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Certain Operations on Bipolar Fuzzy Graph Structures

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Abstract

A graph structure is a useful tool in solving the combinatorial problems in different areas of computer science and computational intelligence systems. A bipolar fuzzy graph structure is a generalization of a bipolar fuzzy graph. In this paper, we present several different types of operations, including composition, Cartesian product, strong product, cross product, and lexicographic product on bipolar fuzzy graph structures. We also investigate some properties of operations.

Keywords: Bipolar fuzzy graph structure (BFGS); Composition; Cartesian product; Strong product; Cross product; Lexicographic product; Join; Union

MSC 2010 No.: 03E72, 68R10, 68R05

1. Introduction

The concepts of graph theory have applications in many areas of computer science (such as data mining, image segmentation, clustering, image capturing, networking, etc.). Graph structures, introduced by Sampathkumar (2006), are a generalization of graphs which is quite useful in studying structures including graphs, signed graphs, and graphs in which every edge is labeled or colored. It helps to study various relations and the corresponding edges, simultaneously.

A fuzzy set, introduced by Zadeh (1965), gives the degree of membership of an object in a given set. Based on the same idea, Zhang (1994) defined the notion of bipolar fuzzy set on a given set X, by saying that a mapping $A : X \longrightarrow [-1, 1]$ was a bipolar fuzzy set where the membership degree $0,$ of an element x, meant that the element x was irrelevant to the corresponding property, the membership degree in $(0,\ 1]$, of an element x, indicated that the element somewhat satisfied the property, and the membership degree in $[-1,\ 0)$, of an element $x,$ indicated that the element somewhat satisfied the implicit counter-property. Rosenfeld (1975) discussed the concept of fuzzy graphs whose basic idea was introduced by Kauffmann (1973). The fuzzy relations between fuzzy sets were also considered by Rosenfeld and he developed the structure of fuzzy graphs, obtaining analogs of several graph theoretical concepts. Bhattacharya (1987) gave some remarks on fuzzy graphs. Several concepts on fuzzy graphs were introduced by Mordeson and Nair (2001). Akram et al. (2011–2016) has introduced several new concepts including bipolar fuzzy graphs, regular bipolar fuzzy graphs, irregular bipolar fuzzy graphs, antipodal bipolar fuzzy graphs and bipolar fuzzy graph structures. In this paper, we present certain operations on bipolar fuzzy graph structures and investigate some of their properties.

2. Preliminaries

We now review some definitions from Dinesh (2011) that are necessary for this paper.

A *graph structure* $G^* = (U,\ E_1,\ E_2,\ ...,\ E_k)$ consists of a non-empty set U together with mutually disjoint, irreflexive and symmetric relations $E_1,\ E_2,\ ...,\ E_k$ on U. If G_1^* and G_2^* are two *graph structures* given by $(U,\ E_1,\ E_2,\ ...,\ E_k)$ and $(V,\ E_1',\ E_2',\ ...,\ E_k')$ respectively, then *cartesian product* of G_1^* and G_2^*, is denoted by "$G_1^* \times G_2^*$" and given by $G_1^* \times G_2^* = (U \times V,\ E_1 \times E_1',\ E_2 \times E_2',\ ...,\ E_k \times E_k')$ where $E_i \times E_i' = \{(u_1v,\ u_2v) | v \in V,\ u_1u_2 \in E_i\} \cup \{(uv_1,\ uv_2) | u \in U,\ v_1v_2 \in E_i'\},\ i = 1,\ 2,\ ...,\ k$. *Composition* of G_1^* and G_2^* is denoted by "$G_1^* \circ G_2^*$" and given by $G_1^* \circ G_2^* = (U \circ V,\ E_1 \circ E_1',\ E_2 \circ E_2',\ ...,\ E_k \circ E_k')$ where $U \circ V = U \times V$ and $E_i \circ E_i' = \{(u_1v,\ u_2v) |\ v \in V,\ u_1u_2 \in E_i\} \cup \{(uv_1,\ uv_2) |\ u \in U,\ v_1v_2 \in E_i'\} \cup \{(u_1v_1,\ u_2v_2) |\ u_1u_2 \in E_i,\ v_1 \neq v_2\},\ i = 1,\ 2,\ ...,\ k$. *Union* of G_1^* and G_2^* is denoted by "$G_1^* \cup G_2^*$" and given by $G_1^* \cup G_2^* = (U \cup V,\ E_1 \cup E_1',\ E_2 \cup E_2',\ ...,\ E_k \cup E_k')$ and *join* of G_1^* and G_2^* is given by $G_1^* + G_2^* = (U + V,\ E_1 + E_1',\ E_2 + E_2',\ ...,\ E_k + E_k')$ where $U + V = U \cup V$ and $E_i + E_i' = E_i \cup E_i' \cup E$ for $i = 1,\ 2,\ ...,\ k$ such that E is the set consisting of all edges which join vertices of U with vertices of V.

Definition 1. (Dinesh (2011))

Let $G^* = (U,\ E_1,\ E_2,\ ...,\ E_k)$ be a graph structure and let $\nu,\ \rho_1,\ \rho_2,\ ...,\ \rho_k$ be the fuzzy subsets of $U,\ E_1,\ E_2,\ ...,\ E_k$ respectively such that

$$0 \leq \rho_i(xy) \leq \mu(x) \wedge \mu(y)\ \forall\ x,\ y \in V,\ i = 1,\ 2,\ ...,\ k.$$

Then $G = (\nu,\ \rho_1,\ \rho_2,\ ...,\ \rho_k)$ is a fuzzy graph structure of G^*.

Definition 2. (Zhang (1998))

Let X be a nonempty set. A *bipolar fuzzy set* B in X is an object having the form

$$B = \{(x,\ \mu_B^P(x),\ \ \mu_B^N(x)) \,|\, x \in X\},$$

where $\mu_B^P : X \to [0,\ 1]$ and $\mu_B^N : X \to [-1,\ 0]$ are mappings.

We use the positive membership degree $\mu_B^P(x)$ to denote the satisfaction degree of an element x to the property corresponding to a bipolar fuzzy set B, and the negative membership degree $\mu_B^N(x)$ to denote the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar fuzzy set B. If $\mu_B^P(x) \neq 0$ and $\mu_B^N(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for B. If $\mu_B^P(x) = 0$ and $\mu_B^N(x) \neq 0$, it is the situation that x does not satisfy the property of B but somewhat satisfies the counter property of B. It is possible for an element x to be such that $\mu_B^P(x) \neq 0$ and $\mu_B^N(x) \neq 0$ when the membership function of the property overlaps that of its counter property over some portion of X.

For the sake of simplicity, we shall use the symbol $B = (\mu_B^P,\ \mu_B^N)$ for the bipolar fuzzy set

$$B = \{(x,\ \mu_B^P(x),\ \mu_B^N(x)) \,|\, x \in X\}.$$

Definition 3. (Zhang (1998))

Let X be a nonempty set. Then we call a mapping $A = (\mu_A^P,\ \mu_A^N) : X \times X \to [0,\ 1] \times [-1,\ 0]$ a *bipolar fuzzy relation* on X such that $\mu_A^P(x,\ y) \in [0,\ 1]$ and $\mu_A^N(x,\ y) \in [-1,\ 0]$.

Definition 4. (Akram (2011))

A bipolar fuzzy graph $G = (V,\ A,\ B)$ is a non-empty set V together with a pair of functions $A = (\mu_A^P,\ \mu_A^N) : V \to [0,\ 1] \times [-1,\ 0]$ and $B = (\mu_B^P,\ \mu_B^N) : V \times V \to [0,\ 1] \times [-1,\ 0]$ such that for all $x,\ y \in V$,

$$\mu_B^P(x,\ y) \leq \min(\mu_A^P(x),\ \mu_A^P(y)) \quad \text{and} \quad \mu_B^N(x,\ y) \geq \max(\mu_A^N(x),\ \mu_A^N(y)).$$

Notice that $\mu_B^P(x,\ y) > 0$, $\mu_B^N(x,\ y) < 0$ for $(x,\ y) \in V \times V$, $\mu_B^P(x,\ y) = \mu_B^N(x,\ y) = 0$ for $(x,\ y) \notin V \times V$, and B is symmetric relation.

3. Operations on Bipolar Fuzzy Graph Structures

Definition 5. (Akram and Akmal (2016))

$\check{G}_b = (M,\ N_1,\ N_2,\ ...,\ N_n)$ is called a *bipolar fuzzy graph structure (BFGS)* of *a graph structure (GS)* $G^* = (U,\ E_1,\ E_2,\ ...,\ E_n)$ if $M = (\mu_M^P,\ \mu_M^N)$ is a *bipolar fuzzy set on* U and for each $i = 1,\ 2,\ ...,\ n$, $N_i = (\mu_{N_i}^P,\ \mu_{N_i}^N)$ is a *bipolar fuzzy set on* E_i such that

$$\mu_{N_i}^P(xy) \leq \mu_M^P(x) \wedge \mu_M^P(y)\ ,\ \mu_{N_i}^N(xy) \geq \mu_M^N(x) \vee \mu_M^N(y) \ \ \forall\ xy \in E_i \subset U \times U.$$

Note that $\mu_{N_i}^P(xy) = 0 = \mu_{N_i}^N(xy)$ for all $xy \in U \times U \setminus E_i$, $0 < \mu_{N_i}^P(xy) \leq 1$, $-1 \leq \mu_{N_i}^N(xy) < 0$ $\forall\ xy \in E_i$. While U and E_i $(i = 1,\ 2,\ ...,\ n)$ are called *underlying vertex set* and *underlying* i*-edge sets* of $\check{G}_b$, respectively. Note that $x \vee y =$maximum of x and y, $x \wedge y =$minimum of x and y, throughout this paper.

Example 1.

Let $U = \{a_1,\ a_2,\ a_3,\ a_4\}$. Let $E_1 = \{a_1a_2,\ a_2a_3\}$ and $E_2 = \{a_3a_4,\ a_1a_4\}$ be two disjoint symmetric relations on U. Then $G^* = (U,\ E_1,\ E_2)$ is a *graph structure*.

Let $M,\ N_1$ and N_2 be *bipolar fuzzy subsets* of $U,\ E_1$ and E_2, respectively, such that

$$M = \{(a_1,\ 0.5,\ -0.2),\ (a_2,\ 0.7,\ -0.3),\ (a_3,\ 0.4,\ -0.3),\ (a_4,\ 0.7,\ -0.3)\},$$

$$N_1 = \{(a_1a_2,\ 0.5,\ -0.1),\ (a_2a_3,\ 0.4,\ -0.3)\}$$

and $N_2 = \{(a_3a_4,\ 0.4,\ -0.2),\ (a_1a_4,\ 0.1,\ -0.2)\}$. Then, $\check{G}_b = (M,\ N_1,\ N_2)$ is a *BFGS* of G^* as shown in Figure 1.

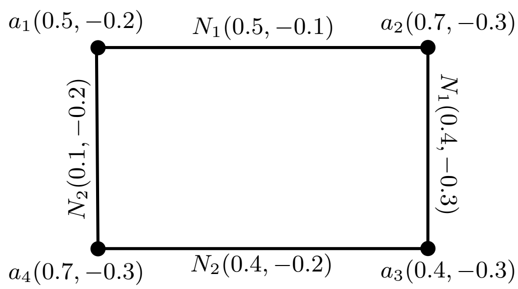


Figure 1: *BFGS* $\check{G}_b = (M,\ N_1,\ N_2)$

Definition 6.

Let $\check{G}_{b1} = (M_1,\ N_{11},\ N_{12},\ ...,\ N_{1n})$ and $\check{G}_{b2} = (M_2,\ N_{21},\ N_{22},\ ...,\ N_{2n})$ be respective *BFGSs* of *GSs* $G_1^* = (U_1,\ E_{11},\ E_{12},\ ...,\ E_{1n})$ and $G_2^* = (U_2,\ E_{21},\ E_{22},\ ...,\ E_{2n})$. The *Cartesian product* $\check{G}_{b1} \times \check{G}_{b2}$ of $\check{G}_{b1}$ and $\check{G}_{b2}$ is then a *BFGS* of $G_1^* \times G_2^* = (U_1 \times U_2,\ E_{11} \times E_{21},\ E_{12} \times E_{22},\ ...,\ E_{1n} \times E_{2n})$ is given by

$$(M_1 \times M_2,\ N_{11} \times N_{21},\ N_{12} \times N_{22},\ ...,\ N_{1n} \times N_{2n})$$

such that

$$\begin{cases} \mu^P_{(M_1\times M_2)}(xy) = (\mu^P_{M_1} \times \mu^P_{M_2})(xy) = \mu^P_{M_1}(x) \wedge \mu^P_{M_2}(y), \\ \mu^N_{(M_1\times M_2)}(xy) = (\mu^N_{M_1} \times \mu^N_{M_2})(xy) = \mu^N_{M_1}(x) \vee \mu^N_{M_2}(y)\ \forall\ xy \in U_1 \times U_2, \end{cases}$$

$$\begin{cases} \mu^P_{(N_{1i}\times N_{2i})}((xy_1)(xy_2)) = (\mu^P_{N_{1i}} \times \mu^P_{N_{2i}})((xy_1)(xy_2)) = \mu^P_{M_1}(x) \wedge \mu^P_{N_{2i}}(y_1y_2), \\ \mu^N_{(N_{1i}\times N_{2i})}((xy_1)(xy_2)) = (\mu^N_{N_{1i}} \times \mu^N_{N_{2i}})((xy_1)(xy_2)) = \mu^N_{M_1}(x) \vee \mu^N_{N_{2i}}(y_1y_2)\ \forall\ x \in u_1,\ y_1y_2 \in E_{2i}, \end{cases}$$

$$\begin{cases} \mu^P_{(N_{1i}\times N_{2i})}((x_1y)(x_2y)) = (\mu^P_{N_{1i}} \times \mu^P_{N_{2i}})((x_1y)(x_2y)) = \mu^P_{M_2}(y) \wedge \mu^P_{N_{1i}}(x_1x_2), \\ \mu^N_{(N_{1i}\times N_{2i})}((x_1y)(x_2y)) = (\mu^N_{N_{1i}} \times \mu^N_{N_{2i}})((x_1y)(x_2y)) = \mu^N_{M_2}(y) \vee \mu^N_{N_{1i}}(x_1x_2)\ \forall\ y \in U_2,\ x_1x_2 \in E_{1i}. \end{cases}$$

Example 2.

