$

The set of all $f$-best simultaneous approximations to an $f$-bounded subset $A$ of $X$ in $K$ is denoted by

$$P_K^f(A) = \left\{ k \in K : F_K(A) = \sup_{a\in A} |f(a- k)| \right\} .$$

The set $K$ is called $f$-simultaneously proximinal $(f-\text{simultaneously Chebyshev})$ if for each $f$-bounded set $A$ in $X$, $P_K^f(A) \neq \phi$ $(\text{singleton})$.

We note that if $f(x) = \|x\|$ $(f(x) = \|x\| + \epsilon)$, then the concept of $f$-best approximation is precisely best approximation, (best $\epsilon$-approximation) (see Khalil, Rezapour, Singer, and others).

A set $K$ is said to be inf-compact at a point $x \in X$ (Pai and Veermani), if each minimizing sequence in $K$ (i.e. $f(x-k_n) \to F_K(x)$) has a convergent subsequence in $K$. The set $K$ is called inf-compact if it is inf-compact at each $x \in X$. A subset $K$ of $X$ is called $f$-compact (Moghaddam) if for every sequence $\{k_n\}$ in $K$, there exist a subsequence $\{k_{n_i}\}$ of $\{k_n\}$ and $k_0 \in K$ such that $f(k_{n_i} - k_0) \to 0$. It is easy to see that if $K$ is $f$-compact or $\inf$-compact, then $K$ is $f$-simultaneously proximinal.

In this paper we introduce and study the concept of $f$-simultaneous approximation of a subspace $K$ of a Hausdorff topological real vector space $X$, existence and uniqueness. Certain results regarding $f$-simultaneous approximation in quotient spaces is obtained by generalizing some of the results in Moghaddam.

Throughout this paper $X$ is a Hausdorff topological real vector space and $f$ is a real valued continuous function on $X$.

## 2. $f$-Simultaneous Approximation

In this section we give some characterization of $f$-proximinal sets in $X$. We begin by the following definitions:

**Definition 3.**

A function $f : X \to \mathbb{R}$ is called

(1) absolutely subadditive if $|f(x+y)| \le |f(x)| + |f(y)|$ for all $x, y \in X$.

(2) absolutely homogeneous if $f(\alpha x) = |\alpha|\, f(x)$, for all $x \in X$ and all $\alpha \in \mathbb{R}$.

**Definition 4.**

A subset $K$ of $X$ is called $f$-closed if for all sequences $\{k_m\}$ of $K$ and for all $x \in X$ such that $f\,(x - k_m) \to 0$ we have $x \in K$.

**Theorem 5.**

Let $K$ be a subset of $X$. Then,

(1) $F_{K+y}(A+y) = F_K(A)$, for all $f$-bounded sets $A \subset X$, $y \in X$.

(2) $P^f_{K+y}(A+y) = P^f_K(A) + y$, for all $f$-bounded sets $A \subset X$, $y \in X$.

(3) $K$ is $f$-simultaneously proximinal $(f$-simultaneously Chebyshev$)$ if and only if $K + y$ is $f$-simultaneously proximinal $(f$-simultaneously Chebyshev$)$ for every $y \in X$.

More over if $f$ is absolutely homogeneous function, then

(4) $F_{\lambda K}(\lambda A) = |\lambda|\, F_K(A)$, for all $f$-bounded sets $A \subset X$ and $\lambda \in \mathbb{R}$.

(5) $P^f_{\lambda K}(\lambda A) = \lambda P^f_K(A)$, for all $f$-bounded sets $A \subset X$ and $\lambda \in \mathbb{R}$.

(6) $K$ is $f$-simultaneously proximinal $(f$-simultaneously Chebyshev$)$ if and only if $\lambda K$ is $f$-simultaneously proximinal $(f$-simultaneously Chebyshev$)$, $\lambda \in \mathbb{R}$.

