On the Stability of a Three Species Syn-Eco-System with Mortality Rate for the Third Species

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Abstract

The system comprises of a commensal (S_1) common to two hosts S_2 and S_3 with mortality rate for the host (S_3). Here all the three species posses limited resources. The model equations constitute a set of three first order non-linear simultaneous coupled differential equations. Criteria for the asymptotic stability of all the eight equilibrium states are established. Trajectories of the perturbations over the equilibrium states are illustrated. Further the global stability of the system is established with the aid of suitably constructed Liapunov's function and the numerical solutions for the growth rate equations are computed using Runge-Kutta fourth order scheme.

1. Introduction

Ecology is a branch of life and environment sciences which asserts the existence of diverse species in the same environment and habitat. It is natural that two or more species living in a common habitat interact in different ways. The Ecological interactions can be broadly classified as Ammensalism, Competition, Commensalism, Neutralism, Mutualism, Predation Parasitism and so on. Lotka (1925) and Volterra (1931) pioneered theoretical ecology significantly and

opened new eras in the field of life and biological sciences. The general concepts of modeling have been discussed by several authors [Colinvaux (1986), Kapur (1985), Kushing (1977), Meyer (1985)]. Srinivas (1991) studied competitive ecosystem of two species and three species with limited and unlimited resources. Later, Narayan et al. (2007) studied prey-predator ecological models with partial cover for the prey and alternate food for the predator. Stability analysis of competitive species was carried out by [Reddy et al. (2007), Sharma et al. (2008)], while Ravindra Reddy (2008) investigated mutualism between two species. Acharyulu et al. (2010, 2011) derived some productive results on various mathematical models of ecological Ammensalism with multifarious resources in the manifold directions. Further Kumar (2010) studied some mathematical models of ecological commensalism. The present author Prasad et al. (2011, 2012, and 2013) investigated on the stability of three and four species syn-ecosystems.

The present investigation is on an analytical study of three species (S_1, S_2, S_3) syn-eco system with mortality rate for the host (S_3). The system comprises of a commensal (S_1), two hosts S_2 and S_3, i.e., S_2 and S_3 both benefit S_1, without getting themselves affected either positively or adversely. Further S_2 is a commensal of S_3 and S_3 is a host of both S_1, S_2.

Commensalism is a symbiotic interaction between two populations where one population (S_1) gets benefit from (S_2) while the other (S_2) is neither harmed nor benefited due to the interaction with (S_1). The benefited species (S_1) is called the commensal and the other (S_2) is called the host. Some real-life examples of commensalism are presented below.

i.   Sucker fish (echeneis) gets attached to the under surface of sharks by its sucker. This provides easy transport for new feeding grounds and also food pieces falling from the sharks prey, to Echeneis.

ii.   A squirrel in an oak tree gets a place to live and food for its survival, while the tree remains neither benefited nor harmed.

iii. A flatworm attached to the horse crab and eating the crab's food, while the crab is not put to any disadvantage.

2. Basic Equations of the Model

The model equations for the three species syn ecosystem is given by the following system of first order non-linear ordinary differential equations employing the following notation:

Notation Adopted

$N_i(t)$: The population strength of S_i at time t, $i = 1, 2, 3$

t: Time instant

d_3:   Natural death rate of S_3

a_i: Natural growth rate of S_i, $i = 1, 2$

a_{ii}:   Self inhibition coefficients of S_i, $i = 1, 2, 3$

a_{12}, a_{13} : Interaction coefficients of S_1 due to S_2 and S_1 due to S_3

a_{23} : Interaction coefficient of S_2 due to S_3

$e_3 = \dfrac{d_3}{a_{33}}$: Extinction coefficient of S_3

$k_i = \dfrac{a_i}{a_{ii}}$: Carrying capacities of S_i, $i = 1, 2$

Further the variables N_1, N_2, N_3 are non-negative and the model parameters $a_1, a_2, d_3, a_{11}, a_{12},$ $a_{22}, a_{33}, a_{13}, a_{23}$ are assumed to be non-negative constants.

The model equations for the growth rates of S_1, S_2, S_3 are:

$$\frac{dN_1}{dt} = N_1 \left(a_1 - a_{11}N_1 + a_{12}N_2 + a_{13}N_3 \right), \tag{2.1}$$

$$\frac{dN_2}{dt} = N_2 \left(a_2 - a_{22}N_2 + a_{23}N_3 \right), \tag{2.2}$$

and

$$\frac{dN_3}{dt} = - N_3 \left(d_3 + a_{33}N_3 \right). \tag{2.3}$$

3. Equilibrium States

The system under investigation has eight equilibrium states given by

$$\frac{dN_i}{dt} = 0, i = 1, 2, 3.$$

(i) Fully washed out state.

