Certain Operations on Bipolar Fuzzy Graph Structures

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Abstract

A graph structure is a useful tool in solving the combinatorial problems in different areas of computer science and computational intelligence systems. A bipolar fuzzy graph structure is a generalization of a bipolar fuzzy graph. In this paper, we present several different types of operations, including composition, Cartesian product, strong product, cross product, and lexicographic product on bipolar fuzzy graph structures. We also investigate some properties of operations.

Keywords: Bipolar fuzzy graph structure (BFGS); Composition; Cartesian product; Strong product; Cross product; Lexicographic product; Join; Union

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1. Introduction

The concepts of graph theory have applications in many areas of computer science (such as data mining, image segmentation, clustering, image capturing, networking, etc.). Graph structures, introduced by Sampathkumar (2006), are a generalization of graphs which is quite useful in studying structures including graphs, signed graphs, and graphs in which every edge is labeled or colored. It helps to study various relations and the corresponding edges, simultaneously.

A fuzzy set, introduced by Zadeh (1965), gives the degree of membership of an object in a
given set. Based on the same idea, Zhang (1994) defined the notion of bipolar fuzzy set on a given set $X$, by saying that a mapping $A : X \rightarrow [-1, 1]$ was a bipolar fuzzy set where the membership degree $0$, of an element $x$, meant that the element $x$ was irrelevant to the corresponding property, the membership degree in $(0, 1]$, of an element $x$, indicated that the element somewhat satisfied the property, and the membership degree in $[-1, 0)$, of an element $x$, indicated that the element somewhat satisfied the implicit counter-property. Rosenfeld (1975) discussed the concept of fuzzy graphs whose basic idea was introduced by Kauffmann (1973). The fuzzy relations between fuzzy sets were also considered by Rosenfeld and he developed the structure of fuzzy graphs, obtaining analogs of several graph theoretical concepts. Bhattacharya (1987) gave some remarks on fuzzy graphs. Several concepts on fuzzy graphs were introduced by Mordeson and Nair (2001). Akram et al. (2011–2016) has introduced several new concepts including bipolar fuzzy graphs, regular bipolar fuzzy graphs, irregular bipolar fuzzy graphs, antipodal bipolar fuzzy graphs and bipolar fuzzy graph structures. In this paper, we present certain operations on bipolar fuzzy graph structures and investigate some of their properties.

2. Preliminaries

We now review some definitions from Dinesh (2011) that are necessary for this paper.

A graph structure $G^* = (U, E_1, E_2, ..., E_k)$ consists of a non-empty set $U$ together with mutually disjoint, irreflexive and symmetric relations $E_1, E_2, ..., E_k$ on $U$. If $G^*_1$ and $G^*_2$ are two graph structures given by $(U, E_1, E_2, ..., E_k)$ and $(V, E'_1, E'_2, ..., E'_k)$ respectively, then cartesian product of $G^*_1$ and $G^*_2$, is denoted by “$G^*_1 \times G^*_2$” and given by $G^*_1 \times G^*_2 = (U \times V, E_1 \times E'_1, E_2 \times E'_2, ..., E_k \times E'_k)$ where $E_i \times E'_i = \{(u,v) \in U \times V | u \in E_i, v \in E'_i\}$ for $i = 1, 2, ..., k$. Composition of $G^*_1$ and $G^*_2$ is denoted by “$G^*_1 \circ G^*_2$” and given by $G^*_1 \circ G^*_2 = (U \circ V, E_1 \circ E'_1, E_2 \circ E'_2, ..., E_k \circ E'_k)$ where $U \circ V = U \times V$ and $E_i \circ E'_i = \{(u,v) \in U \times V | u \in E_i, v \in E'_i\}$ for $i = 1, 2, ..., k$. Union of $G^*_1$ and $G^*_2$ is denoted by “$G^*_1 \cup G^*_2$” and given by $G^*_1 \cup G^*_2 = (U \cup V, E_1 \cup E'_1, E_2 \cup E'_2, ..., E_k \cup E'_k)$ and join of $G^*_1$ and $G^*_2$ is given by $G^*_1 + G^*_2 = (U + V, E_1 + E'_1, E_2 + E'_2, ..., E_k + E'_k)$ where $U + V = U \cup V$ and $E_i + E'_i = E_i \cup E'_i$ for $i = 1, 2, ..., k$ such that $E$ is the set consisting of all edges which join vertices of $U$ with vertices of $V$.

**Definition 1.** (Dinesh (2011))

Let $G^* = (U, E_1, E_2, ..., E_k)$ be a graph structure and let $\nu, \rho_1, \rho_2, ..., \rho_k$ be the fuzzy subsets of $U, E_1, E_2, ..., E_k$ respectively such that

$$0 \leq \rho_i(xy) \leq \mu(x) \wedge \mu(y) \quad \forall x, y \in V, i = 1, 2, ..., k.$$

Then $G = (\nu, \rho_1, \rho_2, ..., \rho_k)$ is a fuzzy graph structure of $G^*$.

**Definition 2.** (Zhang (1998))

Let $X$ be a nonempty set. A bipolar fuzzy set $B$ in $X$ is an object having the form

$$B = \{(x, \mu_B^P(x), \mu_B^N(x)) | x \in X\},$$
where \( \mu^P_B : X \to [0, 1] \) and \( \mu^N_B : X \to [-1, 0] \) are mappings.

We use the positive membership degree \( \mu^P_B(x) \) to denote the satisfaction degree of an element \( x \) to the property corresponding to a bipolar fuzzy set \( B \), and the negative membership degree \( \mu^N_B(x) \) to denote the satisfaction degree of an element \( x \) to some implicit counter-property corresponding to a bipolar fuzzy set \( B \). If \( \mu^P_B(x) \neq 0 \) and \( \mu^N_B(x) = 0 \), it is the situation that \( x \) is regarded as having only positive satisfaction for \( B \). If \( \mu^P_B(x) = 0 \) and \( \mu^N_B(x) \neq 0 \), it is the situation that \( x \) does not satisfy the property of \( B \) but somewhat satisfies the counter property of \( B \). It is possible for an element \( x \) to be such that \( \mu^P_B(x) \neq 0 \) and \( \mu^N_B(x) \neq 0 \) when the membership function of the property overlaps that of its counter property over some portion of \( X \).

For the sake of simplicity, we shall use the symbol \( B = (\mu^P_B, \mu^N_B) \) for the bipolar fuzzy set

\[
B = \{(x, \mu^P_B(x), \mu^N_B(x)) \mid x \in X\}.
\]

**Definition 3.** (Zhang (1998))

Let \( X \) be a nonempty set. Then we call a mapping \( A = (\mu^P_A, \mu^N_A) : X \times X \to [0, 1] \times [-1, 0] \) a bipolar fuzzy relation on \( X \) such that \( \mu^P_A(x, y) \in [0, 1] \) and \( \mu^N_A(x, y) \in [-1, 0] \).