***Proof:***

(1) Let $A \subset X$, $f$-bounded set. Then

$$F_{K+y}(A+y) = \inf_{w \in K} \sup_{a \in A} |f\,((a+y) - (w+y))| = F_K(A).$$

(2) The equation

$$\sup_{a \in A} |f(a - k_0)| = \inf_{k \in K} \sup_{a \in A} |f((a+y) - (k+y))| = \inf_{k \in K} \sup_{a \in A} |f(a-k)|\,,$$

implies that $k_0 + y \ \in P^f_{K+y}(A+y)$ if and only if $k_0 \in P^f_K(A)$. Thus,

$$P^f_{K+y}(A+y) = P^f_K(A) + y.$$

(3) Follows immediately from part two.

(4) Let $A \subset X$, be $f$-bounded set, $\lambda \in \mathbb{R}$. Then,

$$F_{\lambda K}(\lambda A) = \inf_{k\in K}\sup_{a\in A} |f(\lambda a - \lambda k)| = |\lambda| \inf_{k\in K}\sup_{a\in A} |f(a-k)| = |\lambda| \, F_K(A).$$

(5) If $\lambda = 0$, we are done. If $\lambda \neq 0$ and $k_0 \in P^f_{\lambda K}(\lambda A)$, then $k_0 \in \lambda K$ and

$$\sup_{a\in A} |f(\lambda a - k_0)| = \inf_{k\in K}\sup_{a\in A} |f(\lambda a - \lambda k)| \, .$$

This implies that

$$\sup_{a\in A} \left| f(a - \frac{1}{\lambda}k_0) \right| = F_K(A),$$

which implies that $\frac{1}{\lambda}k_0 \in P^f_K(A)$.

(6) Follows immediately from part 5. $\square$

**Theorem 6.**

Let $f$ be an absolutely homogeneous real valued function on $X$ and $M$ be a subspace of $X$. Then

(1) $F_M(\lambda A) = |\lambda| \, F_M(A)$, for all for all $f$-bounded sets $A \subset X$ and $\lambda \in \mathbb{R} - \{0\}$.

(2) $P^f_M(\lambda A) = \lambda P^f_M(A)$, for all for all $f$-bounded sets $A \subset X$ and $\lambda \in \mathbb{R} - \{0\}$.

***Proof:***

(1) Let $A \subset X$ be an $f$-bounded set and $\lambda \neq 0 \in \mathbb{R}$. Then,

$$F_M(\lambda A) = \inf_{m\in M}\sup_{a\in A} |f(\lambda a - m)| = |\lambda| \inf_{m'\in M}\sup_{a\in A} \left| f(a - m') \right| = |\lambda| \, F_M(A).$$

(2) Let $\ m_0 \in P^f_M(\lambda A)$. Then,

$$
\begin{aligned}
\sup_{a\in A}|\lambda|\left|f(a-\frac{1}{\lambda}m_0)\right| &= \sup_{a\in A}|f(\lambda a-m_0)| \\
&= \inf_{m\in M}\sup_{a\in A}|f(\lambda a-m)| \\
&= \inf_{m'\in M}\sup_{a\in A}\ |\lambda|\ \left|f(a-m^{'})\right|.
\end{aligned}
$$

Therefore,

$$
\sup_{a\in A}\left|f(a-\frac{1}{\lambda}m_0)\right| = \inf_{m'\in M}\sup_{a\in A}\left|f(a-m^{'})\right| \ = F_M(A),
$$

for all $\lambda\in\mathbb{R}-\{0\}$, which implies that $\frac{1}{\lambda}m_0\in P_M^f(A)$, and so $m_0\in\lambda\ P_M^f(A)$. $\square$

For a subset $K$ of $X$, let us define $\widehat{K_F}$ be such that

$$
\widehat{K_F}=\left\{A\subset X : F_K(A)=\sup_{a\in A}f(a)\right\}.
$$

Using this we prove the following theorem characterizing $f$-simultaneously proximinal sets.

**Theorem 7.**

Let $K$ be a subspace of $X$. Then $K$ is $f$-simultaneously proximinal in $X$ if and only if every $f$-bounded subset $A$ of $X$ can be written as $B+k$ for some $k\in K$ and $B\in\widehat{K_F}$.