$E_1 : \overline{N}_1 = 0, \overline{N}_2 = 0, \overline{N}_3 = 0$.

(ii) States in which only one of the tree species is survives while the other two are not.

$E_2 : \overline{N}_1 = 0, \overline{N}_2 = 0, \overline{N}_3 = -e_3$,
$E_3 : \overline{N}_1 = 0, \overline{N}_2 = k_2, \overline{N}_3 = 0$,
$E_4 : \overline{N}_1 = k_1, \overline{N}_2 = 0, \overline{N}_3 = 0$.

(iii) States in which only two of the tree species are survives while the other one is not.

$$E_5 : \bar{N}_1 = 0, \bar{N}_2 = k_2 - \frac{a_{23}e_3}{a_{22}}, \bar{N}_3 = -e_3,$$

$$E_6 : \bar{N}_1 = k_1 - \frac{a_{13}e_3}{a_{11}}, \bar{N}_2 = 0, \bar{N}_3 = -e_3,$$

$$E_7 : \bar{N}_1 = k_1 + \frac{a_{12}k_2}{a_{11}}, \bar{N}_2 = k_2, \bar{N}_3 = 0.$$

(iv)   The co-existent state (or) normal steady state.

$$E_8 : \bar{N}_1 = \left(k_1 + \frac{a_{12}k_2}{a_{11}} \right) - \left(a_{13} + \frac{a_{12}a_{23}}{a_{22}} \right) \frac{e_3}{a_{11}}, \bar{N}_2 = k_2 - \frac{a_{23}e_3}{a_{22}}, \bar{N}_3 = -e_3.$$

4.   Stability of the Equilibrium States

Let

$$N = (N_1, N_2, N_3) = \bar{N} + U, \tag{4.1}$$

where $U = (u_1, u_2, u_3)^T$ is a small perturbation over the equilibrium state $\bar{N} = (\bar{N}_1, \bar{N}_2, \bar{N}_3)$.

The basic equations (2.1), (2.2) and (2.3) are quasi-linearized to obtain the equations for the perturbed state as,

$$\frac{dU}{dt} = AU, \tag{4.2}$$

with

$$A = \begin{bmatrix} a_1 - 2a_{11}\bar{N}_1 + a_{12}\bar{N}_2 + a_{13}\bar{N}_3 & a_{12}\bar{N}_1 & a_{13}\bar{N}_1 \\ 0 & a_2 - 2a_{22}\bar{N}_2 + a_{23}\bar{N}_3 & a_{23}\bar{N}_2 \\ 0 & 0 & -d_3 - 2a_{33}\bar{N}_3 \end{bmatrix}. \tag{4.3}$$

The characteristic equation for the system is

$$\det [A - \lambda I] = 0. \tag{4.4}$$

The equilibrium state is stable, if all the roots of the equation (4.4) are negative in case they are real or have negative real parts, in case they are complex.

4.1. Fully washed out state

In this case, we have

$$A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & -d_3 \end{bmatrix}. \tag{4.5}$$

The characteristic equation is

$$(\lambda - a_1)(\lambda - a_2)(\lambda + d_3) = 0, \tag{4.6}$$

The characteristic roots of (4.6) are $a_1, a_2, -d_3$. Since two of these three are positive. Hence, the fully washed out state is **unstable** and the solutions of the equations (4.2) are:

$$u_i = u_{i0} \, e^{a_i t}; \; u_3 = u_{30} \, e^{-d_3 t}, \quad i = 1, 2, \tag{4.7}$$

where u_{10}, u_{20}, u_{30} are the initial values of u_1, u_2, u_3, respectively.

Trajectories of perturbations

The trajectories in $u_1 - u_2$ and $u_2 - u_3$ planes are

$$\left(\frac{u_1}{u_{10}} \right)^{\frac{1}{a_1}} = \left(\frac{u_2}{u_{20}} \right)^{\frac{1}{a_2}} = \left(\frac{u_3}{u_{30}} \right)^{-\frac{1}{d_3}}. \tag{4.8}$$

4.2. Equilibrium state $E_2 : \bar{N}_1 = 0, \, \bar{N}_2 = 0, \, \bar{N}_3 = -e_3$

In this case, we have

$$A = \begin{bmatrix} a_1 - a_{13}e_3 & 0 & 0 \\ 0 & a_2 - a_{23}e_3 & 0 \\ 0 & 0 & d_3 \end{bmatrix}. \tag{4.9}$$

The characteristic roots are: $a_1 - a_{13}e_3$, $a_2 - a_{23}e_3$ and d_3. Since one of these three roots is positive, hence the state is **unstable**. The equations (4.2) yield the solutions,

$$u_1 = u_{10}e^{(a_1 - a_{13}e_3)t}; u_2 = u_{20} \, e^{(a_2 - a_{23}e_3) \, t}; u_3 = u_{30}e^{d_3 t}. \tag{4.10}$$