Let $\check{G_{b1}} = (M_1,\ N_{11},\ N_{12})$ and $\check{G_{b2}} = (M_2,\ N_{21},\ N_{22})$ be respective *BFGSs* of *graph structures* $G_1^* = (U_1,\ E_{11},\ E_{12})$ and $G_2^* = (U_2,\ E_{21},\ E_{22})$ such that $U_1 = \{\ a_1,\ a_2,\ a_3,\ a_4\}$, $U_2 = \{b_1,\ b_2,\ b_3,\ b_4\}, E_{11} = \{a_1a_2\}, E_{12} = \{a_3a_4\},\ E_{21} = \{b_1b_2\}$ and $E_{22} = \{b_3b_4\}$. $\check{G_{b1}}$ and $\check{G_{b2}}$ are shown in Figure 2,

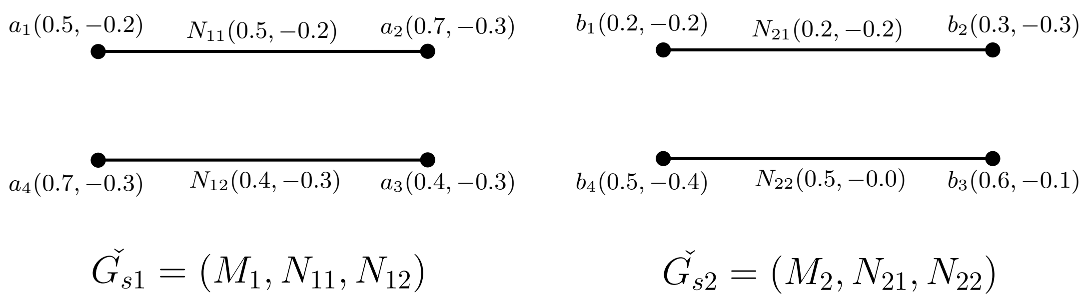


Figure 2: Bipolar Fuzzy Graph Structures

and *Cartesian product* $\check{G_{b1}} \times \check{G_{b2}} = (M_1 \times M_2,\ N_{11} \times N_{21},\ N_{12} \times N_{22})$ is shown in Figure 3.

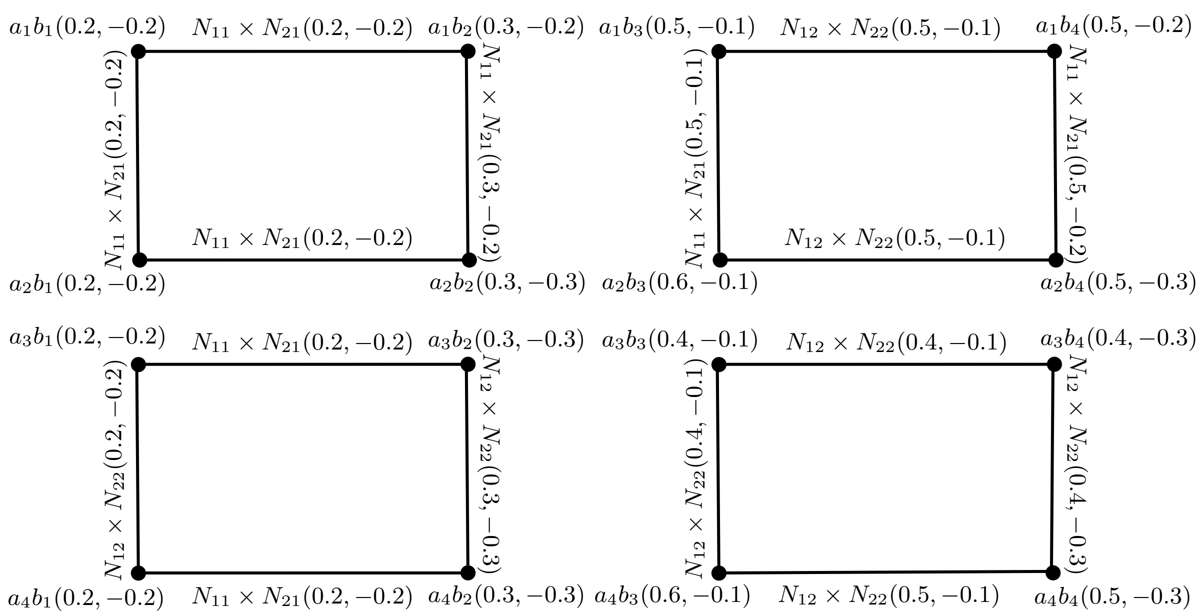
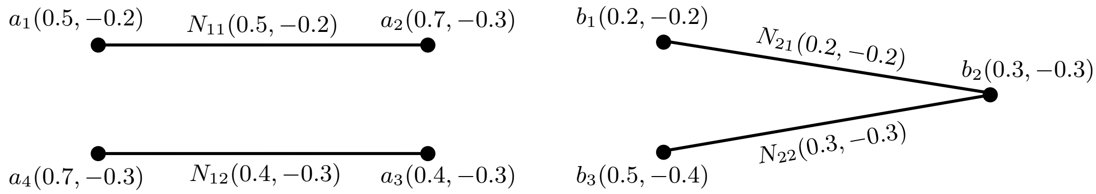


Figure 3: Cartesian Product of Two *BFGSs*

Example 3.

Consider $\check{G_{b1}} = (M_1,\ N_{11},\ N_{12})$ and $\check{G_{b2}} = (M_2,\ N_{21},\ N_{22})$ as shown in Figure 4 and let they be respective *BFGSs* of graph structures $G_1^* = (U_1,\ E_{11},\ E_{12})$ and $G_2^* = (U_2,\ E_{21},\ E_{22})$ such that $U_1 = \{a_1,\ a_2,\ a_3,\ a_4\},\ U_2 = \{b_1,\ b_2,\ b_3\},\ E_{11} = \{a_1a_2\},\ E_{12} = \{a_3a_4\},\ E_{21} = \{b_1b_2\}$

and $E_{22} = \{b_2 b_3\}$.



$\check{G_{s1}} = (M_1, N_{11}, N_{12})$ $\check{G_{s2}} = (M_2, N_{21}, N_{22})$

Figure 4: Bipolar Fuzzy Graph Structures

And their *Cartesian product* given by $\check{G_{b1}} \times \check{G_{b2}} = (M_1 \times M_2,\ N_{11} \times N_{21},\ N_{12} \times N_{22})$ is shown in Figure 5.

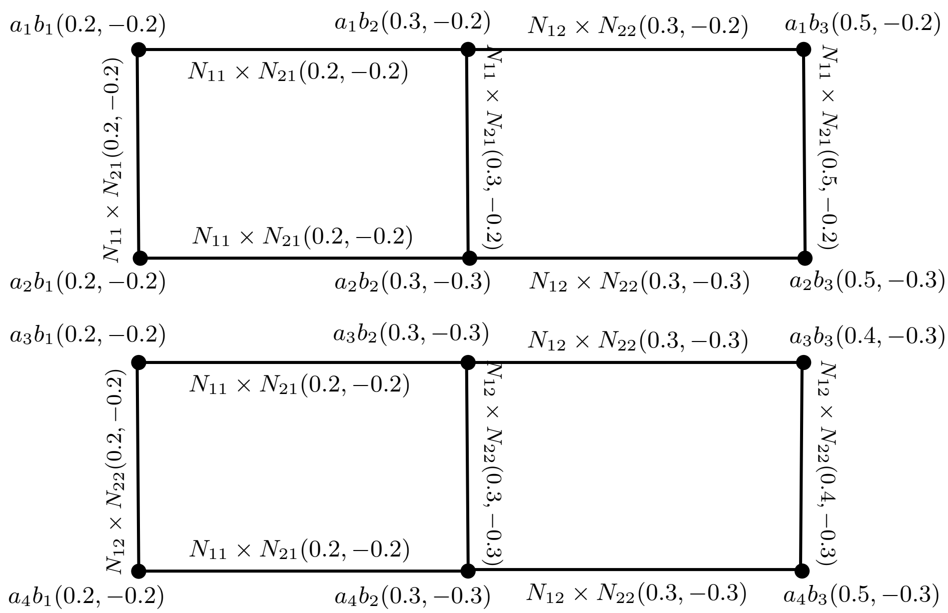


Figure 5: Cartesian Product of Two *BFGSs*

Theorem 1.

Let $G^* = (U_1 \times U_2,\ E_{11} \times E_{21},\ E_{12} \times E_{22},\ \ldots,\ E_{1n} \times E_{2n})$ be Cartesian product of GSs $G_1^* = (U_1,\ E_{11},\ E_{12},\ \ldots,\ E_{1n})$ and $G_2^* = (U_2,\ E_{21},\ E_{22},\ \ldots,\ E_{2n})$. Let $\check{G_{b1}} = (M_1,\ N_{11},\ N_{12},\ \ldots,\ N_{1n})$ and $\check{G_{b2}} = (M_2,\ N_{21},\ N_{22},\ \ldots,\ N_{2n})$ be respective BFGSs of G_1^* and G_2^*. Then, $(M_1 \times M_2,\ N_{11} \times N_{21},\ N_{12} \times N_{22},\ \ldots,\ N_{1n} \times N_{2n})$ is a BFGS of G^*.

Proof:

Case 1. When $u \in U_1,\ b_1b_2 \in\ E_{2i}$

$$\begin{aligned}\mu^P_{(N_{1i}\times N_{2i})}((ub_1)(ub_2)) &= \mu^P_{M_1}(u)\wedge\mu^P_{N_{2i}}(b_1b_2)\\ &\le \mu^P_{M_1}(u)\wedge[\mu^P_{M_2}(b_1)\wedge\mu^P_{M_2}(b_2)]\\ &= [\mu^P_{M_1}(u)\wedge\mu^P_{M_2}(b_1)]\wedge[\mu^P_{M_1}(u)\wedge\mu^P_{M_2}(b_2)]\\ &= \mu^P_{(M_1\times M_2)}(ub_1)\wedge\mu^P_{(M_1\times M_2)}(ub_2),\end{aligned}$$

$$\begin{aligned}\mu^N_{(N_{1i}\times N_{2i})}((ub_1)(ub_2)) &= \mu^N_{M_1}(u)\vee\mu^N_{N_{2i}}(b_1b_2)\\ &\ge \mu^N_{M_1}(u)\vee[\mu^N_{M_2}(b_1)\vee\mu^N_{M_2}(b_2)]\\ &= [\mu^N_{M_1}(u)\vee\mu^N_{M_2}(b_1)]\vee[\mu^N_{M_1}(u)\vee\mu^N_{M_2}(b_2)]\\ &= \mu^N_{(M_1\times M_2)}(ub_1)\vee\mu^N_{(M_1\times M_2)}(ub_2),\end{aligned}$$

for $ub_1,\ ub_2 \in U_1\times U_2$.

Case 2. When $u \in U_2,\ b_1b_2 \in\ E_{1i}$

$$\begin{aligned}\mu^P_{(N_{1i}\times N_{2i})}((b_1u)(b_2u)) &= \mu^P_{M_2}(u)\wedge\mu^P_{N_{1i}}(b_1b_2)\\ &\le \mu^P_{M_2}(u)\wedge[\mu^P_{M_1}(b_1)\wedge\mu^P_{M_1}(b_2)]\\ &= [\mu^P_{M_2}(u)\wedge\mu^P_{M_1}(b_1)]\wedge[\mu^P_{M_2}(u)\wedge\mu^P_{M_1}(b_2)]\\ &= \mu^P_{(M_1\times M_2)}(b_1u)\wedge\mu^P_{(M_1\times M_2)}(b_2u),\end{aligned}$$

$$\begin{aligned}\mu^N_{(N_{1i}\times N_{2i})}((b_1u)(b_2u)) &= \mu^N_{M_2}(u)\vee\mu^N_{N_{1i}}(b_1b_2)\\ &\ge \mu^N_{M_2}(u)\vee[\mu^N_{M_1}(b_1)\vee\mu^N_{M_1}(b_2)]\\ &= [\mu^N_{M_2}(u)\vee\mu^N_{M_1}(b_1)]\vee[\mu^N_{M_2}(u)\vee\mu^N_{M_1}(b_2)]\\ &= \mu^N_{(M_1\times M_2)}(b_1u)\vee\mu^N_{(M_1\times M_2)}(b_2u),\end{aligned}$$

for $b_1u,\ b_2u \in U_1\times U_2$.

Both cases hold for $i = 1,\ 2,\ ...,\ n$. This completes the proof. □

Definition 7.

Let $\check{G_{b1}} = (M_1,\ N_{11},\ N_{12},\ ...,\ N_{1n})$ and $\check{G_{b2}} = (M_2,\ N_{21},\ N_{22},\ ...,\ N_{2n})$ be respective *BFGSs* of *GSs* $G_1^* = (U_1,\ E_{11},\ E_{12},\ ...,\ E_{1n})$ and $G_2^* = (U_2,\ E_{21},\ E_{22},\ ...,\ E_{2n})$. The *cross product* $\check{G_{b1}}*\check{G_{b2}}$ of $\check{G_{b1}}$ and $\check{G_{b2}}$ is a *BFGS* of $G_1^*{*}G_2^* = (U_1*U_2,\ E_{11}*E_{21},\ E_{12}*E_{22},\ ...,\ E_{1n}*E_{2n})$ is given by

$$(M_1*M_2,\ N_{11}*N_{21},\ N_{12}*N_{22},\ ...,\ N_{1n}*N_{2n})$$

such that

$$\begin{cases}\mu^P_{(M_1*M_2)}(xy) = (\mu^P_{M_1}*\mu^P_{M_2})(xy) = \mu^P_{M_1}(x)\wedge\mu^P_{M_2}(y),\\ \mu^N_{(M_1*M_2)}(xy) = (\mu^N_{M_1}*\mu^N_{M_2})(xy) = \mu^N_{M_1}(x)\vee\mu^N_{M_2}(y)\ \forall\ xy\in U_1\times U_2,\end{cases}$$

$$\begin{cases} \mu^P_{(N_{1i}*N_{2i})}((x_1y_1)(x_2y_2)) &= (\mu^P_{N_{1i}} * \mu^P_{N_{2i}})((x_1y_1)(x_2y_2)) = \mu^P_{N_{2i}}(y_1y_2) \wedge \mu^P_{N_{1i}}(x_1x_2), \\ \mu^N_{(N_{1i}*N_{2i})}((x_1y_1)(x_2y_2)) &= (\mu^N_{N_{1i}} * \mu^N_{N_{2i}})((x_1y_1)(x_2y_2)) = \mu^N_{N_{2i}}(y_1y_2) \vee \mu^N_{N_{1i}}(x_1x_2) \\ & \forall\ y_1y_2 \in E_{2i},\ x_1x_2 \in E_{1i}. \end{cases}$$

Example 4.