**Definition 4.** (Akram (2011))

A bipolar fuzzy graph \( G = (V, A, B) \) is a non-empty set \( V \) together with a pair of functions \( A = (\mu^P_A, \mu^N_A) : V \to [0, 1] \times [-1, 0] \) and \( B = (\mu^P_B, \mu^N_B) : V \times V \to [0, 1] \times [-1, 0] \) such that for all \( x, y \in V \),

\[
\mu^P_B(x, y) \leq \min(\mu^P_A(x), \mu^P_A(y)) \quad \text{and} \quad \mu^N_B(x, y) \geq \max(\mu^N_A(x), \mu^N_A(y)).
\]

Notice that \( \mu^P_B(x, y) > 0, \mu^N_B(x, y) < 0 \) for \( (x, y) \in V \times V \), \( \mu^P_B(x, y) = \mu^N_B(x, y) = 0 \) for \( (x, y) \not\in V \times V \), and \( B \) is symmetric relation.

### 3. Operations on Bipolar Fuzzy Graph Structures

**Definition 5.** (Akram and Akmal (2016))

\( \tilde{G}_b = (M, N_1, N_2, ..., N_n) \) is called a bipolar fuzzy graph structure (BFGS) of a graph structure (GS) \( G^* = (U, E_1, E_2, ..., E_n) \) if \( M = (\mu^P_M, \mu^N_M) \) is a bipolar fuzzy set on \( U \) and for each \( i = 1, 2, ..., n \), \( N_i = (\mu^P_{N_i}, \mu^N_{N_i}) \) is a bipolar fuzzy set on \( E_i \) such that

\[
\mu^P_{N_i}(xy) \leq \mu^P_M(x) \land \mu^P_M(y), \quad \mu^N_{N_i}(xy) \geq \mu^N_M(x) \lor \mu^N_M(y) \quad \forall \ xy \in E_i \subset U \times U.
\]

Note that \( \mu^P_{N_i}(xy) = 0 = \mu^N_{N_i}(xy) \) for all \( xy \in U \times U \setminus E_i \), \( 0 < \mu^N_{N_i}(xy) \leq 1, -1 \leq \mu^N_{N_i}(xy) < 0 \) \( \forall \ xy \in E_i \). While \( U \) and \( E_i \) \( (i = 1, 2, ..., n) \) are called underlying vertex set and underlying i-edge sets of \( \tilde{G}_b \), respectively. Note that \( x \lor y = \max \) of \( x \) and \( y \), \( x \land y = \min \) of \( x \) and \( y \), throughout this paper.
**Example 1.**

Let $U = \{a_1, a_2, a_3, a_4\}$. Let $E_1 = \{a_1a_2, a_2a_3\}$ and $E_2 = \{a_3a_4, a_1a_4\}$ be two disjoint symmetric relations on $U$. Then $G^* = (U, E_1, E_2)$ is a graph structure.

Let $M, N_1$ and $N_2$ be bipolar fuzzy subsets of $U$, $E_1$ and $E_2$, respectively, such that

$M = \{(a_1, 0.5, -0.2), (a_2, 0.7, -0.3), (a_3, 0.4, -0.3), (a_4, 0.7, -0.3)\}$

$N_1 = \{(a_1a_2, 0.5, -0.1), (a_2a_3, 0.4, -0.3)\}$

and $N_2 = \{(a_3a_4, 0.4, -0.2), (a_1a_4, 0.1, -0.2)\}$. Then, $\tilde{G}_b = (M, N_1, N_2)$ is a BFGS of $G^*$ as shown in Figure 1.

![Figure 1: BFGS $\tilde{G}_b = (M, N_1, N_2)$](image)

**Definition 6.**

Let $\tilde{G}_{b_1} = (M_1, N_{11}, N_{12}, ..., N_{1n})$ and $\tilde{G}_{b_2} = (M_2, N_{21}, N_{22}, ..., N_{2n})$ be respective BFGSs of GSs $\tilde{G}_1^* = (U_1, E_{11}, E_{12}, ..., E_{1n})$ and $\tilde{G}_2^* = (U_2, E_{21}, E_{22}, ..., E_{2n})$. The Cartesian product $\tilde{G}_{b_1} \times \tilde{G}_{b_2}$ of $\tilde{G}_{b_1}$ and $\tilde{G}_{b_2}$ is then a BFGS of $G_1^* \times G_2^* = (U_1 \times U_2, E_{11} \times E_{21}, E_{12} \times E_{22}, ..., E_{1n} \times E_{2n})$ is given by

$$(M_1 \times M_2, N_{11} \times N_{21}, N_{12} \times N_{22}, ..., N_{1n} \times N_{2n})$$

such that

$$
\begin{align*}
\mu^{P}_{(M_1 \times M_2)}(xy) &= (\mu^{P}_{M_1} \times \mu^{P}_{M_2})(xy) = \mu^{P}_{M_1}(x) \land \mu^{P}_{M_2}(y), \\
\mu^{N}_{(M_1 \times M_2)}(xy) &= (\mu^{N}_{M_1} \times \mu^{N}_{M_2})(xy) = \mu^{N}_{M_1}(x) \lor \mu^{N}_{M_2}(y) \ \forall \ xy \in U_1 \times U_2,
\end{align*}
$$

$$
\begin{align*}
\mu^{P}_{(N_{1i} \times N_{2i})}(xy_1)(xy_2) &= (\mu^{P}_{N_{1i}} \times \mu^{P}_{N_{2i}})((xy_1)(xy_2)) = \mu^{P}_{N_{1i}}(x) \land \mu^{P}_{N_{2i}}(y_1y_2), \\
\mu^{N}_{(N_{1i} \times N_{2i})}(xy_1)(xy_2) &= (\mu^{N}_{N_{1i}} \times \mu^{N}_{N_{2i}})((xy_1)(xy_2)) = \mu^{N}_{N_{1i}}(x) \lor \mu^{N}_{N_{2i}}(y_1y_2) \ \forall \ x \in u_{1i}, \ y_1y_2 \in E_{2i},
\end{align*}
$$

$$
\begin{align*}
\mu^{P}_{(N_{1i} \times N_{2i})}(x_1y)(x_2y) &= (\mu^{P}_{N_{1i}} \times \mu^{P}_{N_{2i}})((x_1y)(x_2y)) = \mu^{P}_{N_{2i}}(y) \land \mu^{P}_{N_{1i}}(x_1x_2), \\
\mu^{N}_{(N_{1i} \times N_{2i})}(x_1y)(x_2y) &= (\mu^{N}_{N_{1i}} \times \mu^{N}_{N_{2i}})((x_1y)(x_2y)) = \mu^{N}_{N_{2i}}(y) \lor \mu^{N}_{N_{1i}}(x_1x_2) \ \forall \ y \in U_2, \ x_1x_2 \in E_{1i}.
\end{align*}
$$
Example 2.