***Proof:***

Suppose the condition hold. Let $A\subset X$ be an $f$-bounded subset of $X$. By assumption there exists $k_0\in K$ and $B\in\widehat{K_F}$ such that $A=B+k_0$. Hence $A-k_0\in\widehat{K_F}$. Therefore,

$$
\begin{aligned}
\sup_{a\in A}|f(a-k_0)| &= F_K(A-k_0) \\
&= \inf_{k\in K}\sup_{a\in A}|f(a-k_0-k)| \\
&= \inf_{k'\in K}\sup_{a\in A}|f(a-k')| = F_K(A).
\end{aligned}
$$

Hence $K$ is $f$-simultaneously proximinal.

Conversely, suppose $K$ is $f$-simultaneously proximinal and $A\subset X$ be an $f$-bounded subset of $X$. Then there exists $k_0\in K$ such that

$$
\sup_{a\in A}|f(a-k_0)| = \inf_{k\in K}\sup_{a\in A}|f(a-k)| = \inf_{k'\in K}\sup_{a\in A}\left|f(a-(k^{'}+k_0))\right|.
$$

where $k = k' + k_0$. Hence,

$$\sup_{a\in A} |f(a-k_0)| = F_K(A-k_0).$$

Consequently $A - k_0 \in \widehat{K_F}$. So there exists $B \in \widehat{K_F}$ such that $A - k_0 = B$ or $A = B + k_0$. $\square$

**Theorem 8.**

Let $f$ be a real valued continuous function on $X$ such that $x = 0$ iff $f(x) = 0$. If $K$ is $f$-simultaneously proximinal, then $K$ is $f$-closed.

***Proof:***

Let $\{k_m\}$ be a sequence of $K$ and $x \in X$, such that $f(x - k_m) \to 0$. Taking $A = \{x\}$, we have

$$F_K(A) = \inf_{k\in K} \sup_{a\in A} |f(a-k)| \leq |f(x-k_m)| \to 0.$$

Since $K$ is $f$-simultaneously proximinal, there exists $k_0 \in K$ such that

$$F_K(A) = |f(x- k_0)| = 0.$$

Hence, $f(x - k_0) = 0$. Using assumption it follows that $x - k_0 = 0$. Hence $x = k_0 \in K$ and $K$ is $f$-closed. $\square$

## 3. $f$-Simultaneous Approximation in Quotient Space

Let $M$ a closed subspace of $X$. Then a function $\widetilde{f} : (X/M) \to \mathbb{R}$ can be defined as follows:

$$\widetilde{f}(x+M) = \inf_{y\in M} |f(x+y)|.$$

**Proposition 9.**

Let $M$ a closed subspace of $X$. If $A$ is $f$-bounded in $X$, then $A/M$ is $\widetilde{f}$-bounded in $X/M$.

***Proof:***

Let $A$ be an $f$-bounded subset in $X$. Since $M$ is a subspace, for $x + M \in A/M$

$$\left|\widetilde{f}(x+M)\right| = \inf_{y\in M} |f(x+y)| \leq |f(x)|.$$

Consequently since $A$ is an $f$-bounded subset of $X$, it follows that $A/M$ is $\widetilde{f}$-bounded in $X/M$. $\square$

**Theorem 10.**

Let $M$ a closed subspace of $X$. If $B$ is$\widetilde{f}$-bounded in $X/M$, then there exists an $f$-bounded subset $A$ of $X$ such that $B = A/M$.

***Proof:***

Let $B$ be a non empty $\widetilde{f}$-bounded in $X/M$. Let $C = \bigcup\limits_{b\in B} b$. Claim $B = \{\overline{x} = x + M : x \in C\}$. Indeed if $b \in B$, then $b = x_b + M$ for some $x_b \in X$. But $M$ is a subspace. Thus $x_b = x_b + 0 \in x_b + M \subset C$. Hence $b = x_b + M \in \{\overline{x} = x + M : x \in C\}$ and $B \subseteq \{\overline{x} = x + M : x \in C\}$. Similarly if $x \in C$, then $x \in b_x + M$ for some $b_x + M \in B$. This implies that $x = b_x + m_x$ for some $m_x \in M$. Hence $x + M = b_x + m_x + M = b_x + M \in B$. Therefore $\{\overline{x} = x + F : x \in C\} \subseteq B$.