Trajectories of perturbations

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are given by

$$\left(\frac{u_1}{u_{10}} \right)^{\frac{1}{a_1 - a_{13}e_3}} = \left(\frac{u_2}{u_{20}} \right)^{\frac{1}{a_2 - a_{23}e_3}} = \left(\frac{u_3}{u_{30}} \right)^{\frac{1}{d_3}}. \tag{4.11}$$

4.3. Equilibrium state $E_3 : \bar{N}_1 = 0, \, \bar{N}_2 = k_2, \, \bar{N}_3 = 0$

In this case, we have

$$A = \begin{bmatrix} a_1 + a_{12}k_2 & 0 & 0 \\ 0 & -a_2 & a_{23}k_2 \\ 0 & 0 & -d_3 \end{bmatrix}. \tag{4.12}$$

The characteristic roots are: $a_1 + a_{12}k_2$, $-a_2$ and $-d_3$. Since one of these three roots is positive, hence the state is **unstable**. The equations (4.2) yield the solution curves,

$$u_1 = u_{10}e^{(a_1 + a_{12}k_2)t}; u_2 = (u_{20} - \gamma_2)e^{-a_2 t} + \gamma_2 e^{-d_3 t}; u_3 = u_{30}e^{-d_3 t}, \tag{4.13}$$

where

$$\gamma_2 = \frac{a_{23}k_2 u_{30}}{a_2 - d_3} \; ; a_2 \neq d_3. \tag{4.14}$$

Trajectories of perturbations

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are given by

$$u_2 = (u_{20} - \gamma_2)\left(\frac{u_1}{u_{10}}\right)^{\frac{-a_2}{a_1 + a_{12}k_2}} + \gamma_2 \left(\frac{u_1}{u_{10}}\right)^{\frac{-d_3}{a_1 + a_{12}k_2}} \; ; u_2 = (u_{20} - \gamma_2)\left(\frac{u_3}{u_{30}}\right)^{\frac{a_2}{d_3}} + \frac{u_3\gamma_2}{u_{30}}. \tag{4.15}$$

4.4. Equilibrium state $E_4 : \bar{N}_1 = k_1, \bar{N}_2 = 0, \bar{N}_3 = 0$

In this case, we get

$$A = \begin{bmatrix} -a_1 & a_{12}k_1 & a_{13}k_1 \\ 0 & a_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}. \tag{4.16}$$

The characteristic roots are: $-a_1, a_2, d_3$. Since two of these three roots are positive, hence the state is **unstable**. The equations (4.2) yield the solutions,

$$u_1 = (u_{10} - m_1 - m_2)e^{-a_1 t} + m_1 e^{a_2 t} + m_2 e^{d_3 t}; u_2 = u_{20}e^{a_2 t}; u_3 = u_{30}e^{d_3 t}, \tag{4.17}$$

where

$$m_1 = \frac{a_{12}k_1 u_{20}}{a_1 + a_2} > 0; m_2 = \frac{a_{13}k_1 u_{30}}{a_1 + d_3} > 0. \tag{4.18}$$

Trajectories of perturbations

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are given by

$$u_1 = \left(u_{20} - m_1 - m_2\right)\left(\frac{u_2}{u_{20}}\right)^{\frac{-a_1}{a_2}} + \frac{u_2 m_1}{u_{20}} + m_2\left(\frac{u_2}{u_{20}}\right)^{\frac{d_3}{a_2}} \text{ and } \left(\frac{u_2}{u_{20}}\right)^{d_3} = \left(\frac{u_3}{u_{30}}\right)^{a_2}. \qquad (4.19)$$

4.5. Equilibrium state $E_5 : \bar{N}_1 = 0, \bar{N}_2 = k_2 - \dfrac{a_{23}e_3}{a_{22}}, \bar{N}_3 = -e_3$

In this case, we get

$$A = \begin{bmatrix} a_1 + a_{12}k_2 - b_1 & 0 & 0 \\ 0 & a_{23}e_3 - a_2 & a_{23}k_2 - \dfrac{a_{23}^2 e_3}{a_{22}} \\ 0 & 0 & d_3 \end{bmatrix}, \qquad (4.20)$$

where

$$b_1 = a_{13}e_3 + \frac{a_{12}a_{23}e_3}{a_{22}} > 0. \qquad (4.21)$$

The characteristic roots are: $a_1 + a_{12}k_2 - b_1$, $a_{23}e_3 - a_2$ and d_3. Since one of these three roots is positive, hence the state is **unstable**. The equations (4.2) yield the solution curves,

$$u_1 = u_{10}e^{(a_1 + a_{12}k_2 - b_1)t}; u_2 = \left(u_{20} - b_2\right)e^{(a_{23}e_3 - a_2)t} + b_2 e^{d_3 t}; u_3 = u_{30}e^{d_3 t}, \qquad (4.22)$$

where

$$b_2 = \frac{a_{23}\left(a_{23}e_3 - a_2\right)u_{30}}{a_{22}\left[a_{23}e_3 - \left(a_2 + d_3\right)\right]} ; a_{23}e_3 \neq \left(a_2 + d_3\right). \qquad (4.23)$$