Let $\check{G_{b1}}$ and $\check{G_{b2}}$ be *BFGSs* as shown in Figure 2 and *cross product* $\check{G_{b1}} * \check{G_{b2}} = (M_1 * M_2,\ N_{11} * N_{21},\ N_{12} * N_{22})$ is as shown in Figure 6.

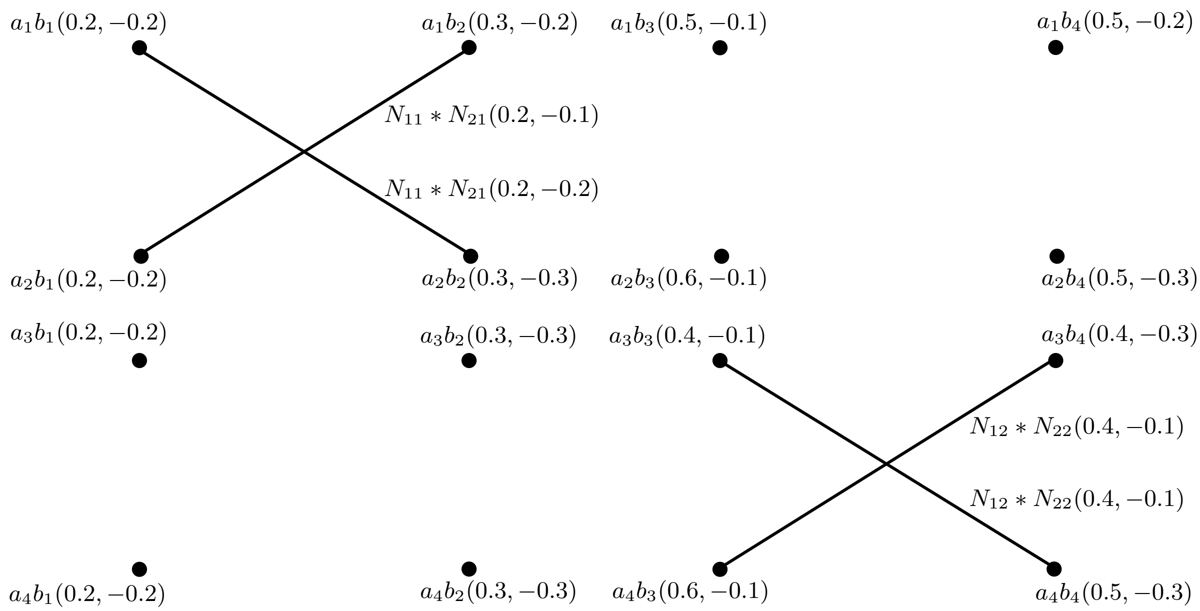


Figure 6: Cross Product of Two *BFGSs*

Example 5.

$\check{G_{b1}} = (M_1,\ N_{11},\ N_{12})$ and $\check{G_{b2}} = (M_2,\ N_{21},\ N_{22})$ be *BFGSs* as shown in Figure 4 and their *cross product* given by $\check{G_{b1}} * \check{G_{b2}} = (M_1 * M_2,\ N_{11} * N_{21},\ N_{12} * N_{22})$ is shown in Figure 7.

Theorem 2.

Let $G^* = (U_1 * U_2,\ E_{11} * E_{21},\ E_{12} * E_{22},\ ...,\ E_{1n} * E_{2n})$ be cross product of GSs $G_1^* = (U_1,\ E_{11},\ E_{12},\ ...,\ E_{1n})$ and $G_2^* = (U_2,\ E_{21},\ E_{22},\ ...,\ E_{2n})$. Let $\check{G_{b1}} = (M_1,\ N_{11},\ N_{12},\ ...,\ N_{1n})$ and $\check{G_{b2}} = (M_2,\ N_{21},\ N_{22},\ ...,\ N_{2n})$ be respective BFGSs of G_1^* and G_2^*. Then $(M_1 * M_2,\ N_{11} * N_{21},\ N_{12} * N_{22},\ ...,\ N_{1n} * N_{2n})$ is a BFGS of G^*.

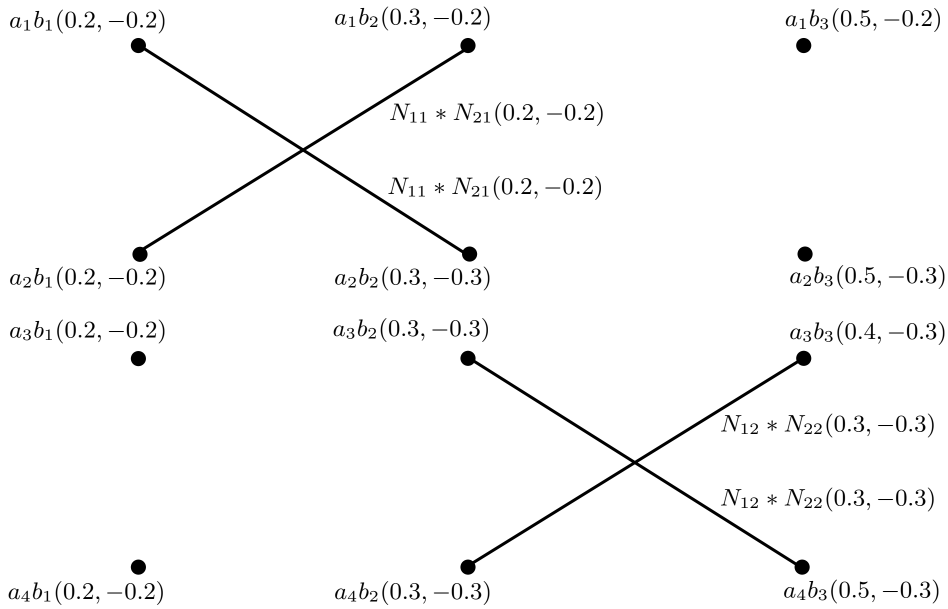


Figure 7: Cross Product of Two *BFGSs*

Proof:

For all $b_1u_1,\ b_2u_2 \in U_1 * U_2$

$$\begin{aligned}
\mu^P_{(N_{1i}*N_{2i})}((b_1u_1)(b_2u_2)) &= \mu^P_{N_{2i}}(u_1u_2) \wedge \mu^P_{N_{1i}}(b_1b_2) \\
&\leq [\mu^P_{M_2}(u_1) \wedge \mu^P_{M_2}(u_2)] \wedge [\mu^P_{M_1}(b_1) \wedge \mu^P_{M_1}(b_2)] \\
&= [\mu^P_{M_2}(u_1) \wedge \mu^P_{M_1}(b_1)] \wedge [\mu^P_{M_2}(u_2) \wedge \mu^P_{M_1}(b_2)] \\
&= \mu^P_{(M_1*M_2)}(b_1u_1) \wedge \mu^P_{(M_1*M_2)}(b_2u_2),
\end{aligned}$$

$$\begin{aligned}
\mu^N_{(N_{1i}*N_{2i})}((b_1u_1)(b_2u_2)) &= \mu^N_{N_{2i}}(u_1u_2) \vee \mu^N_{N_{1i}}(b_1b_2) \\
&\geq [\mu^N_{M_2}(u_1) \vee \mu^N_{M_2}(u_2)] \vee [\mu^N_{M_1}(b_1) \vee \mu^N_{M_1}(b_2)] \\
&= [\mu^N_{M_2}(u_1) \vee \mu^N_{M_1}(b_1)] \vee [\mu^N_{M_2}(u_2) \vee \mu^N_{M_1}(b_2)] \\
&= \mu^N_{(M_1*M_2)}(b_1u_1) \vee \mu^N_{(M_1*M_2)}(b_2u_2),
\end{aligned}$$

for $i = 1,\ 2,\ ...,\ n$. This completes the proof. □

Definition 8.

Let $\check{G_{b1}} = (M_1,\ N_{11},\ N_{12},\ ...,\ N_{1n})$ and $\check{G_{b2}} = (M_2,\ N_{21},\ N_{22},\ ...,\ N_{2n})$ be respective *BFGSs* of *GSs* $G_1^* = (U_1,\ E_{11},\ E_{12},\ ...,\ E_{1n})$ and $G_2^* = (U_2,\ E_{21},\ E_{22},\ ...,\ E_{2n})$. The *lexicographic product* $\check{G_{b1}} \bullet \check{G_{b2}}$ of $\check{G_{b1}}$ and $\check{G_{b2}}$ is a *BFGS* of $G_1^* \bullet G_2^* = (U_1 \bullet U_2,\ E_{11} \bullet E_{21},\ E_{12} \bullet E_{22},\ ...,\ E_{1n} \bullet E_{2n})$ is given by

$$(M_1 \bullet M_2,\ N_{11} \bullet N_{21},\ N_{12} \bullet N_{22},\ ...,\ N_{1n} \bullet N_{2n})$$

such that

$$\begin{cases} \mu^P_{(M_1\bullet M_2)}(xy) = (\mu^P_{M_1}\bullet\mu^P_{M_2})(xy) = \mu^P_{M_1}(x)\wedge\mu^P_{M_2}(y), \\ \mu^N_{(M_1\bullet M_2)}(xy) = (\mu^N_{M_1}\bullet\mu^N_{M_2})(xy) = \mu^N_{M_1}(x)\vee\mu^N_{M_2}(y),\ \forall\ xy\in U_1\times U_2 \end{cases}$$

$$\begin{cases} \mu^P_{(N_{1i}\bullet N_{2i})}((xy_1)(xy_2)) = (\mu^P_{N_{1i}}\bullet\mu^P_{N_{2i}})((xy_1)(xy_2)) = \mu^P_{M_1}(x)\wedge\mu^P_{N_{2i}}(y_1y_2), \\ \mu^N_{(N_{1i}\bullet N_{2i})}((xy_1)(xy_2)) = (\mu^N_{N_{1i}}\bullet\mu^N_{N_{2i}})((xy_1)(xy_2)) = \mu^N_{M_1}(x)\vee\mu^N_{N_{2i}}(y_1y_2), \\ \forall\ x\in u_1,\ y_1y_2\in E_{2i} \end{cases}$$

$$\begin{cases} \mu^P_{(N_{1i}\bullet N_{2i})}((x_1y_1)(x_2y_2)) = (\mu^P_{N_{1i}}\bullet\mu^P_{N_{2i}})((x_1y_1)(x_2y_2)) = \mu^P_{N_{2i}}(y_1y_2)\wedge\mu^P_{N_{1i}}(x_1x_2), \\ \mu^N_{(N_{1i}\bullet N_{2i})}((x_1y_1)(x_2y_2)) = (\mu^N_{N_{1i}}\bullet\mu^N_{N_{2i}})((x_1y_1)(x_2y_2)) = \mu^N_{N_{2i}}(y_1y_2)\vee\mu^N_{N_{1i}}(x_1x_2), \\ \forall\ y_1y_2\in E_{2i},\ x_1x_2\in E_{1i}. \end{cases}$$

Example 6.

Let $\check{G_{b1}}$ and $\check{G_{b2}}$ be *BFGSs* shown in Figure 2 and *lexicographic product* $\check{G_{b1}}\bullet\check{G_{b2}} = (M_1\bullet M_2,\ N_{11}\bullet N_{21},\ N_{12}\bullet N_{22})$ is as shown in Figure 8.

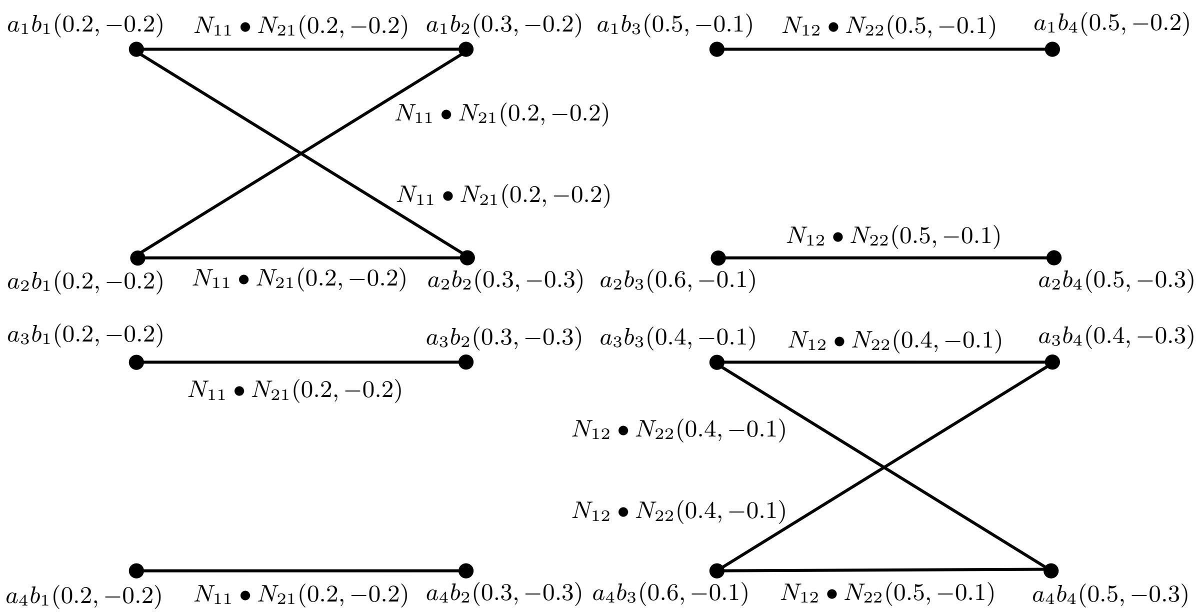


Figure 8: Lexicographic Product of Two *BFGSs*

Example 7.

$\check{G_{b1}} = (M_1,\ N_{11},\ N_{12})$ and $\check{G_{b2}} = (M_2,\ N_{21},\ N_{22})$ be *BFGSs* as shown in Figure 4 and their *lexicographic product* given by $\check{G_{b1}}\bullet\check{G_{b2}} = (M_1\bullet M_2,\ N_{11}\bullet N_{21},\ N_{12}\bullet N_{22})$ is shown in Figure 9.