Let $G_{b1} = (M_1, N_{11}, N_{12})$ and $G_{b2} = (M_2, N_{21}, N_{22})$ be respective BFGSs of graph structures $G_1^* = (U_1, E_{11}, E_{12})$ and $G_2^* = (U_2, E_{21}, E_{22})$ such that $U_1 = \{a_1, a_2, a_3, a_4\}$, $U_2 = \{b_1, b_2, b_3\}$, $E_{11} = \{a_1a_2\}$, $E_{12} = \{a_3a_4\}$, $E_{21} = \{b_1b_2\}$ and $E_{22} = \{b_3\}$. $G_{b1}$ and $G_{b2}$ are shown in Figure 2.

- $G_{s1}^* = (M_1, N_{11}, N_{12})$
- $G_{s2}^* = (M_2, N_{21}, N_{22})$

Figure 2: Bipolar Fuzzy Graph Structures

and Cartesian product $G_{b1} \times G_{b2} = (M_1 \times M_2, N_{11} \times N_{21}, N_{12} \times N_{22})$ is shown in Figure 3.

Example 3.

Consider $G_{b1} = (M_1, N_{11}, N_{12})$ and $G_{b2} = (M_2, N_{21}, N_{22})$ as shown in Figure 4 and let they be respective BFGSs of graph structures $G_1^* = (U_1, E_{11}, E_{12})$ and $G_2^* = (U_2, E_{21}, E_{22})$ such that $U_1 = \{a_1, a_2, a_3, a_4\}$, $U_2 = \{b_1, b_2, b_3\}$, $E_{11} = \{a_1a_2\}$, $E_{12} = \{a_3a_4\}$, $E_{21} = \{b_1b_2\}$
and \( E_{22} = \{b_2b_3\} \).

\[
\begin{align*}
\mathcal{G}_{s1} &= (M_1, N_{11}, N_{12}) \\
\mathcal{G}_{s2} &= (M_2, N_{21}, N_{22})
\end{align*}
\]

Figure 4: Bipolar Fuzzy Graph Structures

And their Cartesian product given by \( \mathcal{G}_{b1} \times \mathcal{G}_{b2} = (M_1 \times M_2, N_{11} \times N_{21}, N_{12} \times N_{22}) \) is shown in Figure 5.

\[
\begin{align*}
\mathcal{G}_{s1} &= (M_1, N_{11}, N_{12}) \\
\mathcal{G}_{s2} &= (M_2, N_{21}, N_{22})
\end{align*}
\]

Figure 5: Cartesian Product of Two BFGSs

**Theorem 1.**

Let \( G^* = (U_1 \times U_2, E_{11} \times E_{21}, E_{12} \times E_{22}, \ldots, E_{1n} \times E_{2n}) \) be Cartesian product of GSs \( G^*_1 = (U_1, E_{11}, E_{12}, \ldots, E_{1n}) \) and \( G^*_2 = (U_2, E_{21}, E_{22}, \ldots, E_{2n}) \). Let \( \mathcal{G}_{b1} = (M_1, N_{11}, N_{12}, \ldots, N_{1n}) \) and \( \mathcal{G}_{b2} = (M_2, N_{21}, N_{22}, \ldots, N_{2n}) \) be respective BFGSs of \( G^*_1 \) and \( G^*_2 \). Then, \((M_1 \times M_2, N_{11} \times N_{21}, N_{12} \times N_{22}, \ldots, N_{1n} \times N_{2n})\) is a BFGS of \( G^* \).
Proof:

Case 1. When \( u \in U_1, \ b_1b_2 \in E_{2i} \)

\[
\mu_{(N_1 \times N_2)}^P((ub_1)(ub_2)) = \mu_{M_1}^P(u) \land \mu_{N_2}^P(b_1b_2)
\leq \mu_{M_1}^P(u) \land [\mu_{M_2}^P(b_1) \lor \mu_{M_2}^P(b_2)]
= [\mu_{M_1}^P(u) \land \mu_{M_2}^P(b_1)] \lor [\mu_{M_1}^P(u) \land \mu_{M_2}^P(b_2)]
= \mu_{(M_1 \times M_2)}^P(ub_1) \land \mu_{(M_1 \times M_2)}^P(ub_2),
\]

\[
\mu_{(N_1 \times N_2)}^N((ub_1)(ub_2)) = \mu_{M_1}^N(u) \lor \mu_{N_2}^N(b_1b_2)
\geq \mu_{M_1}^N(u) \lor [\mu_{M_2}^N(b_1) \lor \mu_{M_2}^N(b_2)]
= [\mu_{M_1}^N(u) \lor \mu_{M_2}^N(b_1)] \lor [\mu_{M_1}^N(u) \lor \mu_{M_2}^N(b_2)]
= \mu_{(M_1 \times M_2)}^N(ub_1) \lor \mu_{(M_1 \times M_2)}^N(ub_2),
\]

for \( ub_1, \ ub_2 \in U_1 \times U_2. \)

Case 2. When \( u \in U_2, \ b_1b_2 \in E_{1i} \)

\[
\mu_{(N_1 \times N_2)}^P((b_1u)(b_2u)) = \mu_{M_2}^P(u) \land \mu_{N_1}^P(b_1b_2)
\leq \mu_{M_2}^P(u) \land [\mu_{M_1}^P(b_1) \lor \mu_{M_1}^P(b_2)]
= [\mu_{M_2}^P(u) \land \mu_{M_1}^P(b_1)] \lor [\mu_{M_2}^P(u) \land \mu_{M_1}^P(b_2)]
= \mu_{(M_1 \times M_2)}^P(b_1u) \land \mu_{(M_1 \times M_2)}^P(b_2u),
\]

\[
\mu_{(N_1 \times N_2)}^N((b_1u)(b_2u)) = \mu_{M_2}^N(u) \lor \mu_{N_1}^N(b_1b_2)
\geq \mu_{M_2}^N(u) \lor [\mu_{M_1}^N(b_1) \lor \mu_{M_1}^N(b_2)]
= [\mu_{M_2}^N(u) \lor \mu_{M_1}^N(b_1)] \lor [\mu_{M_2}^N(u) \lor \mu_{M_1}^N(b_2)]
= \mu_{(M_1 \times M_2)}^N(b_1u) \lor \mu_{(M_1 \times M_2)}^N(b_2u),
\]

for \( b_1u, \ b_2u \in U_1 \times U_2. \)

Both cases hold for \( i = 1, 2, ..., n. \) This completes the proof. \( \Box \)

Definition 7.