Now clearly $C$ is not bounded unless $M$ is trivial. Note that $B$ is $\widetilde{f}-$bounded. So there exists $K > 0$ such that $\left|\widetilde{f}(b)\right| \leq K$ for all $b \in B$. Consider the set $A = \{x \in C : |f(x)| \leq K + 1\} \subseteq C$. Now we claim that for all $x \in C$,

$$\overline{x} \cap A = (x + M) \cap A \neq \phi.$$

Given $x \in C$. Since

$$\left|\widetilde{f}(x + M)\right| = \inf_{m\in M} |f\,(x + m)| \leq K,$$

there exists $m_x \in M$ such that $|f\,(x + m_x)| < K+1$. But $x+m_x \in x+M \subseteq C$. Hence $x+m_x \in (x + M) \cap A \neq \phi$. Claim $B = A/M$. Since $A \subseteq C$, we have $A/F \subseteq \{\overline{x} = x + F : x \in C\} = B$. To show the other inclusion, let $b \in B = \{\overline{x} = x + M : x \in C\}$. Then $b = x_b + M$ for some $x_b \in C$. But $(x_b + M) \cap A \neq \phi$. Thus there exists $a \in A$ such that $a = x_b + m_a \in x_b + M$. Therefore $b = x_b + M = (x_b + m_a) + M = a + M \in A/M$. Hence $B \subseteq A/M$. Consequently $A/M = B$. $\square$

**Theorem 11.**

Let $K$ be a subspace of $X$ and $M$ be a closed $f$-proximinal subspace of $K$. If $k_0$ is a point of $f$-best simultaneous approximation to $A \subset X$ in $K$, then $k_\circ + M$ is an $\widetilde{f}$-best simultaneous approximation to $A/M$ in $K/M$.

***Proof:***

Suppose $k_0 + M$ is not $\widetilde{f}$-best simultaneous approximation to $A/M$ in $K/M$. Then for at least $k \in K$, say $k_1 \in K$, we have

$$\sup_{a\in A}\widetilde{f}(a - k_1 + M) < \sup_{a\in A}\widetilde{f}(a - k_0 + M).$$

Since

$$\sup_{a\in A}\widetilde{f}(a - k_0 + M) = \sup_{a\in A}\inf_{m\in M} |f(a - k_0 + m)| \leq \sup_{a\in A} |f(a - k_0)|,$$

we have

$$\sup_{a\in A}\widetilde{f}(a-k_1+M)=\sup_{a\in A}\inf_{m\in M}|f(a-k_1+m)|<\sup_{a\in A}|f(a-k_0)|.$$

But $M$ is $f$-proximinal, so for some $m_0\in M$ we have

$$\sup_{a\in A}|f(a-k_1+m_0)|=\sup_{a\in A}\inf_{m\in M}|f(a-k_1+m)|<\sup_{a\in A}|f(a-k_0)|.$$

Since $M\subset K$, it follows that $k_1-m_0\in K$. Therefore $k_0$ not $f$-best simultaneous approximation to $A$ in $K$, which is a contradiction. □

**Corollary 12.**

Let $K$ be a subspace of $X$ and $M$ is a closed $f$-proximinal subspace of $K$. If $K$ is $f$-simultaneously proximinal in $X$, then $K/M$ is $\widetilde{f}$-simultaneously proximinal in $X/M$.