Trajectories of perturbations

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are given by

$$u_2 = \left(u_{20} - b_2\right)\left(\frac{u_1}{u_{10}}\right)^{\frac{a_{23}e_3 - a_2}{a_1 + a_{12}k_2 - b_1}} + b_2\left(\frac{u_1}{u_{10}}\right)^{\frac{d_3}{a_1 + a_{12}k_2 - b_1}} ; u_2 = \left(u_{20} - b_2\right)\left(\frac{u_3}{u_{30}}\right)^{\frac{a_{23}e_3 - a_2}{d_3}} + \frac{u_3 b_2}{u_{30}}. \qquad (4.24)$$

4.6. Equilibrium state $E_6 : \bar{N}_1 = k_1 - \dfrac{a_{13}e_3}{a_{11}}, \bar{N}_2 = 0, \bar{N}_3 = -e_3$

In this case, we get

$$A = \begin{bmatrix} a_{13}e_3 - a_1 & a_{12}k_1 - \dfrac{a_{12}a_{13}e_3}{a_{11}} & a_{13}k_1 - \dfrac{a_{13}^2 e_3}{a_{11}} \\ 0 & a_2 - a_{23}e_3 & 0 \\ 0 & 0 & d_3 \end{bmatrix}. \tag{4.25}$$

The characteristic roots are: $a_{13}e_3 - a_1$, $a_2 - a_{23}e_3$ and d_3. Since one of these three roots is positive, hence the state is **unstable**. The equations (4.2) yield the solution curves,

$$u_1 = \left(u_{10} - \phi_1 - \phi_2\right) e^{(a_{13}e_3 - a_1)t} + \phi_1 e^{(a_2 - a_{23}e_3)t} + \phi_2 e^{d_3 t}; \, u_2 = u_{20} e^{(a_2 - a_{23}e_3)t}; \, u_3 = u_{30} e^{d_3 t}, \tag{4.26}$$

where

$$\phi_1 = \frac{a_{12}\left(a_{13}e_3 - a_1\right)u_{20}}{a_{11}\left[e_3\left(a_{13} + a_{23}\right) - \left(a_1 + a_2\right)\right]} \, ; \phi_2 = \frac{a_{13}\left(a_{13}e_3 - a_1\right)u_{30}}{a_{11}\left[a_{13}e_3 - \left(a_1 + d_3\right)\right]}, \tag{4.27}$$

with

$$e_3\left(a_{13} + a_{23}\right) \neq \left(a_1 + a_2\right) \, ; a_{13}e_3 \neq \left(a_1 + d_3\right).$$

Trajectories of perturbations

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are

$$u_1 = \left(u_{10} - \phi_1 - \phi_2\right)\left(\frac{u_2}{u_{20}}\right)^{\frac{a_{13}e_3 - a_1}{a_2 - a_{23}e_3}} + \frac{\phi_1 u_2}{u_{20}} + \phi_2 \left(\frac{u_2}{u_{20}}\right)^{\frac{d_3}{a_2 - a_{23}e_3}} \, ; \, \left(\frac{u_2}{u_{20}}\right)^{d_3} = \left(\frac{u_3}{u_{30}}\right)^{a_2 - a_{23}e_3}. \tag{4.28}$$

4.7. Equilibrium state $E_7 : \bar{N}_1 = k_1 + \dfrac{a_{12}k_2}{a_{11}}, \, \bar{N}_2 = k_2, \, \bar{N}_3 = 0$

In this case, we get

$$A = \begin{bmatrix} -\left(a_1 + a_{12}k_2\right) & k_1 a_{12} + \dfrac{a_{12}^2 k_2}{a_{11}} & k_1 a_{13} + \dfrac{a_{12}a_{13}k_2}{a_{11}} \\ 0 & -a_2 & a_{23}k_2 \\ 0 & 0 & -d_3 \end{bmatrix}. \tag{4.29}$$