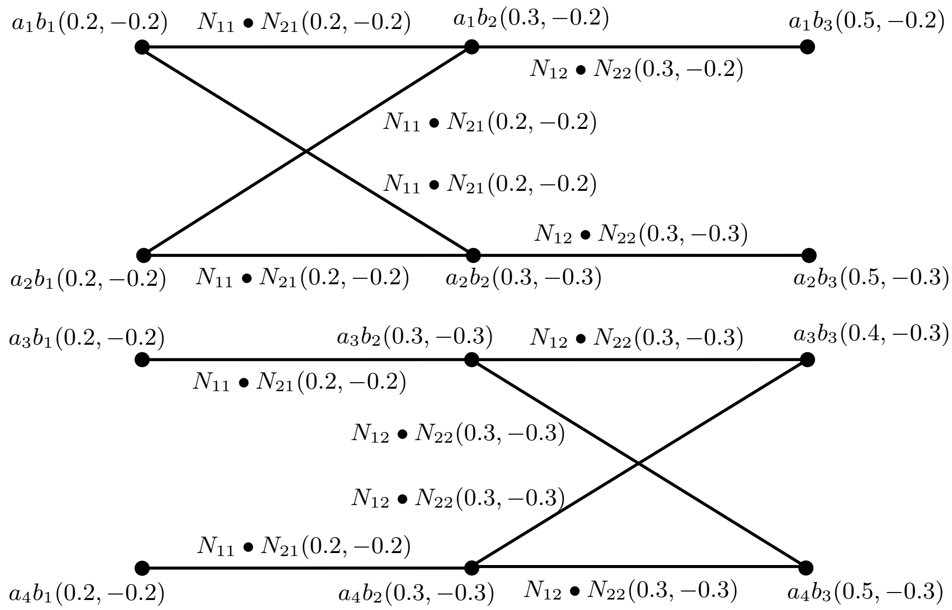


Figure 9: Lexicographic Product of Two *BFGSs*

Theorem 3.

Let $G^* = (U_1 \bullet U_2,\ E_{11} \bullet E_{21},\ E_{12} \bullet E_{22},\ ...,\ E_{1n} \bullet E_{2n})$ be lexicographic product of GSs $G_1^* = (U_1,\ E_{11},\ E_{12},\ ...,\ E_{1n})$ and $G_2^* = (U_2,\ E_{21},\ E_{22},\ ...,\ E_{2n})$. Let $\check{G_{b1}} = (M_1,\ N_{11},\ N_{12},\ ...,\ N_{1n})$ and $\check{G_{b2}} = (M_2,\ N_{21},\ N_{22},\ ...,\ N_{2n})$ be respective BFGSs of G_1^* and G_2^*. Then $(M_1 \bullet M_2,\ N_{11} \bullet N_{21},\ N_{12} \bullet N_{22},\ ...,\ N_{1n} \bullet N_{2n})$ is a BFGS of G^*.

Proof:

Case 1. When $u \in U_1,\ b_1b_2 \in\ E_{2i}$

$$\begin{aligned}\mu^P_{(N_{1i}\bullet N_{2i})}((ub_1)(ub_2)) &= \mu^P_{M_1}(u) \wedge \mu^P_{N_{2i}}(b_1b_2)\\ &\leq \mu^P_{M_1}(u) \wedge [\mu^P_{M_2}(b_1) \wedge \mu^P_{M_2}(b_2)]\\ &= [\mu^P_{M_1}(u) \wedge \mu^P_{M_2}(b_1)] \wedge [\mu^P_{M_1}(u) \wedge \mu^P_{M_2}(b_2)]\\ &= \mu^P_{(M_1\bullet M_2)}(ub_1) \wedge \mu^P_{(M_1\bullet M_2)}(ub_2),\end{aligned}$$

$$\begin{aligned}\mu^N_{(N_{1i}\bullet N_{2i})}((ub_1)(ub_2)) &= \mu^N_{M_1}(u) \vee \mu^N_{N_{2i}}(b_1b_2)\\ &\geq \mu^N_{M_1}(u) \vee [\mu^N_{M_2}(b_1) \vee \mu^N_{M_2}(b_2)]\\ &= [\mu^N_{M_1}(u) \vee \mu^N_{M_2}(b_1)] \vee [\mu^N_{M_1}(u) \vee \mu^N_{M_2}(b_2)]\\ &= \mu^N_{(M_1\bullet M_2)}(ub_1) \vee \mu^N_{(M_1\bullet M_2)}(ub_2),\end{aligned}$$

for $ub_1,\ ub_2 \in U_1 \bullet U_2$.

Case 2. When $u_1u_2 \in \; E_{2i},\; b_1b_2 \in \; E_{1i}$

$$\begin{aligned}\mu^P_{(N_{1i}\bullet N_{2i})}((b_1u_1)(b_2u_2)) &= \mu^P_{N_{2i}}(u_1u_2)\wedge\mu^P_{N_{1i}}(b_1b_2)\\ &\leq [\mu^P_{M_2}(u_1)\wedge\mu^P_{M_2}(u_2)]\wedge[\mu^P_{M_1}(b_1)\wedge\mu^P_{M_1}(b_2)]\\ &= [\mu^P_{M_2}(u_1)\wedge\mu^P_{M_1}(b_1)]\wedge[\mu^P_{M_2}(u_2)\wedge\mu^P_{M_1}(b_2)]\\ &= \mu^P_{(M_1\bullet M_2)}(b_1u_1)\wedge\mu^P_{(M_1\bullet M_2)}(b_2u_2),\end{aligned}$$

$$\begin{aligned}\mu^N_{(N_{1i}\bullet N_{2i})}((b_1u_1)(b_2u_2)) &= \mu^N_{N_{2i}}(u_1u_2)\vee\mu^N_{N_{1i}}(b_1b_2)\\ &\geq [\mu^N_{M_2}(u_1)\vee\mu^N_{M_2}(u_2)]\vee[\mu^N_{M_1}(b_1)\vee\mu^N_{M_1}(b_2)]\\ &= [\mu^N_{M_2}(u_1)\vee\mu^N_{M_1}(b_1)]\vee[\mu^N_{M_2}(u_2)\vee\mu^N_{M_1}(b_2)]\\ &= \mu^N_{(M_1\bullet M_2)}(b_1u_1)\vee\mu^N_{(M_1\bullet M_2)}(b_2u_2),\end{aligned}$$

for $b_1u_1,\; b_2u_2 \in U_1 \bullet U_2$.

Both cases hold for $i = 1,\; 2,\; ...,\; n$. This completes the proof. □

Definition 9.

Let $\check{G_{b1}} = (M_1,\; N_{11},\; N_{12},\; ...,\; N_{1n})$ and $\check{G_{b2}} = (M_2,\; N_{21},\; N_{22},\; ...,\; N_{2n})$ be respective *BFGSs* of *GSs* $G^*_1 = (U_1,\; E_{11},\; E_{12},\; ...,\; E_{1n})$ and $G^*_2 = (U_2,\; E_{21},\; E_{22},\; ...,\; E_{2n})$. The *strong product* $\check{G_{b1}}\boxtimes\check{G_{b2}}$ of $\check{G_{b1}}$ and $\check{G_{b2}}$ is then a *BFGS* of $G^*_1\boxtimes G^*_2 = (U_1\boxtimes U_2,\; E_{11}\boxtimes E_{21},\; E_{12}\boxtimes E_{22},\; ...,\; E_{1n}\boxtimes E_{2n})$ is given by

$$(M_1 \boxtimes M_2,\; N_{11} \boxtimes N_{21},\; N_{12} \boxtimes N_{22},\; ...,\; N_{1n} \boxtimes N_{2n})$$

such that

$$\begin{cases}\mu^P_{(M_1\boxtimes M_2)}(xy) = (\mu^P_{M_1}\boxtimes\mu^P_{M_2})(xy) = \mu^P_{M_1}(x)\wedge\mu^P_{M_2}(y),\\ \mu^N_{(M_1\boxtimes M_2)}(xy) = (\mu^N_{M_1}\boxtimes\mu^N_{M_2})(xy) = \mu^N_{M_1}(x)\vee\mu^N_{M_2}(y),\;\forall\; xy\in U_1\times U_2\end{cases}$$

$$\begin{cases}\mu^P_{(N_{1i}\boxtimes N_{2i})}((xy_1)(xy_2)) = (\mu^P_{N_{1i}}\boxtimes\mu^P_{N_{2i}})((xy_1)(xy_2)) = \mu^P_{M_1}(x)\wedge\mu^P_{N_{2i}}(y_1y_2),\\ \mu^N_{(N_{1i}\boxtimes N_{2i})}((xy_1)(xy_2)) = (\mu^N_{N_{1i}}\boxtimes\mu^N_{N_{2i}})((xy_1)(xy_2)) = \mu^N_{M_1}(x)\vee\mu^N_{N_{2i}}(y_1y_2),\\ \qquad\forall\; x\in u_1,\; y_1y_2\in E_{2i}\end{cases}$$

$$\begin{cases}\mu^P_{(N_{1i}\boxtimes N_{2i})}((x_1y)(x_2y)) = (\mu^P_{N_{1i}}\boxtimes\mu^P_{N_{2i}})((x_1y)(x_2y)) = \mu^P_{M_2}(y)\wedge\mu^P_{N_{1i}}(x_1x_2),\\ \mu^N_{(N_{1i}\boxtimes N_{2i})}((x_1y)(x_2y)) = (\mu^N_{N_{1i}}\boxtimes\mu^N_{N_{2i}})((x_1y)(x_2y)) = \mu^N_{M_2}(y)\vee\mu^N_{N_{1i}}(x_1x_2),\\ \qquad\forall\; y\in U_2,\; x_1x_2\in E_{1i}.\end{cases}$$

$$\begin{cases}\mu^P_{(N_{1i}\boxtimes N_{2i})}((x_1y_1)(x_2y_2)) = (\mu^P_{N_{1i}}\boxtimes\mu^P_{N_{2i}})((x_1y_1)(x_2y_2)) = \mu^P_{N_{2i}}(y_1y_2)\wedge\mu^P_{N_{1i}}(x_1x_2),\\ \mu^N_{(N_{1i}\boxtimes N_{2i})}((x_1y_1)(x_2y_2)) = (\mu^N_{N_{1i}}\boxtimes\mu^N_{N_{2i}})((x_1y_1)(x_2y_2)) = \mu^N_{N_{2i}}(y_1y_2)\vee\mu^N_{N_{1i}}(x_1x_2),\\ \qquad\forall\; y_1y_2\in E_{2i},\; x_1x_2\in E_{1i}.\end{cases}$$

Example 8.

Let $\check{G_{b1}}$ and $\check{G_{b2}}$ be *BFGSs* as shown in Figure 2 and *strong product* $\check{G_{b1}} \boxtimes \check{G_{b2}} = (M_1 \boxtimes M_2,\; N_{11} \boxtimes N_{21},\; N_{12} \boxtimes N_{22})$ is shown in Figure 10.

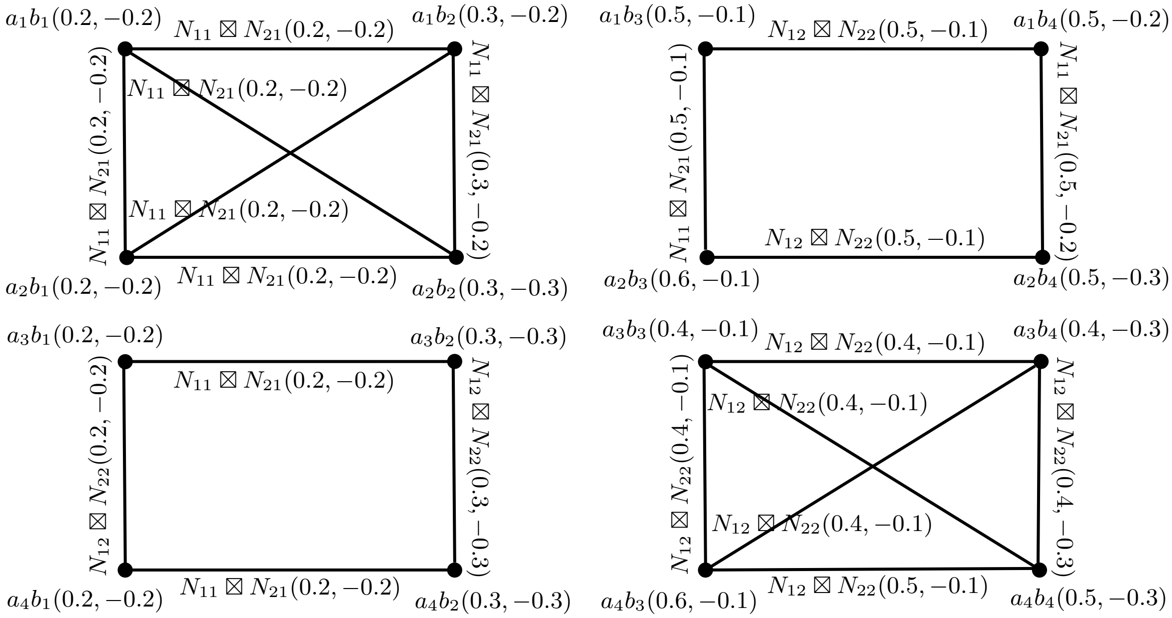


Figure 10: Strong Product of Two *BFGSs*

Example 9.

$\check{G_{b1}} = (M_1,\ N_{11},\ N_{12})$ and $\check{G_{b2}} = (M_2,\ N_{21},\ N_{22})$ be *BFGSs* as shown in Figure 4 and their *strong product* given by $\check{G_{b1}} \boxtimes \check{G_{b2}} = (M_1 \boxtimes M_2,\ N_{11} \boxtimes N_{21},\ N_{12} \boxtimes N_{22})$ is as shown in Figure 11.

Theorem 4.