***Proof:***

Let $B$ be an $\widetilde{f}$-bounded subset of $X/M$. Then by Theorem 10, there exists $f$-bounded subset $A\subset X$ such that $B=A/M$. If $K$ is $f-$simultaneously proximinal in $X$, then there exists at least $k_0\in K$ such that $k_0$ is $f$-best simultaneous approximation to $A$ in $K$. By Theorem 11, $k_0+M$ is an $\widetilde{f}$-best simultaneous approximation to $A/M$ in $K/M$, so $K/M$ is $\widetilde{f}$-simultaneously proximinal in $X/M$. □

**Theorem 13.**

Let $K$ be a subspace of $X$ and $M$ is a closed $f$-proximinal subspace of $K$. If $K/M$ is $\widetilde{f}$-simultaneously proximinal in $X/M$, then $K$ is $f$-simultaneously proximinal in $X$.

***Proof:***

Let $A$ be an $f$- bounded subset of $X$. By Proposition 9, $A/M$ is $\widetilde{f}-$bounded in $X/M$. Since $K/M$ is $\widetilde{f}$-simultaneously proximinal in $X/M$, then there exists $k_0+M\in K/M$ such that $k_0+M$ is $\widetilde{f}$-best simultaneous approximation to $A/M$ from $K/M$, so

$$\begin{aligned}\sup_{a\in A}\widetilde{f}(a-k_0+M) &= \inf_{k\in K}\sup_{a\in A}\widetilde{f}(a-k+M)\\ &= \inf_{k\in K}\sup_{a\in A}\inf_{m\in M}|f(a-k+m)|\\ &\leq \inf_{k\in K}\sup_{a\in A}|f(a-k+m)|\\ &= \inf_{k\in K}\sup_{a\in A}|f(a-k')|, \qquad (1)\end{aligned}$$

where, $k'=k-m\in K$. Since $M$ is $f$-proximinal, there exists $m_0\in M$ such that

$$\sup_{a\in A}|f(a-k_0-m_0)| = \sup_{a\in A}\inf_{m\in M}|f(a-k_0+m)| = \sup_{a\in A}\widetilde{f}(a-k_0+M). \tag{2}$$

Consequently, combining (1) and (2) since $M \subset K$, it follows that

$$\begin{aligned}\sup_{a\in A}|f(a-k_0-m_0)| &\leq \inf_{k'\in K}\sup_{a\in A}|f(a-k')|\\ &\leq \sup_{a\in A}|f(a-k_0-m_0)|\end{aligned}$$

Hence,

$$\sup_{a\in A}|f(a-k_0+m_0)| = \inf_{k\in K}\sup_{a\in A}|f(a-k')|$$

So $k_0+m_0$ is an $f$-best simultaneous approximation to $A$ from $K$ and $K$ is $f$-simultaneously proximinal in $X$. □

**Theorem 14.**

Let $W$ and $M$ be two subspaces of $X$. If $M$ is a closed $f$-proximinal subspace of $X$, then the following assertions are equivalent:
(1) $W/M$ is $\widetilde{f}$-simultaneously proximinal in $X/M$,
(2) $W+M$ is $f$-simultaneously proximinal in $X$.

***Proof:***

$(1) \Rightarrow (2)$. Since $(W+M)/M = W/M$ and $M$ are $f$-simultaneously proximinal, using Theorem 13, it follows that $W+M$ is $f$-simultaneously proximinal in $X$.

$(2) \Rightarrow (1)$. Since $W+M$ is $f$-simultaneously proximinal and $M \subseteq W+M$, by Corollary 12, $(W+M)/M = W/M$ is simultaneously $f$-proximinal. □

**Theorem 15.**

Let $K, M$ be two subspaces of $X$ such that, $M \subset K$. If $M$ is closed $f$-simultaneously proximinal in $X$ and $K$ is $f$-simultaneously Chebyshev in $X$, then $K/M$ is $\widetilde{f}$-simultaneously Chebyshev in $X/M$.