The characteristic roots are: $-\left(a_1 + a_{12}k_2\right), -a_2$ and $-d_3$, and these are all negative, hence the state is **stable**. The equations (4.2) yield the solutions:

$$
\left.\begin{array}{l}
u_1 = \left(u_{10} - \sigma_1 - \sigma_2\right)e^{-\left(a_1 + a_{12}k_2\right)t} + \sigma_1 e^{-a_2 t} + \sigma_2 e^{-d_3 t}; \\[2mm]
u_2 = \left(u_{20} - \sigma_3\right)e^{-a_2 t} + \sigma_3 e^{-d_3 t} \; ; u_3 = u_{30}e^{-d_3 t},
\end{array}\right\}
\tag{4.30}
$$

where

$$
\sigma_1 = \frac{a_{12}\left(a_1 + a_{12}k_2\right)\left(u_{20} - \sigma_3\right)}{a_{11}\left(a_1 + a_{12}k_2 - a_2\right)} \; ; \sigma_2 = \frac{\left(a_1 + a_{12}k_2\right)\left(a_{12}\sigma_3 + a_{13}u_{30}\right)}{a_{11}\left(a_1 + a_{12}k_2 - d_3\right)} ; \sigma_3 = \frac{a_{23}k_2 u_{30}}{a_2 - d_3} \quad . \tag{4.31}
$$

It can be noticed that $u_1 \to 0, u_2 \to 0$ and $u_3 \to 0$ as $t \to \infty$.

Trajectories of perturbations

The trajectories in $u_1 - u_3$ and $u_2 - u_3$ planes are given by

$$
u_1 = \left(u_{10} - \sigma_1 - \sigma_2\right)\left(\frac{u_3}{u_{30}}\right)^{\frac{a_1 + a_{12}k_2}{d_3}} + \sigma_1 \left(\frac{u_3}{u_{30}}\right)^{\frac{a_2}{d_3}} + \frac{\sigma_2 u_3}{u_{30}} \; ; u_2 = \left(u_{20} - \sigma_3\right)\left(\frac{u_3}{u_{30}}\right)^{\frac{a_2}{d_3}} + \frac{\sigma_3 u_3}{u_{30}}. \tag{4.32}
$$

4.8. The normal steady state $E_8\left(\overline{N}_1, \overline{N}_2, \overline{N}_3\right)$

In this case, we get

$$
A = \begin{bmatrix}
\tau_1 & -\dfrac{a_{12}\tau_1}{a_{11}} & -\dfrac{a_{13}\tau_1}{a_{11}} \\[3mm]
0 & a_{23}e_3 - a_2 & a_{23}k_2 - \dfrac{a_{23}^2 e_3}{a_{22}} \\[3mm]
0 & 0 & d_3
\end{bmatrix},
\tag{4.33}
$$

where

$$
\tau_1 = \tau - \left(a_1 + a_{12}k_2\right) \; ; \; \tau = a_{13}e_3 + \frac{a_{12}a_{23}e_3}{a_{22}} > 0. \tag{4.34}
$$

The characteristic roots are; $\tau_1, a_{23}e_3 - a_2$ and d_3. Since one of these three roots is positive, hence the state is **unstable**. The equations (4.2) yield the solution curves,

$$
\left.\begin{array}{l}
u_1 = \left(u_{10} - \psi_1 - \psi_2\right)e^{\tau_1 t} + \psi_1 e^{\left(a_{23}e_3 - a_2\right)t} - \psi_2 e^{d_3 t}; \\[2mm]
u_2 = \left(u_{20} - \psi_3\right)e^{\left(a_{23}e_3 - a_2\right)t} + \psi_3 e^{d_3 t}; u_3 = u_{30}e^{d_3 t},
\end{array}\right\}
\tag{4.35}
$$

where

$$\psi_1 = \frac{a_{12}\tau_1(\psi_3 - u_{20})}{a_{11}(a_{23}e_3 - a_2 - \tau_1)}, \psi_2 = \frac{\tau_1(a_{12}\psi_3 + a_{13}u_{30})}{a_{11}(\tau_1 - d_3)}, \psi_3 = \frac{a_{23}(a_{23}e_3 - a_2)u_{30}}{a_{22}(a_{23}e_3 - a_2 - d_3)}, \qquad (4.36)$$

with

$$a_{23}e_3 \neq a_2 + \tau_1 ; \tau_1 \neq d_3 \; ; a_{23}e_3 \neq a_2 + d_3 .$$

Trajectories of perturbations

Trajectories in $u_1 - u_3$ and $u_2 - u_3$ planes are given by

$$\left.\begin{aligned}
u_1 &= (u_{10} - \psi_1 - \psi_2)\left(\frac{u_3}{u_{30}}\right)^{\frac{\tau_1}{d_3}} + \psi_1\left(\frac{u_3}{u_{30}}\right)^{\frac{a_{23}e_3 - a_2}{d_3}} + \frac{u_3\psi_2}{u_{30}}, \\
u_2 &= (u_{20} - \psi_3)\left(\frac{u_3}{u_{30}}\right)^{\frac{a_{23}e_3 - a_2}{d_3}} + \frac{u_3\psi_3}{u_{30}}.
\end{aligned}\right\} \qquad (4.37)$$

5.   Liapunov's function for global stability

In section 4 we discussed the local stability of all eight equilibrium states. From which only the state, $E_7(\bar{N}_1, \bar{N}_2, 0)$ is **stable** and rest of them are **unstable**. We now examine the global stability of dynamical system (2.1), (2.2) and (2.3) at this state by suitable Liapunov's function.