Let \( G_{b_1} = (M_1, N_{11}, N_{12}, ..., N_{1n}) \) and \( G_{b_2} = (M_2, N_{21}, N_{22}, ..., N_{2n}) \) be respective BFGSs of GSs \( G_1^* = (U_1, E_{11}, E_{12}, ..., E_{1n}) \) and \( G_2^* = (U_2, E_{21}, E_{22}, ..., E_{2n}). \) The cross product \( G_{b_1} \ast G_{b_2} \) of \( G_{b_1} \) and \( G_{b_2} \) is a BFGS of \( G_1^* \ast G_2^* = (U_1 \ast U_2, E_{11} \ast E_{21}, E_{12} \ast E_{22}, ..., E_{1n} \ast E_{2n}) \) is given by

\[
(M_1 \ast M_2, N_{11} \ast N_{21}, N_{12} \ast N_{22}, ..., N_{1n} \ast N_{2n})
\]
such that

\[
\left\{ \begin{array}{l}
\mu_{(M_1 \ast M_2)}^P(x) = (\mu_{M_1}^P \ast \mu_{M_2}^P)(xy) = \mu_{M_1}^P(x) \land \mu_{M_2}^P(y), \\
\mu_{(M_1 \ast M_2)}^N(x) = (\mu_{M_1}^N \ast \mu_{M_2}^N)(xy) = \mu_{M_1}^N(x) \lor \mu_{M_2}^N(y) \forall xy \in U_1 \times U_2,
\end{array} \right.
\]

[72x-2663]Definition 7.
Example 4.

Let $\tilde{G}_{b1}$ and $\tilde{G}_{b2}$ be BFGSs as shown in Figure 2 and cross product $\tilde{G}_{b1} \ast \tilde{G}_{b2} = (M_1 \ast M_2, \ N_{11} \ast N_{21}, \ N_{12} \ast N_{22})$ is as shown in Figure 6.

**Figure 6: Cross Product of Two BFGSs**

Example 5.

$\tilde{G}_{b1} = (M_1, \ N_{11}, \ N_{12})$ and $\tilde{G}_{b2} = (M_2, \ N_{21}, \ N_{22})$ be BFGSs as shown in Figure 4 and their cross product given by $\tilde{G}_{b1} \ast \tilde{G}_{b2} = (M_1 \ast M_2, \ N_{11} \ast N_{21}, \ N_{12} \ast N_{22})$ is shown in Figure 7.

Theorem 2.

Let $G^* = (U_1 \ast U_2, \ E_{11} \ast E_{21}, \ E_{12} \ast E_{22}, \ ..., \ E_{1n} \ast E_{2n})$ be cross product of GSs $G^*_1 = (U_1, \ E_{11}, \ E_{12}, \ ..., \ E_{1n})$ and $G^*_2 = (U_2, \ E_{21}, \ E_{22}, \ ..., \ E_{2n})$. Let $\tilde{G}_{b1} = (M_1, \ N_{11}, \ N_{12}, \ ..., \ N_{1n})$ and $\tilde{G}_{b2} = (M_2, \ N_{21}, \ N_{22}, \ ..., \ N_{2n})$ be respective BFGSs of $G^*_1$ and $G^*_2$. Then $(M_1 \ast M_2, \ N_{11} \ast N_{21}, \ N_{12} \ast N_{22}, \ ..., \ N_{1n} \ast N_{2n})$ is a BFGS of $G^*$. 

\[
\begin{align*}
\left\{ \begin{array}{c}
\mu^p_{(N_{11} \ast N_{21})}((x_1y_1)(x_2y_2)) = (\mu^p_{N_{11}} \ast \mu^p_{N_{21}})((x_1y_1)(x_2y_2)) = \mu^p_{N_{21}}(y_1y_2) \land \mu^p_{N_{11}}(x_1x_2), \\
\mu^N_{(N_{11} \ast N_{21})}((x_1y_1)(x_2y_2)) = (\mu^N_{N_{11}} \ast \mu^N_{N_{21}})((x_1y_1)(x_2y_2)) = \mu^N_{N_{21}}(y_1y_2) \lor \mu^N_{N_{11}}(x_1x_2)
\end{array} \right. \forall \ y_1y_2 \in E_{21}, \ x_1x_2 \in E_{11}.
\end{align*}
\]
Definition 8.

Let $G_{b1} = (M_1, N_{11}, N_{12}, ..., N_{1n})$ and $G_{b2} = (M_2, N_{21}, N_{22}, ..., N_{2n})$ be respective BFGSs of GSs $G_1^* = (U_1, E_{11}, E_{12}, ..., E_{1n})$ and $G_2^* = (U_2, E_{21}, E_{22}, ..., E_{2n})$. The lexicographic product $G_{b1} \circ G_{b2}$ of $G_{b1}$ and $G_{b2}$ is a BFGS of $G_1^* \circ G_2^* = (U_1 \circ U_2, E_{11} \circ E_{21}, E_{12} \circ E_{22}, ..., E_{1n} \circ E_{2n})$ is given by

$$\{(M_1 \circ M_2, N_{11} \circ N_{21}, N_{12} \circ N_{22}, ..., N_{1n} \circ N_{2n})\}$$
such that
\[
\begin{align*}
\mu^p_{(M_1 \cdot M_2)}(xy) &= (\mu^p_{M_1} \cdot \mu^p_{M_2})(xy) = \mu^p_{M_1}(x) \wedge \mu^p_{M_2}(y), \\
\mu^N_{(M_1 \cdot M_2)}(xy) &= (\mu^N_{M_1} \cdot \mu^N_{M_2})(xy) = \mu^N_{M_1}(x) \vee \mu^N_{M_2}(y), \quad \forall \ xy \in U_1 \times U_2
\end{align*}
\]

\[
\begin{align*}
\mu^p_{(N_{11} \cdot N_{21})}((xy_1)(xy_2)) &= (\mu^p_{N_{11}} \cdot \mu^p_{N_{21}})((xy_1)(xy_2)) = \mu^p_{N_{11}}(x) \wedge \mu^p_{N_{21}}(y_1y_2), \\
\mu^N_{(N_{11} \cdot N_{21})}((xy_1)(xy_2)) &= (\mu^N_{N_{11}} \cdot \mu^N_{N_{21}})((xy_1)(xy_2)) = \mu^N_{N_{11}}(x) \vee \mu^N_{N_{21}}(y_1y_2), \\
\forall \ x \in u_1, \ y_1y_2 \in E_{2i}
\end{align*}
\]

\[
\begin{align*}
\mu^p_{(N_{11} \cdot N_{21})}((x_1y_1)(x_2y_2)) &= (\mu^p_{N_{11}} \cdot \mu^p_{N_{21}})((x_1y_1)(x_2y_2)) = \mu^p_{N_{11}}(y_1y_2) \wedge \mu^p_{N_{21}}(x_1x_2), \\
\mu^N_{(N_{11} \cdot N_{21})}((x_1y_1)(x_2y_2)) &= (\mu^N_{N_{11}} \cdot \mu^N_{N_{21}})((x_1y_1)(x_2y_2)) = \mu^N_{N_{11}}(y_1y_2) \vee \mu^N_{N_{21}}(x_1x_2), \\
\forall \ y_1y_2 \in E_{2i}, \ x_1x_2 \in E_{1i}
\end{align*}
\]

Example 6.

Let $G_{b_1}$ and $G_{b_2}$ be BFGSs shown in Figure 2 and lexicographic product $G_{b_1} \cdot G_{b_2} = (M_1 \cdot M_2, N_{11} \cdot N_{21}, N_{12} \cdot N_{22})$ is as shown in Figure 8.