Let $G^* = (U_1 \boxtimes U_2,\ E_{11} \boxtimes E_{21},\ E_{12} \boxtimes E_{22},\ ...,\ E_{1n} \boxtimes E_{2n})$ be strong product of GSs $G_1^* = (U_1,\ E_{11},\ E_{12},\ ...,\ E_{1n})$ and $G_2^* = (U_2,\ E_{21},\ E_{22},\ ...,\ E_{2n})$. Let $\check{G_{b1}} = (M_1,\ N_{11},\ N_{12},\ ...,\ N_{1n})$ and $\check{G_{b2}} = (M_2,\ N_{21},\ N_{22},\ ...,\ N_{2n})$ be respective BFGSs of G_1^* and G_2^*. Then $(M_1 \boxtimes M_2,\ N_{11} \boxtimes N_{21},\ N_{12} \boxtimes N_{22},\ ...,\ N_{1n} \boxtimes N_{2n})$ is a BFGS of G^*.

Proof:

Case 1. When $u \in U_1,\ b_1b_2 \in\ E_{2i}$

$$\begin{aligned}
\mu^P_{(N_{1i}\boxtimes N_{2i})}((ub_1)(ub_2)) &= \mu^P_{M_1}(u) \wedge \mu^P_{N_{2i}}(b_1b_2)\\
&\leq \mu^P_{M_1}(u) \wedge [\mu^P_{M_2}(b_1) \wedge \mu^P_{M_2}(b_2)]\\
&= [\mu^P_{M_1}(u) \wedge \mu^P_{M_2}(b_1)] \wedge [\mu^P_{M_1}(u) \wedge \mu^P_{M_2}(b_2)]\\
&= \mu^P_{(M_1\boxtimes M_2)}(ub_1) \wedge \mu^P_{(M_1\boxtimes M_2)}(ub_2),
\end{aligned}$$

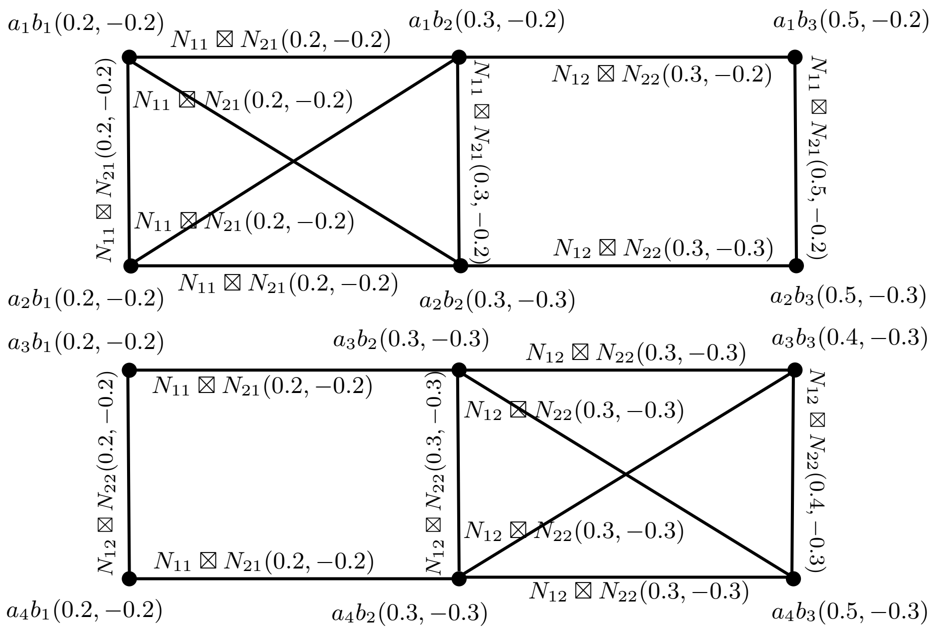


Figure 11: Strong Product of Two *BFGSs*

$$\begin{aligned}\mu^N_{(N_{1i}\boxtimes N_{2i})}((ub_1)(ub_2)) &= \mu^N_{M_1}(u)\vee\mu^N_{N_{2i}}(b_1b_2)\\ &\geq \mu^N_{M_1}(u)\vee[\mu^N_{M_2}(b_1)\vee\mu^N_{M_2}(b_2)]\\ &= [\mu^N_{M_1}(u)\vee\mu^N_{M_2}(b_1)]\vee[\mu^N_{M_1}(u)\vee\mu^N_{M_2}(b_2)]\\ &= \mu^N_{(M_1\boxtimes M_2)}(ub_1)\vee\mu^N_{(M_1\boxtimes M_2)}(ub_2),\end{aligned}$$

for $ub_1,\ ub_2\in U_1\boxtimes U_2$.

Case 2. When $u\in U_2,\ b_1b_2\in\ E_{1i}$

$$\begin{aligned}\mu^P_{(N_{1i}\boxtimes N_{2i})}((b_1u)(b_2u)) &= \mu^P_{M_2}(u)\wedge\mu^P_{N_{1i}}(b_1b_2)\\ &\leq \mu^P_{M_2}(u)\wedge[\mu^P_{M_1}(b_1)\wedge\mu^P_{M_1}(b_2)]\\ &= [\mu^P_{M_2}(u)\wedge\mu^P_{M_1}(b_1)]\wedge[\mu^P_{M_2}(u)\wedge\mu^P_{M_1}(b_2)]\\ &= \mu^P_{(M_1\boxtimes M_2)}(b_1u)\wedge\mu^P_{(M_1\boxtimes M_2)}(b_2u),\end{aligned}$$

$$\begin{aligned}\mu^N_{(N_{1i}\boxtimes N_{2i})}((b_1u)(b_2u)) &= \mu^N_{M_2}(u)\vee\mu^N_{N_{1i}}(b_1b_2)\\ &\geq \mu^N_{M_2}(u)\vee[\mu^N_{M_1}(b_1)\vee\mu^N_{M_1}(b_2)]\\ &= [\mu^N_{M_2}(u)\vee\mu^N_{M_1}(b_1)]\vee[\mu^N_{M_2}(u)\vee\mu^N_{M_1}(b_2)]\\ &= \mu^N_{(M_1\boxtimes M_2)}(b_1u)\vee\mu^N_{(M_1\boxtimes M_2)}(b_2u),\end{aligned}$$

for $b_1u,\ b_2u\in U_1\boxtimes U_2$.

Case 3. When $u_1u_2 \in E_{2i},\ b_1b_2 \in E_{1i}$

$$\begin{aligned}
\mu^P_{(N_{1i}\boxtimes N_{2i})}((b_1u_1)(b_2u_2)) &= \mu^P_{N_{2i}}(u_1u_2)\wedge \mu^P_{N_{1i}}(b_1b_2)\\
&\leq [\mu^P_{M_2}(u_1)\wedge \mu^P_{M_2}(u_2)]\wedge[\mu^P_{M_1}(b_1)\wedge \mu^P_{M_1}(b_2)]\\
&= [\mu^P_{M_2}(u_1)\wedge \mu^P_{M_1}(b_1)]\wedge[\mu^P_{M_2}(u_2)\wedge \mu^P_{M_1}(b_2)]\\
&= \mu^P_{(M_1\boxtimes M_2)}(b_1u_1)\wedge \mu^P_{(M_1\boxtimes M_2)}(b_2u_2),
\end{aligned}$$

$$\begin{aligned}
\mu^N_{(N_{1i}\boxtimes N_{2i})}((b_1u_1)(b_2u_2)) &= \mu^N_{N_{2i}}(u_1u_2)\vee \mu^N_{N_{1i}}(b_1b_2)\\
&\geq [\mu^N_{M_2}(u_1)\vee \mu^N_{M_2}(u_2)]\vee[\mu^N_{M_1}(b_1)\vee \mu^N_{M_1}(b_2)]\\
&= [\mu^N_{M_2}(u_1)\vee \mu^N_{M_1}(b_1)]\vee[\mu^N_{M_2}(u_2)\vee \mu^N_{M_1}(b_2)]\\
&= \mu^N_{(M_1\boxtimes M_2)}(b_1u_1)\vee \mu^N_{(M_1\boxtimes M_2)}(b_2u_2),
\end{aligned}$$

for $b_1u_1,\ b_2u_2 \in U_1 \boxtimes U_2$.

All three cases hold for $i = 1,\ 2,\ ...,\ n$. This completes the proof. $\square$

Definition 10.

Let $\check{G}_{b1} = (M_1,\ N_{11},\ N_{12},\ ...,\ N_{1n})$ and $\check{G}_{b2} = (M_2,\ N_{21},\ N_{22},\ ...,\ N_{2n})$ be respective *BFGSs* of *GSs* $G^*_1 = (U_1,\ E_{11},\ E_{12},\ ...,\ E_{1n})$ and $G^*_2 = (U_2,\ E_{21},\ E_{22},\ ...,\ E_{2n})$. The *composition* $\check{G}_{b1}\circ\check{G}_{b2}$, of $\check{G}_{b1}$ and $\check{G}_{b2}$ is then a *BFGS* of $G^*_1\circ G^*_2 = (U_1\circ U_2,\ E_{11}\circ E_{21},\ E_{12}\circ E_{22},\ ...,\ E_{1n}\circ E_{2n})$ is given by

$$(M_1 \circ M_2,\ N_{11} \circ N_{21},\ N_{12} \circ N_{22},\ ...,\ N_{1n} \circ N_{2n})$$

such that

$$\begin{cases}
\mu^P_{(M_1\circ M_2)}(xy) = (\mu^P_{M_1}\circ \mu^P_{M_2})(xy) = \mu^P_{M_1}(x)\wedge \mu^P_{M_2}(y),\\
\mu^N_{(M_1\circ M_2)}(xy) = (\mu^N_{M_1}\circ \mu^N_{M_2})(xy) = \mu^N_{M_1}(x)\vee \mu^N_{M_2}(y),\ \forall\ xy \in U_1\times U_2
\end{cases}$$

$$\begin{cases}
\mu^P_{(N_{1i}\circ N_{2i})}((xy_1)(xy_2)) = (\mu^P_{N_{1i}}\circ \mu^P_{N_{2i}})((xy_1)(xy_2)) = \mu^P_{M_1}(x)\wedge \mu^P_{N_{2i}}(y_1y_2),\\
\mu^N_{(N_{1i}\circ N_{2i})}((xy_1)(xy_2)) = (\mu^N_{N_{1i}}\circ \mu^N_{N_{2i}})((xy_1)(xy_2)) = \mu^N_{M_1}(x)\vee \mu^N_{N_{2i}}(y_1y_2),\ \forall\ x\in U_1,\ y_1y_2\in E_{2i}
\end{cases}$$

$$\begin{cases}
\mu^P_{(N_{1i}\circ N_{2i})}((x_1y)(x_2y)) = (\mu^P_{N_{1i}}\circ \mu^P_{N_{2i}})((x_1y)(x_2y)) = \mu^P_{M_2}(y)\wedge \mu^P_{N_{1i}}(x_1x_2),\\
\mu^N_{(N_{1i}\circ N_{2i})}((x_1y)(x_2y)) = (\mu^N_{N_{1i}}\circ \mu^N_{N_{2i}})((x_1y)(x_2y)) = \mu^N_{M_2}(y)\vee \mu^N_{N_{1i}}(x_1x_2),\\
\qquad \forall\ y\in U_2,\ x_1x_2\in E_{1i}
\end{cases}$$

$$\begin{cases}
\mu^P_{(N_{1i}\circ N_{2i})}((x_1y_1)(x_2y_2)) = (\mu^P_{N_{1i}}\circ \mu^P_{N_{2i}})((x_1y_1)(x_2y_2)) = \mu^P_{M_2}(y_1)\wedge \mu^P_{M_2}(y_2)\wedge \mu^P_{N_{1i}}(x_1x_2),\\
\mu^N_{(N_{1i}\circ N_{2i})}((x_1y_1)(x_2y_2)) = (\mu^N_{N_{1i}}\circ \mu^N_{N_{2i}})((x_1y_1)(x_2y_2))\\
\qquad = \mu^N_{M_2}(y_1)\vee \mu^N_{M_2}(y_2)\vee \mu^N_{N_{1i}}(x_1x_2),\\
\qquad \forall\ x_1x_2\in E_{1i},\ y_1,\ y_2\in U_2\ \textit{such that}\ y_1\neq y_2.
\end{cases}$$

Example 10.

Consider $\check{G}_{b1}$ and $\check{G}_{b2}$ as shown in Figure 2. Their *composition* represented by $\check{G}_{b1}\circ\check{G}_{b2} = (M_1 \circ M_2,\ N_{11}\circ N_{21},\ N_{12}\circ N_{22})$ is shown in Figure 12.

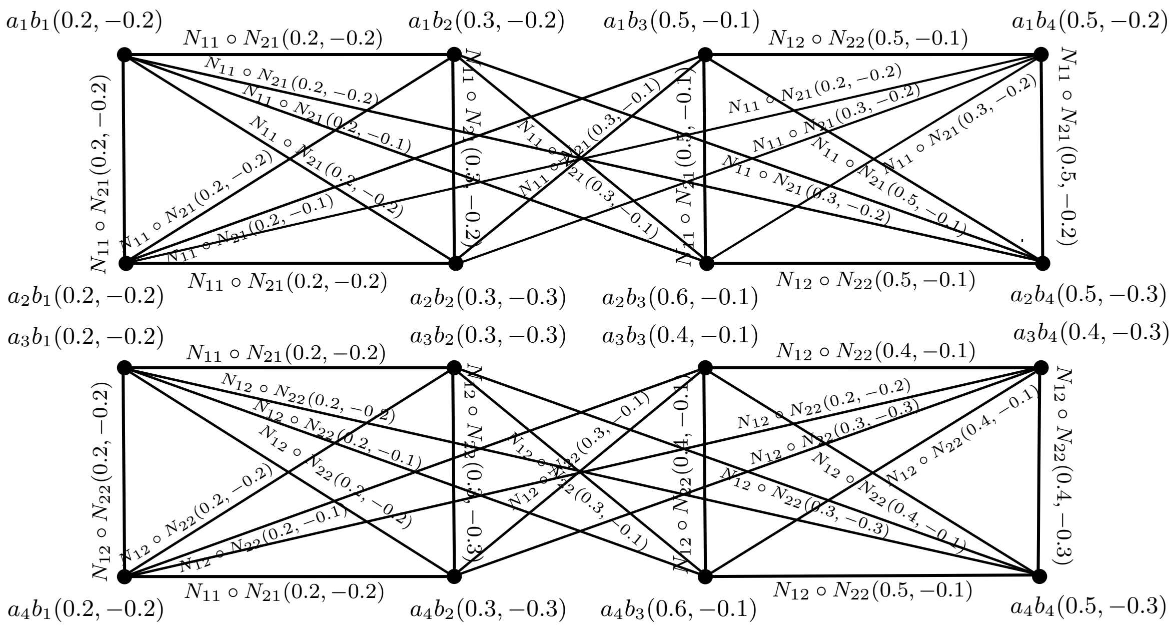


Figure 12: Composition of Two *BFGSs*

Example 11.