Theorem.

The equilibrium state $E_7\left(k_1 + \frac{a_{12}k_2}{a_{11}}, k_2, 0\right)$ is globally asymptotically stable.

Proof :

Let us consider the following Liapunov's function

$$V(N_1, N_2) = N_1 - \bar{N}_1 - \bar{N}_1 \ln\left(\frac{N_1}{\bar{N}_1}\right) + l_1\left[N_2 - \bar{N}_2 - \bar{N}_2 \ln\left(\frac{N_2}{\bar{N}_2}\right)\right], \qquad (5.1)$$

where l_1 is a suitable constant to be determined as in the subsequent steps.

Now, the time derivative of V, along with solutions of (2.1) and (2.2) can be written as

$$\frac{dV}{dt} = \left(\frac{N_1 - \bar{N}_1}{N_1}\right)\frac{dN_1}{dt} + l_1\left(\frac{N_2 - \bar{N}_2}{N_2}\right)\frac{dN_2}{dt}, \tag{5.2}$$

$$= (N_1 - \bar{N}_1)(a_1 - a_{11}N_1 + a_{12}N_2) + l_1(N_2 - \bar{N}_2)(a_2 - a_{22}N_2),$$

$$= -a_{11}(N_1 - \bar{N}_1)^2 + a_{12}(N_1 - \bar{N}_1)(N_2 - \bar{N}_2) + l_1\left[-a_{22}(N_2 - \bar{N}_2)^2\right],$$

$$\frac{dV}{dt} = -\left[\sqrt{a_{11}}(N_1 - \bar{N}_1) + \sqrt{l_1 a_{22}}(N_2 - \bar{N}_2)\right]^2 - \left(2\sqrt{l_1 a_{11} a_{22}} - a_{12}\right)(N_1 - \bar{N}_1)(N_2 - \bar{N}_2). \tag{5.3}$$

The positive constant l_1 is so chosen that the coefficient of $(N_1 - \bar{N}_1)(N_2 - \bar{N}_2)$ in (5.3) is to vanish.

Then, we have $l_1 = \dfrac{a_{12}^2}{4a_{11}a_{22}} > 0$ and, with this choice of the constant l_1,

$$V(N_1, N_2) = N_1 - \bar{N}_1 - \bar{N}_1 \ln\left(\frac{N_1}{\bar{N}_1}\right) + \frac{a_{12}^2}{4a_{11}a_{22}}\left[N_2 - \bar{N}_2 - \bar{N}_2 \ln\left(\frac{N_2}{\bar{N}_2}\right)\right]. \tag{5.4}$$

$$\frac{dV}{dt} = -\left[\sqrt{a_{11}}(N_1 - \bar{N}_1) - \frac{a_{12}}{2\sqrt{a_{11}}}(N_2 - \bar{N}_2)\right]^2, \tag{5.5}$$

which is negative definite. Hence, the state is **globally asymptotically stable.**

6. A numerical approach of the growth rate equations

The numerical solutions of the growth rate equations (2.1), (2.2) and (2.3) computed employing the fourth order Runge-Kutta method for specific values of the various parameters that characterize the model and the initial conditions. The results are illustrated in Figures 6.1 to 6.6.

Example 1. Let $a_1 = 3.51, a_2 = 0.504, d_3 = 0.432, a_{11} = 2.538, a_{22} = 0.252, a_{33} = 0.972, a_{12} = 0.432,$
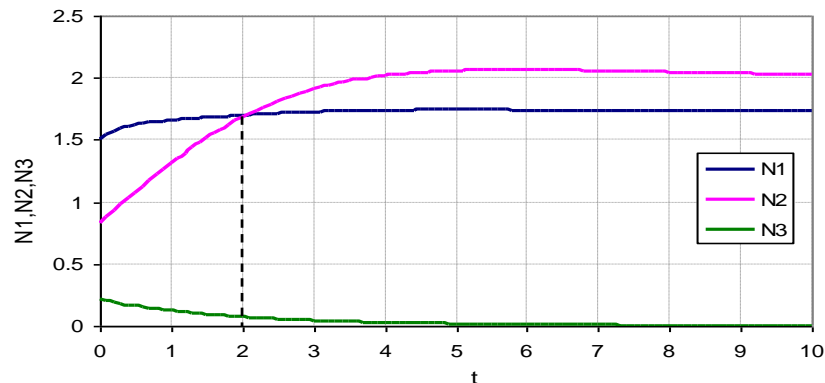$a_{13} = 1.273, a_{23} = 1.35$