![Figure 8: Lexicographic Product of Two BFGSs](attachment:figure8.png)

Example 7.

$G_{b_1} = (M_1, N_{11}, N_{12})$ and $G_{b_2} = (M_2, N_{21}, N_{22})$ be BFGSs as shown in Figure 4 and their lexicographic product given by $G_{b_1} \cdot G_{b_2} = (M_1 \cdot M_2, N_{11} \cdot N_{21}, N_{12} \cdot N_{22})$ is shown in Figure 9.
Proof:

Case 1. When \( u \in U_1, b_1b_2 \in E_{2i} \)

\[
\mu_{(N_1 \cdot N_2)}^P(\langle ub_1)(ub_2) \rangle) = \mu_{N_1}^P(u) \wedge \mu_{N_2}^P(b_1b_2) \\
\leq \mu_{M_1}^P(u) \wedge [\mu_{M_2}^P(b_1) \wedge \mu_{M_2}^P(b_2)] \\
= [\mu_{M_1}^P(u) \wedge \mu_{M_2}^P(b_1)] \wedge [\mu_{M_1}^P(u) \wedge \mu_{M_2}^P(b_2)] \\
= \mu_{(M_1 \cdot M_2)}^P(ub_1) \wedge \mu_{(M_1 \cdot M_2)}^P(ub_2),
\]

\[
\mu_{(N_1 \cdot N_2)}^N(\langle ub_1)(ub_2) \rangle) = \mu_{M_1}^N(u) \vee \mu_{N_2}^N(b_1b_2) \\
\geq \mu_{M_1}^N(u) \vee [\mu_{M_2}^N(b_1) \vee \mu_{M_2}^N(b_2)] \\
= [\mu_{M_1}^N(u) \vee \mu_{M_2}^N(b_1)] \vee [\mu_{M_1}^N(u) \vee \mu_{M_2}^N(b_2)] \\
= \mu_{(M_1 \cdot M_2)}^N(ub_1) \vee \mu_{(M_1 \cdot M_2)}^N(ub_2),
\]

for \( ub_1, ub_2 \in U_1 \cdot U_2 \).
Case 2. When \( u_1 u_2 \in E_{21}, \ b_1 b_2 \in E_{1i} \)

\[
\mu_{(N_1 \bullet N_2)}((b_1 u_1)(b_2 u_2)) = \mu_{N_2}^P((b_1 u_2)(b_2 u_2)) \land \mu_{N_1}^P((b_1 b_2))
\leq [\mu_{M_2}^P(u_1) \land \mu_{M_2}^P(u_2)] \land [\mu_{M_1}^P(b_1) \land \mu_{M_1}^P(b_2)]
= [\mu_{M_2}^P(u_1) \land \mu_{M_2}^P(b_1)] \land [\mu_{M_2}^P(u_2) \land \mu_{M_2}^P(b_2)]
= \mu_{M_1 \bullet M_2}^P(b_1 u_1) \land \mu_{M_1 \bullet M_2}^P(b_2 u_2),
\]

\[
\mu_{(N_1 \bullet N_2)}((b_1 u_1)(b_2 u_2)) = \mu_{N_2}^N((b_1 u_2)(b_2 u_2)) \lor \mu_{N_1}^N((b_1 b_2))
\geq [\mu_{N_2}^N(u_1) \lor \mu_{N_2}^N(u_2)] \lor [\mu_{N_1}^N(b_1) \lor \mu_{N_1}^N(b_2)]
= [\mu_{N_2}^N(u_1) \lor \mu_{N_2}^N(b_1)] \lor [\mu_{N_2}^N(u_2) \lor \mu_{N_2}^N(b_2)]
= \mu_{N_1 \bullet N_2}^N(b_1 u_1) \lor \mu_{N_1 \bullet N_2}^N(b_2 u_2),
\]

for \( b_1 u_1, \ b_2 u_2 \in U_1 \bullet U_2 \).

Both cases hold for \( i = 1, 2, ..., n \). This completes the proof. \( \square \)

**Definition 9.**

Let \( G_{b1} = (M_1, N_{11}, N_{12}, ..., N_{1n}) \) and \( G_{b2} = (M_2, N_{21}, N_{22}, ..., N_{2n}) \) be respective BFGSs of GSs \( G_1^* = (U_1, E_{11}, E_{12}, ..., E_{1n}) \) and \( G_2^* = (U_2, E_{21}, E_{22}, ..., E_{2n}) \). The **strong product** \( G_{b1} \boxtimes G_{b2} \) of \( G_{b1} \) and \( G_{b2} \) is then a BFGS of \( G_1^* \boxtimes G_2^* = (U_1 \boxtimes U_2, E_{11} \boxtimes E_{21}, E_{12} \boxtimes E_{22}, ..., E_{1n} \boxtimes E_{2n}) \) is given by

\[
(M_1 \boxtimes M_2, N_{11} \boxtimes N_{21}, N_{12} \boxtimes N_{22}, ..., N_{1n} \boxtimes N_{2n})
\]

such that

\[
\begin{align*}
\mu_{(M_1 \boxtimes M_2)}^P((xy)) &= (\mu_{M_1}^P \boxtimes \mu_{M_2}^P)((xy)) = \mu_{M_1}^P(x) \land \mu_{M_2}^P(y), \\
\mu_{(N_1 \boxtimes N_2)}^N((xy)) &= (\mu_{N_1}^N \boxtimes \mu_{N_2}^N)((xy)) = \mu_{N_1}^N(x) \lor \mu_{N_2}^N(y),
\end{align*}
\]

\[
\begin{align*}
\mu_{(M_1 \boxtimes M_2)}^P(((xy_1)(xy_2))) &= (\mu_{M_1}^P \boxtimes \mu_{M_2}^P)(((xy_1)(xy_2))) = \mu_{M_1}^P(x) \land \mu_{M_2}^P(y), \\
\mu_{(N_1 \boxtimes N_2)}^N(((xy_1)(xy_2))) &= (\mu_{N_1}^N \boxtimes \mu_{N_2}^N)(((xy_1)(xy_2))) = \mu_{N_1}^N(x) \lor \mu_{N_2}^N(y), \\
\end{align*}
\]

\[
\begin{align*}
\mu_{(M_1 \boxtimes M_2)}^P(((xy_1)(xy_2))) &= (\mu_{M_1}^P \boxtimes \mu_{M_2}^P)(((xy_1)(xy_2))) = \mu_{M_1}^P(x) \land \mu_{M_2}^P(y), \\
\mu_{(N_1 \boxtimes N_2)}^N(((xy_1)(xy_2))) &= (\mu_{N_1}^N \boxtimes \mu_{N_2}^N)(((xy_1)(xy_2))) = \mu_{N_1}^N(x) \lor \mu_{N_2}^N(y),
\end{align*}
\]