***Proof:***

Suppose not. Then there exists $A$, $f$-bounded subset of $X$ such that $A/M \in X/M$ is $\widetilde{f}$-bounded and $k_1+M$, $k_2+M \ \in P^{\widetilde{f}}_{K/M}(A/M)$ such that $k_1+M \neq k_2+M$. Thus $k_1-k_2 \notin M$. Since $M$ is an $f$-simultaneously proximinal in $X$, then

$$P^f_M(A-k_1) \neq \phi, \text{ and } P^f_M(A-k_2) \neq \phi.$$

Let $m_1 \in P_M^f(A-k_1)$, and $m_2 \in P_M^f(A-k_2)$. By Theorem 13, $k_1+ m_1$ and $k_2+m_2$ are $f$-best simultaneous approximations to $A$ from $K$. Since $K$ is $f$-simultaneously Chebyshev in $X$, then $k_1 + m_1 = k_2+m_2$ and hence $k_1-k_2 = m_1-m_2 \in M$, which is a contradiction. □

**Definition 16.**

A subset $K$ of $X$ is called $f$-quasi-simultaneously Chebyshev if $P_K^f(A)$ is non empty and $f$-compact set in $X$, for all $f$-bounded subsets of $X$.

**Theorem 17.**

Let $M$ be a closed $f$-simultaneously proximinal subspace of $X$ and $K$ is $f$-quasi-simultaneously Chebyshev of $X$ such that $M \subset K$. Then $K/M$ is $\widetilde{f}$-quasi-simultaneously Chebyshev in $X/M$.

***Proof:***

Since $K$ is $f$-simultaneously proximinal in $X$, By Corollary 12, $K/M$ is $\widetilde{f}$-simultaneously proximinal in $X/M$. Let $B$ be an $\widetilde{f}-$bounded subset of $X/M$. Then, by Theorem 10, $B = A/M$ for an $f-$bounded subset $A$ of $X$. If $(k_n+M)$ a sequence in $P_{K/M}^{\widetilde{f}}(A/M)$, by the proof of Theorem 13, for every $n$, there exists $m_n \in M$ such that $k_n+m_n = k'_n \in P_K^f(A)$. But since $M$ is a subspace, we have

$$k'_n + M = k_n + m_n + M = k_n + M.$$

Since $K$ is $f$-quasi-simultaneously Chebyshev in $X$, the sequence $\{k_n\}$ has a subsequence $\{k_{ni}\}$ such that $f(k_{ni}-k_0) \to 0$ for some $k_0 \in P_K^f(A)$. But

$$\widetilde{f}(k_{ni}-k_0+M) \leq |f(k_{ni}-k_0)| \to 0.$$

Therefore,

$$\widetilde{f}(k_{ni}-k_0+M) \to 0$$

and

$$\widetilde{f}((k_{ni}+M)-(k_0+M)) \to 0.$$

Hence, $P_{K/M}^{\widetilde{f}}(A/M)$ is $\widetilde{f}$-compact and $K/M$ is $\widetilde{f}$-quasi-simultaneously Chebyshev. This complete the proof. □

**Definition 18.**

A topological vector space $X$ is said to have the $f$- property if every $f$-bounded sequence in $X$ has an $f$-convergent subsequence, where $f$ is a real valued continuous function on $X$.

Note that the space $X = l^2$ has the $f$-property for every projection $f : X \to \mathbb{R}$, and if $f(x) = \|x\|$ , then every finite dimensional Banach space has the $f$-property.

**Proposition 19.**

Let $f$ be an absolutely homogeneous subadditive continuous real valued function on a topological vector space $X$ and $K$ be an $f$-closed subspace of $X$. Then for any $f$-bounded subset $A$ of $X$, $P^f_K(A)$ is $f$-closed.