Figure 6.1. Variation of N_1, N_2, N_3 against time (t) for $N_{10} = 1.5, N_{20} = 0.83, N_{30} = 0.22$
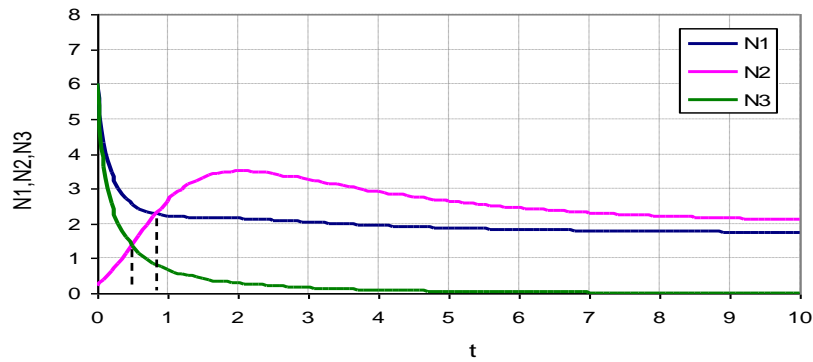
Figure 6.2. Variation of N_1, N_2, N_3 against time (t) for $N_{10} = 6, N_{20} = 0.2, N_{30} = 6$

Example 2. Let $a_1 = 0.43, a_2 = 14.184, d_3 = 2.45, a_{11} = 7.452, a_{22} = 12.798, a_{33} = 0.378, a_{12} = 2.59,$
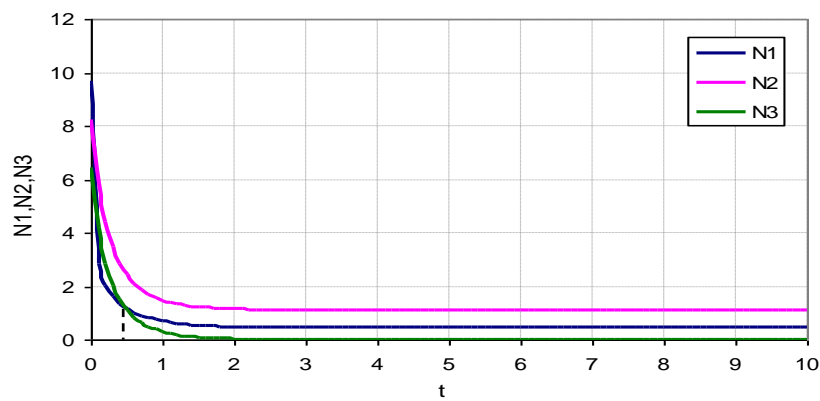$a_{13} = 0.288, a_{23} = 13.842$

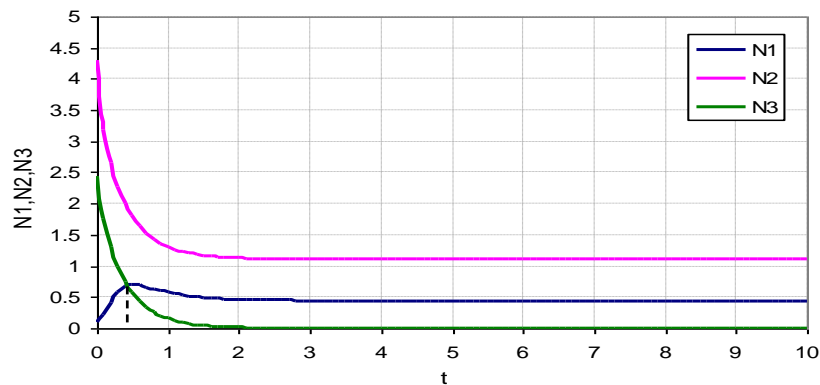**Figure 6.3.** Variation of N_1, N_2, N_3 against time (t) for $N_{10} = 9.684, N_{20} = 8.244, N_{30} = 6.444$

Figure 6.4. Variation of N_1, N_2, N_3 against time (t) for $N_{10} = 0.09, N_{20} = 4.288, N_{30} = 2.448$