\[
\begin{align*}
\mu_{(M_1 \boxtimes M_2)}^P(((xy_1)(xy_2))) &= (\mu_{M_1}^P \boxtimes \mu_{M_2}^P)(((xy_1)(xy_2))) = \mu_{M_1}^P(x) \land \mu_{M_2}^P(y), \\
\mu_{(N_1 \boxtimes N_2)}^N(((xy_1)(xy_2))) &= (\mu_{N_1}^N \boxtimes \mu_{N_2}^N)(((xy_1)(xy_2))) = \mu_{N_1}^N(x) \lor \mu_{N_2}^N(y),
\end{align*}
\]

\[
\begin{align*}
\mu_{(M_1 \boxtimes M_2)}^P(((xy_1)(xy_2))) &= (\mu_{M_1}^P \boxtimes \mu_{M_2}^P)(((xy_1)(xy_2))) = \mu_{M_1}^P(x) \land \mu_{M_2}^P(y), \\
\mu_{(N_1 \boxtimes N_2)}^N(((xy_1)(xy_2))) &= (\mu_{N_1}^N \boxtimes \mu_{N_2}^N)(((xy_1)(xy_2))) = \mu_{N_1}^N(x) \lor \mu_{N_2}^N(y),
\end{align*}
\]

\[
\begin{align*}
\mu_{(M_1 \boxtimes M_2)}^P(((xy_1)(xy_2))) &= (\mu_{M_1}^P \boxtimes \mu_{M_2}^P)(((xy_1)(xy_2))) = \mu_{M_1}^P(x) \land \mu_{M_2}^P(y), \\
\mu_{(N_1 \boxtimes N_2)}^N(((xy_1)(xy_2))) &= (\mu_{N_1}^N \boxtimes \mu_{N_2}^N)(((xy_1)(xy_2))) = \mu_{N_1}^N(x) \lor \mu_{N_2}^N(y),
\end{align*}
\]

\[
\begin{align*}
\mu_{(M_1 \boxtimes M_2)}^P(((xy_1)(xy_2))) &= (\mu_{M_1}^P \boxtimes \mu_{M_2}^P)(((xy_1)(xy_2))) = \mu_{M_1}^P(x) \land \mu_{M_2}^P(y), \\
\mu_{(N_1 \boxtimes N_2)}^N(((xy_1)(xy_2))) &= (\mu_{N_1}^N \boxtimes \mu_{N_2}^N)(((xy_1)(xy_2))) = \mu_{N_1}^N(x) \lor \mu_{N_2}^N(y),
\end{align*}
\]

\[
\begin{align*}
\mu_{(M_1 \boxtimes M_2)}^P(((xy_1)(xy_2))) &= (\mu_{M_1}^P \boxtimes \mu_{M_2}^P)(((xy_1)(xy_2))) = \mu_{M_1}^P(x) \land \mu_{M_2}^P(y), \\
\mu_{(N_1 \boxtimes N_2)}^N(((xy_1)(xy_2))) &= (\mu_{N_1}^N \boxtimes \mu_{N_2}^N)(((xy_1)(xy_2))) = \mu_{N_1}^N(x) \lor \mu_{N_2}^N(y),
\end{align*}
\]

\[
\begin{align*}
\mu_{(M_1 \boxtimes M_2)}^P(((xy_1)(xy_2))) &= (\mu_{M_1}^P \boxtimes \mu_{M_2}^P)(((xy_1)(xy_2))) = \mu_{M_1}^P(x) \land \mu_{M_2}^P(y), \\
\mu_{(N_1 \boxtimes N_2)}^N(((xy_1)(xy_2))) &= (\mu_{N_1}^N \boxtimes \mu_{N_2}^N)(((xy_1)(xy_2))) = \mu_{N_1}^N(x) \lor \mu_{N_2}^N(y),
\end{align*}
\]

**Example 8.**

Let \( G_{b1} \) and \( G_{b2} \) be BFGSs as shown in Figure 2 and strong product \( G_{b1} \boxtimes G_{b2} = (M_1 \boxtimes M_2, N_{11} \boxtimes N_{21}, N_{12} \boxtimes N_{22}) \) is shown in Figure 10.
Example 9.

\( G_{b_1} = (M_1, N_{11}, N_{12}) \) and \( G_{b_2} = (M_2, N_{21}, N_{22}) \) be BFGSs as shown in Figure 4 and their strong product given by \( \bar{G}_{b_1} \otimes G_{b_2} = (M_1 \otimes M_2, N_{11} \otimes N_{21}, N_{12} \otimes N_{22}) \) is as shown in Figure 11.

Theorem 4.

Let \( G^* = (U_1 \otimes U_2, E_{11} \otimes E_{21}, E_{12} \otimes E_{22}, ..., E_{1n} \otimes E_{2n}) \) be strong product of GSs \( G_1^* = (U_1, E_{11}, E_{12}, ..., E_{1n}) \) and \( G_2^* = (U_2, E_{21}, E_{22}, ..., E_{2n}) \). Let \( G_{b_1} = (M_1, N_{11}, N_{12}, ..., N_{1n}) \) and \( G_{b_2} = (M_2, N_{21}, N_{22}, ..., N_{2n}) \) be respective BFGSs of \( G_1^* \) and \( G_2^* \). Then \( (M_1 \otimes M_2, N_{11} \otimes N_{21}, N_{12} \otimes N_{22}, ..., N_{1n} \otimes N_{2n}) \) is a BFGS of \( G^* \).

Proof:

Case 1. When \( u \in U_1, b_1 b_2 \in E_{2i} \)

\[
\mu_{(N_{1i} \otimes N_{2i})}((ub_1)(ub_2)) = \mu_{M_1}^P(u) \land \mu_{N_{2i}}^P(b_1 b_2) \\
\leq \mu_{M_1}^P(u) \land [\mu_{M_2}^P(b_1) \land 1_{M_2}^P(b_2)] \\
= [\mu_{M_1}^P(u) \land \mu_{M_2}^P(b_1)] \land [\mu_{M_1}^P(u) \land 1_{M_2}^P(b_2)] \\
= \mu_{(M_1 \otimes M_2)}^P(ub_1) \land \mu_{(M_1 \otimes M_2)}^P(ub_2),
\]
\[
\mu^N_{(N_1 \boxtimes N_2)}((ub_1)(ub_2)) = \mu^N_{M_1}(u) \lor \mu^N_{N_2}(b_1b_2) \\
\geq \mu^N_{M_1}(u) \lor [\mu^N_{M_2}(b_1) \lor \mu^N_{M_2}(b_2)] \\
= [\mu^N_{M_1}(u) \lor \mu^N_{M_2}(b_1)] \lor [\mu^N_{M_1}(u) \lor \mu^N_{M_2}(b_2)] \\
= \mu^N_{(M_1 \boxtimes M_2)}(ub_1) \lor \mu^N_{(M_1 \boxtimes M_2)}(ub_2),
\]

for \(ub_1, \ ub_2 \in U_1 \boxtimes U_2\).