Let $\check{G_{b1}}$ and $\check{G_{b2}}$ be *BFGSs* as shown in Figure 4. Their *composition* represented by $\check{G_{b1}}\circ\check{G_{b2}} = (M_1\circ M_2,\ N_{11}\circ N_{21},\ N_{12}\circ N_{22})$ is shown in Figure 13.

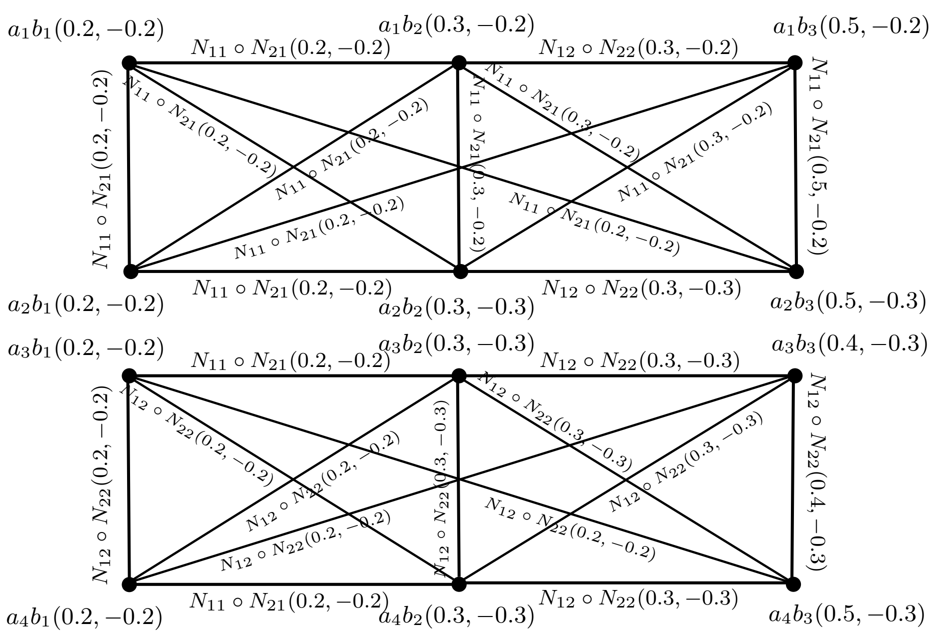


Figure 13: Composition of Two *BFGSs*

Theorem 5.

Let $G^* = (U_1 \circ U_2,\ E_{11} \circ E_{21},\ E_{12} \circ E_{22},\ ...,\ E_{1n} \circ E_{2n})$ be the composition of GSs $G_1^* = (U_1,\ E_{11},\ E_{12},\ ...,\ E_{1n})$ and $G_2^* = (U_2,\ E_{21},\ E_{22},\ ...,\ E_{2n})$. Let $\check{G_{b1}} = (M_1,\ N_{11},\ N_{12},\ ...,\ N_{1n})$ and $\check{G_{b2}} = (M_2,\ N_{21},\ N_{22},\ ...,\ N_{2n})$ be respective BFGSs of G_1^* and G_2^*. Then $\check{G_{b1}} \circ \check{G_{b2}} = (M_1 \circ M_2,\ N_{11} \circ N_{21},\ N_{12} \circ N_{22},\ ...,\ N_{1n} \circ N_{2n})$ is a BFGS of G^*.

Proof:

Case 1. When $u \in U_1,\ b_1b_2 \in\ E_{2i}$

$$\begin{aligned}
\mu^P_{(N_{1i}\circ N_{2i})}((ub_1)(ub_2)) &= \mu^P_{M_1}(u) \wedge \mu^P_{N_{2i}}(b_1b_2)\\
&\leq \mu^P_{M_1}(u) \wedge [\mu^P_{M_2}(b_1) \wedge \mu^P_{M_2}(b_2)]\\
&= [\mu^P_{M_1}(u) \wedge \mu^P_{M_2}(b_1)] \wedge [\mu^P_{M_1}(u) \wedge \mu^P_{M_2}(b_2)]\\
&= \mu^P_{(M_1\circ M_2)}(ub_1) \wedge \mu^P_{(M_1\circ M_2)}(ub_2),
\end{aligned}$$

$$\begin{aligned}
\mu^N_{(N_{1i}\circ N_{2i})}((ub_1)(ub_2)) &= \mu^N_{M_1}(u) \vee \mu^N_{N_{2i}}(b_1b_2)\\
&\geq \mu^N_{M_1}(u) \vee [\mu^N_{M_2}(b_1) \vee \mu^N_{M_2}(b_2)]\\
&= [\mu^N_{M_1}(u) \vee \mu^N_{M_2}(b_1)] \vee [\mu^N_{M_1}(u) \vee \mu^N_{M_2}(b_2)]\\
&= \mu^N_{(M_1\circ M_2)}(ub_1) \vee \mu^N_{(M_1\circ M_2)}(ub_2),
\end{aligned}$$

for $ub_1,\ ub_2 \in U_1 \circ U_2$.

Case 2. When $u \in U_2,\ b_1b_2 \in\ E_{1i}$

$$\begin{aligned}
\mu^P_{(N_{1i}\circ N_{2i})}((b_1u)(b_2u)) &= \mu^P_{M_2}(u) \wedge \mu^P_{N_{1i}}(b_1b_2)\\
&\leq \mu^P_{M_2}(u) \wedge [\mu^P_{M_1}(b_1) \wedge \mu^P_{M_1}(b_2)]\\
&= [\mu^P_{M_2}(u) \wedge \mu^P_{M_1}(b_1)] \wedge [\mu^P_{M_2}(u) \wedge \mu^P_{M_1}(b_2)]\\
&= \mu^P_{(M_1\circ M_2)}(b_1u) \wedge \mu^P_{(M_1\circ M_2)}(b_2u),
\end{aligned}$$

$$\begin{aligned}
\mu^N_{(N_{1i}\circ N_{2i})}((b_1u)(b_2u)) &= \mu^N_{M_2}(u) \vee \mu^N_{N_{1i}}(b_1b_2)\\
&\geq \mu^N_{M_2}(u) \vee [\mu^N_{M_1}(b_1) \vee \mu^N_{M_1}(b_2)]\\
&= [\mu^N_{M_2}(u) \vee \mu^N_{M_1}(b_1)] \vee [\mu^N_{M_2}(u) \vee \mu^N_{M_1}(b_2)]\\
&= \mu^N_{(M_1\circ M_2)}(b_1u) \vee \mu^N_{(M_1\circ M_2)}(b_2u),
\end{aligned}$$

for $b_1u,\ b_2u \in U_1 \circ U_2$.

Case 3. When $b_1b_2 \in\ E_{1i},\ u_1,\ u_2 \in U_2$ *such that* $u_1 \neq u_2$,

$$\begin{aligned}
\mu^P_{(N_{1i}\circ N_{2i})}((b_1u_1)(b_2u_2)) &= \mu^P_{M_2}(u_1) \wedge \mu^P_{M_2}(u_2) \wedge \mu^P_{N_{1i}}(b_1b_2)\\
&\leq \mu^P_{M_2}(u_1) \wedge \mu^P_{M_2}(u_2) \wedge [\mu^P_{M_1}(b_1) \wedge \mu^P_{M_1}(b_2)]\\
&= [\mu^P_{M_2}(u_1) \wedge \mu^P_{M_1}(b_1)] \wedge [\mu^P_{M_2}(u_2) \wedge \mu^P_{M_1}(b_2)]\\
&= \mu^P_{(M_1\circ M_2)}(b_1u_1) \wedge \mu^P_{(M_1\circ M_2)}(b_2u_2),
\end{aligned}$$

$$\begin{aligned}\mu^N_{(N_{1i}\circ N_{2i})}((b_1u_1)(b_2u_2)) &= \mu^N_{M_2}(u_1)\vee\mu^N_{M_2}(u_2)\vee\mu^N_{N_{1i}}(b_1b_2)\\ &\geq \mu^N_{M_2}(u_1)\vee\mu^N_{M_2}(u_2)\vee[\mu^N_{M_1}(b_1)\vee\mu^N_{M_1}(b_2)]\\ &= [\mu^N_{M_2}(u_1)\vee\mu^N_{M_1}(b_1)]\vee[\mu^N_{M_2}(u_2)\vee\mu^N_{M_1}(b_2)]\\ &= \mu^N_{(M_1\circ M_2)}(b_1u_1)\vee\mu^N_{(M_1\circ M_2)}(b_2u_2),\end{aligned}$$

for $b_1u_1,\ b_2u_2\in U_1\circ U_2$.

All three cases hold for $i=1,\ 2,\ ...,\ n$. This completes the proof. □

Definition 11.

Let $\check{G_{b1}}=(M_1,\ N_{11},\ N_{12},\ ...,\ N_{1n})$ and $\check{G_{b2}}=(M_2,\ N_{21},\ N_{22},\ ...,\ N_{2n})$ be respective *BFGSs of GSs* $G_1^*=(U_1,\ E_{11},\ E_{12},\ ...,\ E_{1n})$ and $G_2^*=(U_2,\ E_{21},\ E_{22},\ ...,\ E_{2n})$ and let $U_1\cap U_2=\emptyset$. The *union* $\check{G_{b1}}\cup\check{G_{b2}}$, of $\check{G_{b1}}$ and $\check{G_{b2}}$ is then a *BFGS* of $G_1^*\cup G_2^*=(U_1\cup U_2,\ E_{11}\cup E_{21},\ E_{12}\cup E_{22},\ ...,\ E_{1n}\cup E_{2n})$ is given by

$$(M_1\cup M_2,\ N_{11}\cup N_{21},\ N_{12}\cup N_{22},\ ...,\ N_{1n}\cup N_{2n})$$

such that $M_1\cup M_2$ is defined by

$$\mu^P_{(M_1\cup M_2)}(x)=(\mu^P_{M_1}\cup\mu^P_{M_2})(x)=\mu^P_{M_1}(x)\vee\mu^P_{M_2}(x),$$
$$\mu^N_{(M_1\cup M_2)}(x)=(\mu^N_{M_1}\cup\mu^N_{M_2})(x)=\mu^N_{M_1}(x)\wedge\mu^N_{M_2}(x)\ \forall\ x\in U_1\cup U_2$$

(assuming $\mu^P_{M_j}(x)=0,\ \mu^N_{M_j}(x)=0\ if\ x\notin U_j,\ j=1,\ 2)$

and $N_{1i}\cup N_{2i}$ for $i=1,\ 2,\ ...,\ n,$ is defined by

$$\mu^P_{(N_{1i}\cup N_{2i})}(xy)=(\mu^P_{N_{1i}}\cup\mu^P_{N_{2i}})(xy)=\mu^P_{N_{1i}}(xy)\vee\mu^P_{N_{2i}}(xy),$$
$$\mu^N_{(N_{1i}\cup N_{2i})}(x)=(\mu^N_{N_{1i}}\cup\mu^N_{N_{2i}})(xy)=\mu^N_{N_{1i}}(x)\wedge\mu^N_{N_{2i}}(x)\ \forall\ xy\in E_{1i}\cup E_{2i}$$

(assuming $\mu^P_{N_{ji}}(xy)=0,\ \mu^N_{N_{ji}}(xy)=0\ if\ xy\notin E_{ji},\ j=1,\ 2)$.

Example 12.

Let $\check{G_{b1}}$ and $\check{G_{b2}}$ be *BFGSs* as shown in Figure 2. Their *union* represented by $\check{G_{b1}}\cup\check{G_{b2}}=(M_1\cup M_2,\ N_{11}\cup N_{21},\ N_{12}\cup N_{22})$ is shown in Figure 14.

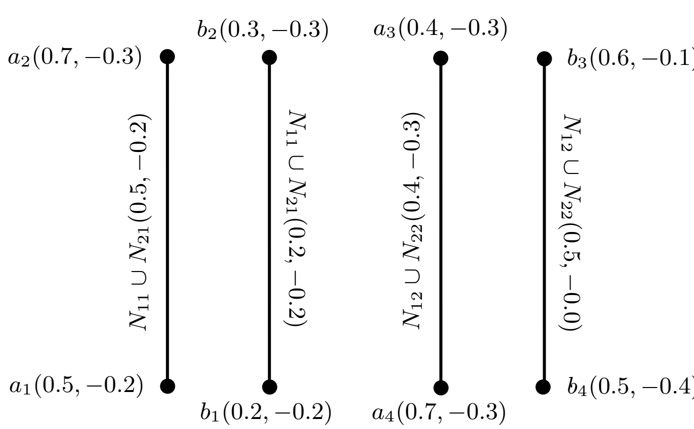


Figure 14: Union of Two *BFGSs*

Example 13.

Let $\breve{G_{b1}}$ and $\breve{G_{b2}}$ be *BFGSs* as shown in Figure 4. Their *union* represented by $\breve{G_{b1}} \cup \breve{G_{b2}} = (M_1 \cup M_2,\ N_{11} \cup N_{21},\ N_{12} \cup N_{22})$ is shown in Figure 15.

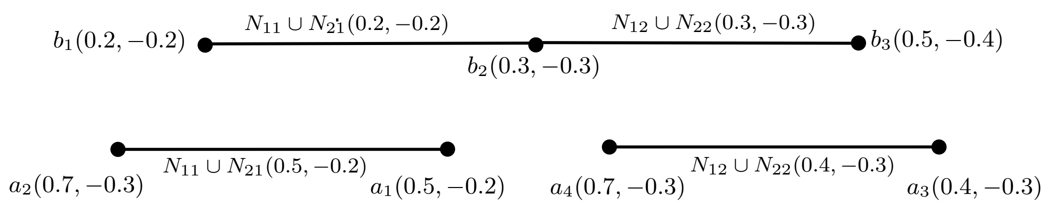


Figure 15: Union of Two *BFGSs*

Theorem 6.