***Proof:***

Let $K$ be an $f$-closed subspace of $X$ and $A$ be an $f$-bounded subset of $X$. If $\{k_m\}$ is a sequence in $P^f_K(A)$ and $x \in X$ such that $f(k_m - x) \to 0$, then $x \in K$ since $K$ is $f$-closed. Further

$$\begin{aligned}\inf_{k\in K}\sup_{a\in A}|f(a-k)| &= \sup_{a\in A}|f(a-k_m)| \\ &= \sup_{a\in A}|f((a-x)-(k_m-x))| \\ &\geq \sup_{a\in A}||f\left(a-x\right)|-|f\left(k_m-x\right)||\,.\end{aligned}$$

Taking the limit as $m \to \infty$, we get

$$\inf_{k\in K}\sup_{a\in A}|f(a-k)| \geq \sup_{a\in A}|f\left(a-x\right)|\,.$$

Consequently,

$$\inf_{k\in K}\sup_{a\in A}|f(a-k)| = \sup_{a\in A}|f\left(a-x\right)|\,.$$

Hence $x \in P^f_K(A)$ and $P^f_K(A)$ is $f$-closed. $\square$

**Theorem 20.**

Let $f$ be a real valued sub-additive continuous function on a topological vector space $X$ that has the $f$-property and $M$ be a closed subspace of $X$. If $W$ is a subspace of $X$ such that $W + M$ is $f$-closed, then the following assertions are equivalent:
(1) $W/M$ is $\widetilde{f}$-simultaneously quasi-Chebyshev in $X/M$.
(2) $W + M$ is $f$-simultaneously quasi-Chebyshev in $X$.

***Proof:***

(1) $\Rightarrow$ (2) Since $M$ is $f$-simultaneously proximinal by Theorem 14, $W + M$ is $f$-simultaneously proximinal in $X$. Let $A$ be an arbitrary $f-$bounded set in $X$. Then $P^f_{W+M}(A) \neq \phi$. Now to show that $P^f_{W+M}(A)$ is $f$-compact, we need to show that every sequence in $P^f_{W+M}(A)$ has an $f$-convergent subsequence. Let $\{g_n\}_{n=1}^{\infty}$ be an arbitrary sequence in $P^f_{W+M}(A)$. Then by Theorem 11, for each $n > 1$, $g_n + M \in P^{\widetilde{f}}_{(W+M)/M}\left(A/M\right)$. Since $P^{\widetilde{f}}_{(W+M)/M}\left(A/M\right)$ is $\widetilde{f}$-compact, one

can choose $g_0 \in W+M$ with $g_0+M \in P^{\widetilde{f}}_{(W+M)/M}(A/M)$ and $\{g_{n_k}+M\}_{k=1}^{\infty}$ is $\widetilde{f}-$convergent to $g_0+M$ for some subsequence $\{g_{n_k}+M\}_{k=1}^{\infty}$ of $\{g_n+M\}_{n=1}^{\infty}$. That means,

$$\widetilde{f}\left(g_0-g_{n_k}+M\right)=\inf_{m\in M}\left|f\left(g_0-g_{n_k}-m\right)\right|\to 0.$$

Now, since $M$ is $f$-proximinal in $X$, there exists $m_{n_k}\in M$ such that $m_{n_k}\in P^f_M\left(g_0-g_{n_k}\right)$, for every $k\geq 1$, and hence

$$\left|f\left(g_0-g_{n_k}-m_{n_k}\right)\right|=\inf_{m\in M}\left|f\left(g_0-g_{n_k}-m\right)\right|.$$

Therefore,

$$\lim_{k\to\infty} f\left(g_0-g_{n_k}-m_{n_k}\right)=0.$$

On the other hand, $\{g_{n_k}\}_{k=1}^{\infty}$ is an $f$-bounded sequence because $g_n\in P^f_{W+M}(A)$. In fact $\left|f\left(g_n\right)\right|\leq 2\sup_{a\in A}\left|f(a)\right|$. Since $M$ has the $f$-property, with out loss of generality, we may assume that for some $m_0\in M$, $f\left(m_{n_k}-m_0\right)\to 0$. Let $g'=g_0-m_0$. Then, $g'\in W+M$ and

$$\begin{aligned} f\left(g'-g_{n_k}\right) &= f\left(g_0-m_0-g_{n_k}\right)\\ &\leq f\left(g_0-g_{n_k}-m_{n_k}\right)+f\left(m_{n_k}-m_0\right), \end{aligned}$$

$\forall\ k\geq 1$. Thus $\lim\limits_{k\to\infty} f\left(g'-g_{n_k}\right)=0$. Since $\{g_{n_k}\}_{k=1}^{\infty}\in P^f_{W+M}(A)$, for every $k\geq 1$, and $P^f_{W+M}(A)$ is $f$-closed, since $W+M$ is $f$-closed by Proposition 19, we conclude that $g'\in P^f_{W+M}(A)$. Hence $P^f_{W+M}(A)$ is $f-$compact.