**Case 2.** When \(u \in U_2, \ b_1b_2 \in E_{1i}\)

\[
\mu^P_{(N_1 \boxtimes N_2)}((b_1u)(b_2u)) = \mu^P_{M_2}(u) \land \mu^P_{N_2}(b_1b_2) \\
\leq \mu^P_{M_2}(u) \land [\mu^P_{M_1}(b_1) \land \mu^P_{M_1}(b_2)] \\
= [\mu^P_{M_2}(u) \land \mu^P_{M_1}(b_1)] \land [\mu^P_{M_2}(u) \land \mu^P_{M_1}(b_2)] \\
= \mu^P_{(M_1 \boxtimes M_2)}(b_1u) \land \mu^P_{(M_1 \boxtimes M_2)}(b_2u),
\]

\[
\mu^N_{(N_1 \boxtimes N_2)}((b_1u)(b_2u)) = \mu^N_{M_2}(u) \lor \mu^N_{N_2}(b_1b_2) \\
\geq \mu^N_{M_2}(u) \lor [\mu^N_{M_1}(b_1) \lor \mu^N_{M_1}(b_2)] \\
= [\mu^N_{M_2}(u) \lor \mu^N_{M_1}(b_1)] \lor [\mu^N_{M_2}(u) \lor \mu^N_{M_1}(b_2)] \\
= \mu^N_{(M_1 \boxtimes M_2)}(b_1u) \lor \mu^N_{(M_1 \boxtimes M_2)}(b_2u),
\]

for \(b_1u, \ b_2u \in U_1 \boxtimes U_2\).
Case 3. When \( u_1 u_2 \in E_{2i}, \ b_1 b_2 \in E_{1i} \)

\[
\mu_{(N_1 \otimes N_2)}^P((b_1 u_1)(b_2 u_2)) = \mu_{N_2}^P(u_1 u_2) \land \mu_{M_1}^P(b_1 b_2) \\
\leq [\mu_{M_2}^P(u_1) \land \mu_{M_2}^P(u_2)] \land [\mu_{M_1}^P(b_1) \land \mu_{M_1}^P(b_2)] \\
= [\mu_{M_2}^P(u_1) \land \mu_{M_1}^P(b_1)] \land [\mu_{M_2}^P(u_2) \land \mu_{M_1}^P(b_2)] \\
= \mu_{(M_1 \otimes M_2)}^P(b_1 u_1) \land \mu_{(M_1 \otimes M_2)}^P(b_2 u_2),
\]

\[
\mu_{(N_1 \otimes N_2)}^N((b_1 u_1)(b_2 u_2)) = \mu_{N_2}^N(u_1 u_2) \lor \mu_{N_1}^N(b_1 b_2) \\
\geq [\mu_{M_2}^N(u_1) \lor \mu_{M_2}^N(u_2)] \lor [\mu_{M_1}^N(b_1) \lor \mu_{M_1}^N(b_2)] \\
= [\mu_{M_2}^N(u_1) \lor \mu_{M_1}^N(b_1)] \lor [\mu_{M_2}^N(u_2) \lor \mu_{M_1}^N(b_2)] \\
= \mu_{(M_1 \otimes M_2)}^N(b_1 u_1) \lor \mu_{(M_1 \otimes M_2)}^N(b_2 u_2),
\]

for \( b_1 u_1, \ b_2 u_2 \in U_1 \boxdot U_2 \).

All three cases hold for \( i = 1, 2, \ldots, n \). This completes the proof. \( \Box \)

**Definition 10.**

Let \( \tilde{G}_{b_1} = (M_1, N_{11}, N_{12}, \ldots, N_{1n}) \) and \( \tilde{G}_{b_2} = (M_2, N_{21}, N_{22}, \ldots, N_{2n}) \) be respective BFGSs of GSs \( G_1^* = (U_1, E_{11}, E_{12}, \ldots, E_{1n}) \) and \( G_2^* = (U_2, E_{21}, E_{22}, \ldots, E_{2n}) \). The composition \( \tilde{G}_{b_1} \circ \tilde{G}_{b_2} \) of \( \tilde{G}_{b_1} \) and \( \tilde{G}_{b_2} \) is then a BFGS of \( G_1^* \circ G_2^* = (U_1 \circ U_2, E_{11} \circ E_{21}, E_{12} \circ E_{22}, \ldots, E_{1n} \circ E_{2n}) \) is given by

\[
(M_1 \circ M_2, N_{11} \circ N_{21}, N_{12} \circ N_{22}, \ldots, N_{1n} \circ N_{2n})
\]

such that

\[
\left\{
\begin{aligned}
\mu_{(M_1 \circ M_2)}^P(xy) &= (\mu_{M_1}^P \circ \mu_{M_2}^P)(xy) = \mu_{M_1}^P(x) \land \mu_{M_2}^P(y), \\
\mu_{(M_1 \circ M_2)}^N(xy) &= (\mu_{M_1}^N \circ \mu_{M_2}^N)(xy) = \mu_{M_1}^N(x) \lor \mu_{M_2}^N(y), \ \forall \ xy \in U_1 \times U_2
\end{aligned}
\right.
\]

\[
\left\{
\begin{aligned}
\mu_{(N_1 \circ N_2)}^P((x_1 y_1)(x_2 y_2)) &= (\mu_{N_1}^P \circ \mu_{N_2}^P)((x_1 y_1)(x_2 y_2)) = \mu_{N_1}^P(x_1) \land \mu_{N_2}^P(y_1), \\
\mu_{(N_1 \circ N_2)}^N((x_1 y_1)(x_2 y_2)) &= (\mu_{N_1}^N \circ \mu_{N_2}^N)((x_1 y_1)(x_2 y_2)) = \mu_{N_1}^N(x_1) \lor \mu_{N_2}^N(y_1), \ \forall \ x_1 \in U_1, \ y_1 y_2 \in E_{2i}
\end{aligned}
\right.
\]

\[
\left\{
\begin{aligned}
\mu_{(N_1 \circ N_2)}^P((x_1 y_1)(x_2 y_2)) &= (\mu_{N_1}^P \circ \mu_{N_2}^P)((x_1 y_1)(x_2 y_2)) = \mu_{N_2}^P(y_2) \land \mu_{N_1}^P(x_1 x_2), \\
\mu_{(N_1 \circ N_2)}^N((x_1 y_1)(x_2 y_2)) &= (\mu_{N_1}^N \circ \mu_{N_2}^N)((x_1 y_1)(x_2 y_2)) = \mu_{N_2}^N(y_2) \lor \mu_{N_1}^N(x_1 x_2), \ \forall \ y \in U_2, \ x_1 x_2 \in E_{1i}
\end{aligned}
\right.
\]

\[
\left\{
\begin{aligned}
\mu_{(N_1 \circ N_2)}^P((x_1 y_1)(x_2 y_2)) &= (\mu_{N_1}^P \circ \mu_{N_2}^P)((x_1 y_1)(x_2 y_2)) = \mu_{N_2}^P(y_1) \land \mu_{N_1}^P(x_1 x_2), \\
\mu_{(N_1 \circ N_2)}^N((x_1 y_1)(x_2 y_2)) &= (\mu_{N_1}^N \circ \mu_{N_2}^N)((x_1 y_1)(x_2 y_2)) = \mu_{N_2}^N(y_1) \lor \mu_{N_1}^N(x_1 x_2), \ \forall \ x_1 x_2 \in E_{1i}, \ y_1, \ y_2 \in U_2 \text{ such that } y_1 \neq y_2.
\end{aligned}
\right.
\]

**Example 10.**

Consider \( \tilde{G}_{b_1} \) and \( \tilde{G}_{b_2} \) as shown in Figure 2. Their composition represented by \( \tilde{G}_{b_1} \circ \tilde{G}_{b_2} = (M_1 \circ M_2, N_{11} \circ N_{21}, N_{12} \circ N_{22}) \) is shown in Figure 12.
Example 11.