Let $G^* = (U_1 \cup U_2,\ E_{11} \cup E_{21},\ E_{12} \cup E_{22},\ ...,\ E_{1n} \cup E_{2n})$ be the union of GSs $G_1^* = (U_1,\ E_{11},\ E_{12},\ ...,\ E_{1n})$ and $G_2^* = (U_2,\ E_{21},\ E_{22},\ ...,\ E_{2n})$. Let $\breve{G_{b1}} = (M_1,\ N_{11},\ N_{12},\ ...,\ N_{1n})$ and $\breve{G_{b2}} = (M_2,\ N_{21},\ N_{22},\ ...,\ N_{2n})$ be respective BFGSs of G_1^* and G_2^*. Then $\breve{G_{b1}} \cup \breve{G_{b2}} = (M_1 \cup M_2,\ N_{11} \cup N_{21},\ N_{12} \cup N_{22},\ ...,\ N_{1n} \cup N_{2n})$ is a BFGS of G^*.

Proof:

Let $u_1u_2 \in E_{1i} \cup E_{2i}$.

Case 1. When $u_1,\ u_2 \in U_1$, then by definition 11

$$\mu^P_{M_2}(u_1) = \mu^P_{M_2}(u_2) = \mu^P_{N_{2i}}(u_1u_2) = 0,\ \mu^N_{M_2}(u_1) = \mu^N_{M_2}(u_2) = \mu^N_{N_{2i}}(u_1u_2) = 0,$$

so we have

$$\begin{aligned}
\mu^P_{(N_{1i} \cup N_{2i})}(u_1u_2) &= \mu^P_{N_{1i}}(u_1u_2) \vee \mu^P_{N_{2i}}(u_1u_2) \\
&= \mu^P_{N_{1i}}(u_1u_2) \vee\ 0 \\
&\leq [\mu^P_{M_1}(u_1) \wedge \mu^P_{M_1}(u_2)] \vee\ 0 \\
&= [\mu^P_{M_1}(u_1) \vee\ 0] \wedge [\mu^P_{M_1}(u_2) \vee\ 0] \\
&= [\mu^P_{M_1}(u_1) \vee \mu^P_{M_2}(u_1)] \wedge [\mu^P_{M_1}(u_2) \vee \mu^P_{M_2}(u_2)] \\
&= \mu^P_{(M_1 \cup M_2)}(u_1) \wedge \mu^P_{(M_1 \cup M_2)}(u_2),
\end{aligned}$$

$$\begin{aligned}
\mu^N_{(N_{1i} \cup N_{2i})}(u_1u_2) &= \mu^N_{N_{1i}}(u_1u_2) \wedge \mu^N_{N_{2i}}(u_1u_2) \\
&= \mu^N_{N_{1i}}(u_1u_2) \wedge\ 0 \\
&\geq [\mu^N_{M_1}(u_1) \vee \mu^N_{M_1}(u_2)] \wedge\ 0 \\
&= [\mu^N_{M_1}(u_1) \wedge\ 0] \vee [\mu^N_{M_1}(u_2) \wedge\ 0] \\
&= [\mu^N_{M_1}(u_1) \wedge \mu^N_{M_2}(u_1)] \vee [\mu^N_{M_1}(u_2) \wedge \mu^N_{M_2}(u_2)] \\
&= \mu^N_{(M_1 \cup M_2)}(u_1) \vee \mu^N_{(M_1 \cup M_2)}(u_2),
\end{aligned}$$

for $u_1,\ u_2 \in U_1 \cup U_2$.

Case 2. When $u_1,\ u_2 \in U_2$, then by definition 11

$$\mu^P_{M_1}(u_1) = \mu^P_{M_1}(u_2) = \mu^P_{N_{1i}}(u_1u_2) = 0,\ \mu^N_{M_1}(u_1) = \mu^N_{M_1}(u_2) = \mu^N_{N_{1i}}(u_1u_2) = 0,$$

so we have

$$\begin{aligned}
\mu^P_{(N_{1i}\cup N_{2i})}(u_1u_2) &= \mu^P_{N_{1i}}(u_1u_2) \vee \mu^P_{N_{2i}}(u_1u_2)\\
&= 0\ \vee \mu^P_{N_{2i}}(u_1u_2)\\
&\leq 0\ \vee [\mu^P_{M_2}(u_1) \wedge \mu^P_{M_2}(u_2)]\\
&= [0 \vee \mu^P_{M_2}(u_1)] \wedge [0 \vee \mu^P_{M_1}(u_2)]\\
&= [\mu^P_{M_1}(u_1) \vee \mu^P_{M_2}(u_1)] \wedge [\mu^P_{M_1}(u_2) \vee \mu^P_{M_2}(u_2)]\\
&= \mu^P_{(M_1\cup M_2)}(u_1) \wedge \mu^P_{(M_1\cup M_2)}(u_2),
\end{aligned}$$

$$\begin{aligned}
\mu^N_{(N_{1i}\cup N_{2i})}(u_1u_2) &= \mu^N_{N_{1i}}(u_1u_2) \wedge \mu^N_{N_{2i}}(u_1u_2)\\
&= 0\ \wedge \mu^N_{N_{2i}}(u_1u_2)\\
&\geq 0\ \wedge [\mu^N_{M_2}(u_1) \vee \mu^N_{M_2}(u_2)]\\
&= [0 \wedge \mu^N_{M_2}(u_1)] \vee [0 \wedge \mu^N_{M_1}(u_2)]\\
&= [\mu^N_{M_1}(u_1) \wedge \mu^N_{M_2}(u_1)] \vee [\mu^N_{M_1}(u_2) \wedge \mu^N_{M_2}(u_2)]\\
&= \mu^N_{(M_1\cup M_2)}(u_1) \vee \mu^N_{(M_1\cup M_2)}(u_2),
\end{aligned}$$

for $u_1,\ u_2 \in U_1 \cup U_2$.

Both cases hold for $i = 1,\ 2,\ ...,\ n$. This completes the proof. □

Theorem 7.

If $G^* = (U_1{\cup}U_2,\ E_{11}{\cup}E_{21},\ E_{12}{\cup}E_{22},\ ...,\ E_{1n}{\cup}E_{2n})$ is the union of GSs $G^*_1 = (U_1,\ E_{11},\ E_{12},\ ...,\ E_{1n})$ and $G^*_2 = (U_2,\ E_{21},\ E_{22},\ ...,\ E_{2n})$. Then every BFGS $\check{G}_b = (M,\ N_1,\ N_2,\ ...,\ N_n)$ of G^* is the union of a BFGS $\check{G_{b1}}$ of G^*_1 and a BFGS $\check{G_{b2}}$ of G^*_2.

Proof:

We define $M_1,\ \ M_2$, N_{1i} and N_{2i} for $i = 1,\ 2,\ ...,\ n$ as

$$\begin{array}{ll}
\mu^P_{M_1}(u) = \mu^P_M(u),\ \mu^N_{M_1}(u) = \mu^N_M(u), & if\ u \in U_1\\
\mu^P_{M_2}(u) = \mu^P_M(u),\ \mu^N_{M_2}(u) = \mu^N_M(u), & if\ u \in U_2\\
\mu^P_{N_{1i}}(u_1u_2) = \mu^P_{N_i}(u_1u_2),\ \mu^N_{N_{1i}}(u_1u_2) = \mu^N_{N_i}(u_1u_2), & if\ u_1u_2 \in E_1i\\
\mu^P_{N_{2i}}(u_1u_2) = \mu^P_{N_i}(u_1u_2),\ \mu^N_{N_{2i}}(u_1u_2) = \mu^N_{N_i}(u_1u_2), & if\ u_1u_2 \in E_2i.
\end{array}$$

Then, $M = M_1 \cup M_2$ and $N_i = N_{1i} \cup N_{2i},\ i = 1,\ 2,\ ...,\ n.$

Now for $u_1u_2 \in E_{ji},\ j = 1,\ 2\ and\ i = 1,\ 2,\ ...,\ n$

$$\mu^P_{N_{ji}}(u_1u_2) = \mu^P_{N_i}(u_1u_2) \leq \mu^P_M(u_1) \wedge \mu^P_M(u_2) = \mu^P_{M_j}(u_1) \wedge \mu^P_{M_j}(u_2)$$
$$\mu^N_{N_{ji}}(u_1u_2) = \mu^N_{N_i}(u_1u_2) \geq \mu^N_M(u_1) \vee \mu^N_M(u_2) = \mu^N_{M_j}(u_1) \vee \mu^N_{M_j}(u_2),$$

i.e.,

$\check{G_{bj}} = (M_j,\ N_{j1},\ N_{j2},\ ...,\ N_{jn})$ is a *BFGS* of $G^*_j,\ j = 1,\ 2.$

Thus, $\check{G}_b = (M,\ N_1,\ N_2,\ ...,\ N_n)$, a *BFGS* of $G^* = G_1 \cup G_2$, is the *union* of a *BFGS* of G_1^* and a *BFGS* of G_2^*. □

Definition 12.

Let $\check{G_{b1}} = (M_1,\ N_{11},\ N_{12},\ ...,\ N_{1n})$ and $\check{G_{b2}} = (M_2,\ N_{21},\ N_{22},\ ...,\ N_{2n})$ be respective *BFGSs* of *GSs* $G_1^* = (U_1,\ E_{11},\ E_{12},\ ...,\ E_{1n})$ and $G_2^* = (U_2,\ E_{21},\ E_{22},\ ...,\ E_{2n})$ and let $U_1 \cap U_2 = \emptyset$. The *join* $\check{G_{b1}} + \check{G_{b2}}$ of $\check{G_{b1}}$ and $\check{G_{b2}}$, is then a *BFGS* of $G_1^* + G_2^* = (U_1 + U_2,\ E_{11} + E_{21},\ E_{12} + E_{22},\ ...,\ E_{1n} + E_{2n})$ is given by

$$(M_1 + M_2,\ N_{11} + N_{21},\ N_{12} + N_{22},\ ...,\ N_{1n} + N_{2n})$$

such that $M_1 + M_2$ is defined by

$$\mu^P_{(M_1+M_2)}(x) = \mu^P_{(M_1 \cup M_2)}(x),$$
$$\mu^N_{(M_1+M_2)}(x) = \mu^N_{(M_1 \cup M_2)}(x)\ \forall\ x \in U_1 \cup U_2,$$

$N_{1i} + N_{2i}$ for $i = 1,\ 2,\ ...,\ n$ is defined by

$$\mu^P_{(N_{1i}+N_{2i})}(xy) = \mu^P_{(N_{1i} \cup N_{2i})}(xy),$$
$$\mu^N_{(N_{1i}+N_{2i})}(x) = \mu^N_{(N_{1i} \cup N_{2i})}(x)\ \forall\ xy \in E_{1i} \cup E_{2i}$$

and

$$\mu^P_{(N_{1i}+N_{2i})}(xy) = (\mu^P_{N_{1i}} + \mu^P_{N_{2i}})(xy) = \mu^P_{M_1}(x) \wedge \mu^P_{M_2}(y),$$
$$\mu^N_{(N_{1i}+N_{2i})}(x) = (\mu^N_{N_{1i}} + \mu^N_{N_{2i}})(xy) = \mu^N_{M_1}(x) \vee \mu^N_{M_2}(y)\ \forall\ x \in U_1,\ y \in U_2.$$

Example 14.

Let $\check{G_{b1}}$ and $\check{G_{b2}}$ be *BFGSs* as shown in Figure 2. Their *join* represented by $\check{G_{b1}} + \check{G_{b2}} = (M_1 + M_2,\ N_{11} + N_{21},\ N_{12} + N_{22})$ is shown in Figure 16.

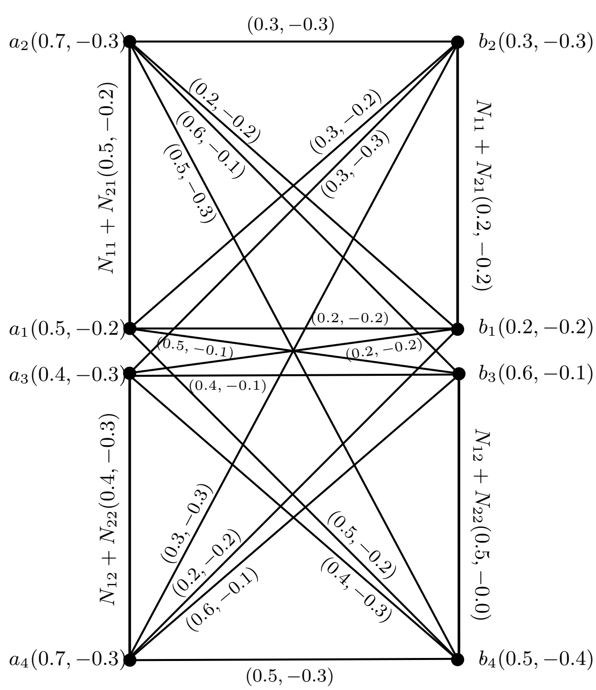


Figure 16: Join of Two *BFGSs*

Example 15.

Let $\check{G_{b1}}$ and $\check{G_{b2}}$ be *BFGSs* as shown in Figure 4. Their *join* represented by $\check{G_{b1}} + \check{G_{b2}} = (M_1 + M_2,\ N_{11} + N_{21},\ N_{12} + N_{22})$ is shown in Figure 17.

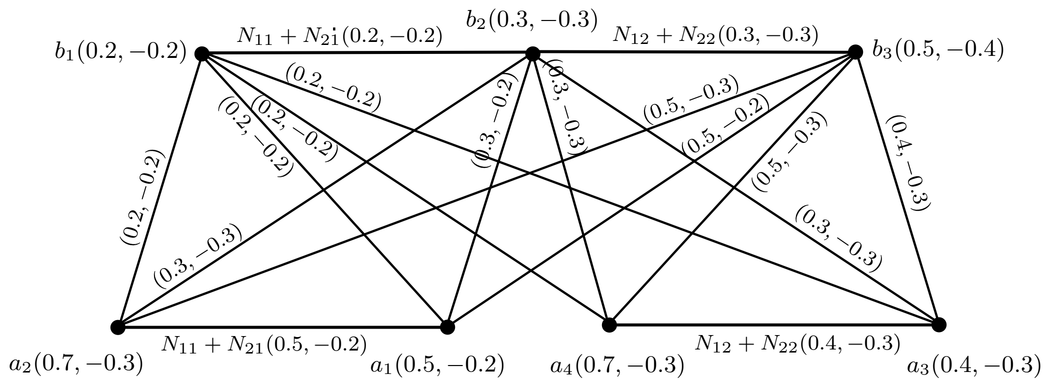


Figure 17: Join of Two *BFGSs*

Theorem 8.