Example 3. Let $a_1 = 0.756, a_2 = 2.78, d_3 = 0.28, a_{11} = 13.39, a_{22} = 3.474, a_{33} = 0.504, a_{12} = 1.404,$
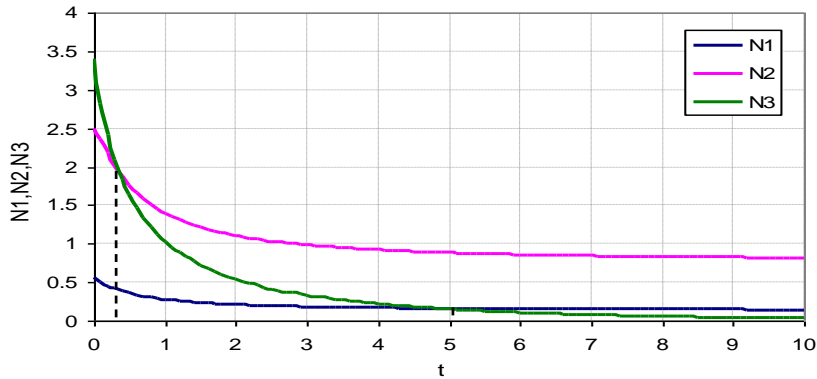$a_{13} = 0.558, a_{23} = 1.656$

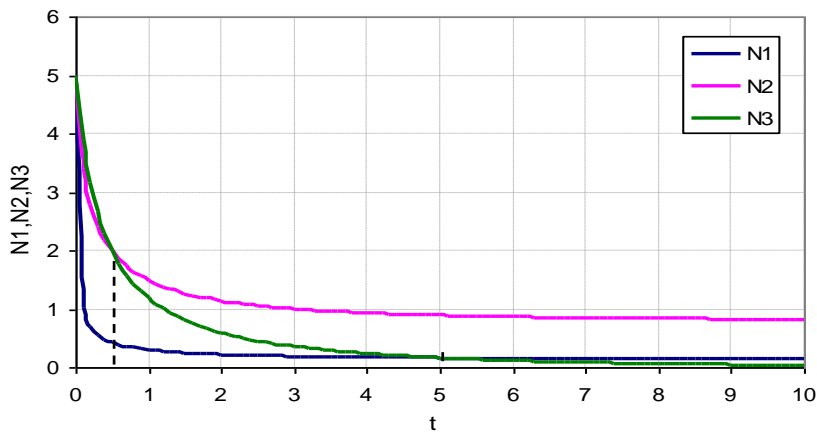**Figure 6.5.** Variation of N_1, N_2, N_3 against time (t) for $N_{10} = 0.558, N_{20} = 2.484, N_{30} = 3.402$

Figure 6.6. Variation of N_1, N_2, N_3 against time (t) for $N_{10} = N_{20} = N_{30} = 5$

7.   Observations of the above graphs

Case 1: In this case the initial conditions of S_1, S_2, S_3 are in decreasing order. The natural growth rate of S_2 and the natural death rate of S_3 are almost equal. Further the first species dominates over the second species up to the time instant $t^* = 2$ after which the dominance is reversed as shown in Figure 6.1.

Case 2: In this case the initial conditions of S_1 and S_3 are identical. Initially the first and third species dominates over the second till the time instant $t^* = 0.85$ and $t^* = 0.61$ respectively and thereafter the dominance is reversed. Further we notice that the second species has the least initial value. (Figure 6.2).

Case 3: This is a situation at the natural growth rate of the host (S_2) is highest. In this case the initial conditions of S_1, S_2, S_3 are in decreasing order. Further it is evident that all the three species asymptotically converge to the equilibrium point. (Figure 6.3).

Case 4: In this case the first species has the least initial value. The S_3 dominates over the S_1 initially up to the time $t* = 0.43$ after which the dominance is reversed. Further the initial conditions of S_1, S_3, S_2 are in increasing order. (Figure 6.4).

Case 5: This is a situation at the self inhibition coefficient of S_1 is highest. In this case the initial conditions of S_1, S_2, S_3 are in increasing order. Initially the S_3 dominates by the S_2 up to the time $t* = 0.32$ and the S_1 up to the time $t* = 5.04$ and the dominances are reversed. (Figure 6.5).

Case 6: This is a situation at the initial conditions of the three species are identical. In this case the self inhibition coefficients of S_1, S_2, S_3 are in decreasing order. Further it is evident that all the three species asymptotically converge to the equilibrium point. (Figure 6.6).

8.   Conclusion

Investigate some relation-chains between the species such as Prey-Predation, Commensalism, Mutualism, Competition and Ammensalism between three species (S_1, S_2, S_3) with the population relations.

The present paper deals with an investigation on the stability of a three species syn eco-system with mortality rate for the host. The system comprises of a commensal (S_1), two hosts S_2 and S_3, i.e., S_2 and S_3 both benefit S_1, without getting themselves affected either positively or adversely. In this paper we established all possible equilibrium states. It is conclude that, in all eight equilibrium states, only one state E_7 is stable. Further the global stability is established with the help of suitable Liapunov's function and the growth rates of the species are numerically estimated using Runge-Kutta fourth order method.

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