(2) $\Rightarrow$ (1) Since $M$ and $W+M$ are $f$-simultaneously proximinal and $M\subseteq W+M$, then $(W+M)/M=W/M$ is $\widetilde{f}$-simultaneously proximinal in $X/M$.

Now, let $A$ be an arbitrary $f$-bounded set in $X$. Then $P^{\widetilde{f}}_{W/M}(A/M)$ is non-empty. So from the hypothesis we have $W+M$ is $f-$simultaneously quasi-Chebyshev in $X$, and hence $P^f_{W+M}(A)$ is $f$-compact in $X$. Using Theorem 11, we conclude that

$$P^{\widetilde{f}}_{(W+M)/M}(A/M)=\pi\left(P^f_{W+M}(A)\right),$$

where $\pi: X\to X/M$, $\pi(x)=x+M$, is continuous. Consequently $P^{\widetilde{f}}_{W/M}(A/M)$ is $\widetilde{f}$-compact. Therefore, $W/M$ is $f$-simultaneously quasi-Chebyshev in $X$. $\square$

Note that Theorem 20 is still true if the restriction $W+M$ is $f$-closed is replaced by the condition that the function $f(x)=0$ if and only if $x=0$ and use Theorem 8 to prove that $W+M$ is $f$-closed.

## 4. Conclusions

In this paper we introduce and study the concept of $f$-simultaneous approximation of a nonempty subset $K$ of Hausdorff topological vector space $X$, existence and uniqueness as a generalization to the problem of simultaneous approximation in the sense that if the function $f$ is taken to be the usual norm, the problem is turned out to be precisely the problem of best approximation in the usual sense. Further we obtain some results regarding $f$-simultaneous approximation in the quotient space.

REFERENCES

Breckner, W. W. and Brosowski, B. (1971). Ein kriterium zur charakter isierung von sonnen, Mathematika, Vol. 13, pp. 181-188.

Govindarajulu, P. and Pai, D. V. (1980) . On properties of sets related to $f$-projections, J. Math. Anal. Appl., Vol. 73, pp. 457-465.

Khalil, R. and Abushamla, W. (2003). $f$-proximinality in function spaces, Dirasat, Vol. 30, pp. 93-96.

Moghaddam, M. A. (2010) . On $f$-best approximation in quotient topological vector spaces, Int. Math. Forum., Vol. 12, pp. 587-595.

Narang, T. D. (1985) . On $f$-best approximation in topological spaces, Arch. Math., Vol. 4, pp. 229-234.

Narang, T. D. (1986) . Approximation relative to an ultra function, Arch. Math., Vol. 4, pp. 181-186.

Pai, D. V. and Veermani, P. (1982) . Applications of fixed point theorems in optimization and best approximation, Nonlinear Anal. Appl. Marcel Dekker, Int. New York, Edited by S.P. Singh and J. H. Barry, pp. 393-400.

Rezapour, Sh. (2003) . $\epsilon$-weakly Chebyshev subspaces of Banach spaces, Anal. Theory Appl., Vol. 19, No. 2, pp. 130-135.

Rezapour, Sh. and Mohebi, H. (2003) . $\epsilon$-weakly Chebyshev subspaces and qoutient spaces, Bulletin of the Iranian Math. Soc., Vol. 29, No. 2, pp. 27-33.

Saidi, F., Hussein, D. and Khalil, R. (2002). Best simultaneous approximation in $L^p(I, X)$, Approx. Theory J., Vol. 116, pp. 369-379.

Singer, I. (1970) . *Best approximation in normed linear spaces by elements of linear subspaces*, Springer-Verlag.