Let $G^* = (U_1 + U_2,\ E_{11} + E_{21},\ E_{12} + E_{22},\ ...,\ E_{1n} + E_{2n})$ be the join of GSs $G_1^* = (U_1,\ E_{11},\ E_{12},\ ...,\ E_{1n})$ and $G_2^* = (U_2,\ E_{21},\ E_{22},\ ...,\ E_{2n})$. Let $\check{G_{b1}} = (M_1,\ N_{11},\ N_{12},\ ...,\ N_{1n})$ and $\check{G_{b2}} = (M_2,\ N_{21},\ N_{22},\ ...,\ N_{2n})$ be respective BFGSs of G_1^* and G_2^*. Then $\check{G_{b1}} + \check{G_{b2}} = (M_1 + M_2,\ N_{11} + N_{21},\ N_{12} + N_{22},\ ...,\ N_{1n} + N_{2n})$ is a BFGS of G^*.

Proof:

Let $u_1 u_2 \in E_{1i} + E_{2i}$.

Case 1. When $u_1,\ u_2 \in U_1$, then by definition 12

$$\mu^P_{M_2}(u_1) = \mu^P_{M_2}(u_2) = \mu^P_{N_{2i}}(u_1u_2) = 0,\ \mu^N_{M_2}(u_1) = \mu^N_{M_2}(u_2) = \mu^N_{N_{2i}}(u_1u_2) = 0,$$

so we have

$$\begin{aligned}
\mu^P_{(N_{1i}+N_{2i})}(u_1u_2) &= \mu^P_{N_{1i}}(u_1u_2) \vee \mu^P_{N_{2i}}(u_1u_2)\\
&= \mu^P_{N_{1i}}(u_1u_2) \vee\ 0\\
&\leq [\mu^P_{M_1}(u_1) \wedge \mu^P_{M_1}(u_2)] \vee\ 0\\
&= [\mu^P_{M_1}(u_1) \vee\ 0] \wedge [\mu^P_{M_1}(u_2) \vee\ 0]\\
&= [\mu^P_{M_1}(u_1) \vee \mu^P_{M_2}(u_1)] \wedge [\mu^P_{M_1}(u_2) \vee \mu^P_{M_2}(u_2)]\\
&= \mu^P_{(M_1+M_2)}(u_1) \wedge \mu^P_{(M_1+M_2)}(u_2),
\end{aligned}$$

$$\begin{aligned}
\mu^N_{(N_{1i}+N_{2i})}(u_1u_2) &= \mu^N_{N_{1i}}(u_1u_2) \wedge \mu^N_{N_{2i}}(u_1u_2)\\
&= \mu^N_{N_{1i}}(u_1u_2) \wedge \ 0\\
&\geq [\mu^N_{M_1}(u_1) \vee \mu^N_{M_1}(u_2)] \wedge \ 0\\
&= [\mu^N_{M_1}(u_1) \wedge \ 0] \vee [\mu^N_{M_1}(u_2) \wedge \ 0]\\
&= [\mu^N_{M_1}(u_1) \wedge \mu^N_{M_2}(u_1)] \vee [\mu^N_{M_1}(u_2) \wedge \mu^N_{M_2}(u_2)]\\
&= \mu^N_{(M_1+M_2)}(u_1) \vee \mu^N_{(M_1+M_2)}(u_2),
\end{aligned}$$

for $u_1,\ u_2 \in U_1 + U_2$.

Case 2. When $u_1,\ u_2 \in U_2$, then by definition 12

$$\mu^P_{M_1}(u_1) = \mu^P_{M_1}(u_2) = \mu^P_{N_{1i}}(u_1u_2) = 0,\ \mu^N_{M_1}(u_1) = \mu^N_{M_1}(u_2) = \mu^N_{N_{1i}}(u_1u_2) = 0,$$

so we have

$$\begin{aligned}
\mu^P_{(N_{1i}+N_{2i})}(u_1u_2) &= \mu^P_{N_{1i}}(u_1u_2) \vee \mu^P_{N_{2i}}(u_1u_2)\\
&= 0 \ \vee \mu^P_{N_{2i}}(u_1u_2)\\
&\leq 0 \ \vee [\mu^P_{M_2}(u_1) \wedge \mu^P_{M_2}(u_2)]\\
&= [0 \vee \mu^P_{M_2}(u_1)] \wedge [0 \vee \mu^P_{M_1}(u_2)]\\
&= [\mu^P_{M_1}(u_1) \vee \mu^P_{M_2}(u_1)] \wedge [\mu^P_{M_1}(u_2) \vee \mu^P_{M_2}(u_2)]\\
&= \mu^P_{(M_1+M_2)}(u_1) \wedge \mu^P_{(M_1+M_2)}(u_2),
\end{aligned}$$

$$\begin{aligned}
\mu^N_{(N_{1i}+N_{2i})}(u_1u_2) &= \mu^N_{N_{1i}}(u_1u_2) \wedge \mu^N_{N_{2i}}(u_1u_2)\\
&= 0 \ \wedge \mu^N_{N_{2i}}(u_1u_2)\\
&\geq 0 \ \wedge [\mu^N_{M_2}(u_1) \vee \mu^N_{M_2}(u_2)]\\
&= [0 \wedge \mu^N_{M_2}(u_1)] \vee [0 \wedge \mu^N_{M_1}(u_2)]\\
&= [\mu^N_{M_1}(u_1) \wedge \mu^N_{M_2}(u_1)] \vee [\mu^N_{M_1}(u_2) \wedge \mu^N_{M_2}(u_2)]\\
&= \mu^N_{(M_1+M_2)}(u_1) \vee \mu^N_{(M_1+M_2)}(u_2),
\end{aligned}$$

for $u_1,\ u_2 \in U_1 + U_2$.

Case 3. When $u_1 \in U_1,\ u_2 \in U_2$ then by definition 12

$$\mu^P_{M_1}(u_2) = \mu^P_{M_2}(u_1) = \mu^N_{M_1}(u_2) = \mu^N_{M_2}(u_1) = 0,$$

and we have

$$\begin{aligned}
\mu^P_{(N_{1i}+N_{2i})}(u_1u_2) &= \mu^P_{M_1}(u_1) \wedge \mu^P_{M_2}(u_2)\\
&= [\mu^P_{M_1}(u_1) \vee \ 0] \wedge [0 \vee \mu^P_{M_2}(u_2)]\\
&= [\mu^P_{M_1}(u_1) \vee \mu^P_{M_2}(u_1)] \wedge [\mu^P_{M_1}(u_2) \vee \mu^P_{M_2}(u_2)]\\
&= \mu^P_{(M_1+M_2)}(u_1) \wedge \mu^P_{(M_1+M_2)}(u_2),
\end{aligned}$$

$$\begin{aligned}\mu^N_{(N_{1i}+N_{2i})}(u_1u_2) &= \mu^N_{M_1}(u_1)\vee\mu^N_{M_2}(u_2)\\ &= [\mu^N_{M_1}(u_1)\wedge\ 0]\vee[0\wedge\mu^N_{M_2}(u_2)]\\ &= [\mu^N_{M_1}(u_1)\wedge\mu^N_{M_2}(u_1)]\vee[\mu^N_{M_1}(u_2)\wedge\mu^N_{M_2}(u_2)]\\ &= \mu^N_{(M_1+M_2)}(u_1)\vee\mu^N_{(M_1+M_2)}(u_2),\end{aligned}$$

for $u_1,\ u_2\in U_1+U_2$.

All three cases hold for $i=1,\ 2,\ ...,\ n$. This completes the proof. □

Theorem 9.

If $G^*=(U_1+U_2,\ E_{11}+E_{21},\ E_{12}+E_{22},\ ...,\ E_{1n}+E_{2n})$ is the join of *GSs* $G_1^*=(U_1,\ E_{11},\ E_{12},\ ...,\ E_{1n})$ and $G_2^*=(U_2,\ E_{21},\ E_{22},\ ...,\ E_{2n})$ and $\check{G}_b=(M,\ N_1,\ N_2,\ ...,\ N_n)$ is a strong BFGS of G^* Then $\check{G}_b$ is the join of $\check{G}_{b1}$, a strong BFGS of G_1^*, and $\check{G}_{b2}$, a strong BFGS of G_2^*.

Proof:

Let define M_j and N_{ji} for $i=1,\ 2,\ ...,\ n$ and $j=1,\ 2$ as

$$\begin{array}{ll}\mu^P_{M_j}(u)=\mu^P_M(u),\ \mu^N_{M_j}(u)=\mu^N_M(u), & if\ u\in U_j\\ \mu^P_{N_{ji}}(u_1u_2)=\mu^P_{N_i}(u_1u_2),\ \mu^N_{N_{ji}}(u_1u_2)=\mu^N_{N_i}(u_1u_2), & if\ u_1u_2\in E_{ji}.\end{array}$$

By similar way as in the proof of Theorem 7, for $u_1u_2\in E_{ji},\ j=1,\ 2\ and\ i=1,\ 2,\ ...,\ n$

$$\mu^P_{N_{ji}}(u_1u_2)=\mu^P_{N_i}(u_1u_2)=\mu^P_M(u_1)\wedge\mu^P_M(u_2)=\mu^P_{M_j}(u_1)\wedge\mu^P_{M_j}(u_2)$$
$$\mu^N_{N_{ji}}(u_1u_2)=\mu^N_{N_i}(u_1u_2)=\mu^N_M(u_1)\vee\mu^N_M(u_2)=\mu^N_{M_j}(u_1)\vee\mu^N_{M_j}(u_2).$$

So $\check{G}_{bj}=(M_j,\ N_{j1},\ N_{j2},\ ...,\ N_{jn})$ is a strong *BFGS* of $G_j^*,\ j=1,\ 2.$

Moreover, $\check{G}_b$ is the join of $\check{G}_{b1}$ and $\check{G}_{b2}$ as shown in the following.

Using definitions 11 and 12, $M=M_1\cup M_2=M_1+M_2$ and

$$N_i=N_{1i}\cup N_{2i}=N_{1i}+N_{2i},\ \forall\ u_1u_2\in E_{1i}\cup E_{2i}.$$

When $u_1u_2\in E_{1i}+E_{2i}\setminus(E_{1i}\cup E_{2i})$, i.e., $u_1\in U_1$ and $u_2\in U_2$

$$\mu^P_{N_i}(u_1u_2)=\mu^P_M(u_1)\wedge\mu^P_M(u_2)=\mu^P_{M_1}(u_1)\wedge\mu^P_{M_2}(u_2)=\mu^P_{N_{1i}+N_{2i}}(u_1u_2)$$
$$\mu^N_{N_i}(u_1u_2)=\mu^N_M(u_1)\vee\mu^N_M(u_2)=\mu^N_{M_1}(u_1)\vee\mu^N_{M_2}(u_2)=\mu^N_{N_{1i}+N_{2i}}(u_1u_2)$$

There are similar calculations when $u_1\in U_2$ and $u_2\in U_1$. This is true for $i=1,\ 2,\ ...,\ n$. This ends the proof. □

4. Conclusions

Graph theoretical concepts are widely used to study and model various applications in different areas. However, in many cases, some aspects of a graph-theoretic problem may be vague or uncertain. It is natural to deal with the vagueness and uncertainty using the methods of fuzzy sets or bipolar fuzzy sets which have shown advantages in handling vagueness and uncertainty than fuzzy sets. So we have applied the concept of bipolar fuzzy sets to graph structures. We have discussed some operations on bipolar fuzzy graph structures. We are extending our work to: (1) Bipolar fuzzy soft graph structures, (2) Soft graph structures, (3) Rough fuzzy soft graph structures, and (4) Roughness in fuzzy graph structures.

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REFERENCES

Akram, M. (2011). Bipolar fuzzy graphs, Information Sciences, Vol. 181, pp. 5548-5564.

Akram, M. (2013). Bipolar fuzzy graphs with applications, Knowledge-Based Systems, Vol. 39, pp. 1-8.

Akram, M. and Dudek, W. A. (2012). Regular bipolar fuzzy graphs, Neural Computing & Applications, Vol. 21, No. 1, pp. 197–205.

Akram, M., Li, S. -G. and Shum, K. P. (2013). Antipodal bipolar fuzzy graphs, Italian Journal of Pure and Applied Mathematics, Vol.31, pp. 97-110.

Akram, M. and Akmal, R. (2016). Application of bipolar fuzzy sets in graph structures, Applied Computational Intelligence and Soft Computing, Article ID 5859080, pp. 1-13.

Bhattacharya, P. (1987). Some remarks on fuzzy graphs, Pattern Recognition Letter, Vol. 6, pp. 297-302.

Deb, M. and De, P.K. (2015). Optimal solution of a fully fuzzy linear fractional programming problem by using graded mean integration representation method, Applications and Applied Mathematics, Vol. 10, No. 1, pp. 571-587.

Dinesh, T. (2011). A study on graph structures, incidence algebras and their fuzzy analogues [Ph.D. thesis], Kannur University, Kannur, India.

Dinesh, T. and Ramakrishnan, T. V. (2011). On generalised fuzzy graph structures, Applied Mathematical Sciences, Vol. 5, No. 4, pp. 173 - 180.

Gulistan, M., Shahzad, M., Ahmed, S. and Ilyas, M. (2015). Characterization of gamma hemirings by generalized fuzzy gamma ideals, Applications and Applied Mathematics, Vol. 10, No. 1, pp. 495-520.

Kauffman, A. (1973). Introduction a la Theorie des Sous-emsembles Flous, Masson et Cie, Vol.1.

Lee, K.-M. (2000). Bipolar-valued fuzzy sets and their basic operations, Proc. Int. Conf., Bangkok, Thailand, pp. 307-317.

Mordeson, J. N. and Nair, P. S. (1998). Fuzzy graphs and fuzzy hypergraphs, Physica Verlag, Heidelberg.

Rosenfeld, A. (1975). Fuzzy graphs, Fuzzy Sets and their Applications (L.A. Zadeh, K.S. Fu, M. Shimura, Eds.), Academic Press, New York, pp. 77-95.

Sampathkumar, E. (2006). Generalized graph structures, Bull. Kerala Math. Assoc., Vol 3, No.

2, pp. 65-123.
Zadeh, L. A. (1965). Fuzzy sets, Information and Control, Vol. 8, pp. 338-353.
Zhang, W.-R. (1998). Bipolar fuzzy sets, Proc. of FUZZ-IEEE, pp. 835-840.