Let $\tilde{G}_{b_1}$ and $\tilde{G}_{b_2}$ be BFGSs as shown in Figure 4. Their composition represented by $\tilde{G}_{b_1} \circ \tilde{G}_{b_2} = (M_1 \circ M_2, N_{11} \circ N_{21}, N_{12} \circ N_{22})$ is shown in Figure 13.
Theorem 5.
Let $G^* = (U_1 \circ U_2, \ E_{11} \circ E_{21}, \ E_{12} \circ E_{22}, \ ... \ , \ E_{1n} \circ E_{2n})$ be the composition of GSs $G_1^* = (U_1, \ E_{11}, \ E_{12}, \ ... , \ E_{1n})$ and $G_2^* = (U_2, \ E_{21}, \ E_{22}, \ ... , \ E_{2n})$. Let $G_{b1} = (M_1, \ N_{i1}, \ N_{i2}, \ ... , \ N_{in})$ and $G_{b2} = (M_2, \ N_{21}, \ N_{22}, \ ... , \ N_{2n})$ be respective BFGSs of $G_1^*$ and $G_2^*$. Then $G_{b1} \circ G_{b2} = (M_1 \circ M_2, \ N_{i1} \circ N_{21}, \ N_{i2} \circ N_{22}, \ ... , \ N_{in} \circ N_{2n})$ is a BFGS of $G^*$.

Proof:

Case 1. When $u \in U_1$, $b_1 b_2 \in E_{2i}$

$$\mu^P_{(N_{i1} \circ N_{2i})}((ub_1)(ub_2)) = \mu^P_{M_i}(u) \wedge \mu^P_{N_{2i}}(b_1 b_2)$$

$$\leq \mu^P_{M_i}(u) \wedge [\mu^P_{M_2}(b_1) \wedge \mu^P_{M_2}(b_2)]$$

$$= [\mu^P_{M_i}(u) \wedge \mu^P_{M_2}(b_1)] \wedge [\mu^P_{M_i}(u) \wedge \mu^P_{M_2}(b_2)]$$

$$= \mu^P_{(M_i \circ M_2)}(ub_1) \wedge \mu^P_{(M_i \circ M_2)}(ub_2),$$

for $ub_1, \ ub_2 \in U_1 \circ U_2$.

Case 2. When $u \in U_2$, $b_1 b_2 \in E_{1i}$

$$\mu^P_{(N_{i1} \circ N_{2i})}((b_1u)(b_2u)) = \mu^P_{M_2}(u) \wedge \mu^P_{N_{i1}}(b_1 b_2)$$

$$\leq \mu^P_{M_2}(u) \wedge [\mu^P_{M_1}(b_1) \wedge \mu^P_{M_1}(b_2)]$$

$$= [\mu^P_{M_2}(u) \wedge \mu^P_{M_1}(b_1)] \wedge [\mu^P_{M_2}(u) \wedge \mu^P_{M_1}(b_2)]$$

$$= \mu^P_{(M_1 \circ M_2)}(b_1 u) \wedge \mu^P_{(M_1 \circ M_2)}(b_2 u),$$

for $b_1 u, \ b_2 u \in U_1 \circ U_2$.

Case 3. When $b_1 b_2 \in E_{1i}$, $u_1, \ u_2 \in U_2$ such that $u_1 \neq u_2$,

$$\mu^P_{(N_{i1} \circ N_{2i})}((b_1 u_1)(b_2 u_2)) = \mu^P_{M_2}(u_1) \wedge \mu^P_{M_2}(u_2) \wedge \mu^P_{N_{i1}}(b_1 b_2)$$

$$\leq \mu^P_{M_2}(u_1) \wedge [\mu^P_{M_1}(b_1) \wedge \mu^P_{M_1}(b_2)]$$

$$= [\mu^P_{M_2}(u_1) \wedge \mu^P_{M_1}(b_1)] \wedge [\mu^P_{M_2}(u_2) \wedge \mu^P_{M_1}(b_2)]$$

$$= \mu^P_{(M_1 \circ M_2)}(b_1 u_1) \wedge \mu^P_{(M_1 \circ M_2)}(b_2 u_2),$$
\[
\begin{align*}
\mu_{(N_1 \cup N_2)}^N((b_1u_1)(b_2u_2)) &= \mu_{M_2}^N(u_1) \lor \mu_{M_2}^N(u_2) \lor \mu_{N_1}^N(b_1b_2) \\
&\geq \mu_{M_2}^N(u_1) \lor \mu_{M_2}^N(u_2) \lor [\mu_{M_1}^N(b_1) \lor \mu_{M_1}^N(b_2)] \\
&= [\mu_{M_2}^N(u_1) \lor \mu_{M_2}^N(u_2)] \lor [\mu_{M_2}^N(u_2) \lor \mu_{M_1}^N(b_2)] \\
&= \mu_{(M_1 \cup M_2)}^N(b_1u_1) \lor \mu_{(M_1 \cup M_2)}^N(b_2u_2),
\end{align*}
\]

for \(b_1u_1, b_2u_2 \in U_1 \circ U_2\).

All three cases hold for \(i = 1, 2, ..., n\). This completes the proof. \(\square\)

**Definition 11.**

Let \(\tilde{G}_{b_1} = (M_1, N_{11}, N_{12}, ..., N_{1n})\) and \(\tilde{G}_{b_2} = (M_2, N_{21}, N_{22}, ..., N_{2n})\) be respective BFGSs of \(G_1 = (U_1, E_{11}, E_{12}, ..., E_{1n})\) and \(G_2 = (U_2, E_{21}, E_{22}, ..., E_{2n})\) and let \(U_1 \cap U_2 = \emptyset\). The union \(\tilde{G}_{b_1} \cup \tilde{G}_{b_2}\) of \(\tilde{G}_{b_1}\) and \(\tilde{G}_{b_2}\) is then a BFGS of \(G_1^* \cup G_2^* = (U_1 \cup U_2, E_{11} \cup E_{21}, E_{12} \cup E_{22}, ..., E_{1n} \cup E_{2n})\) is given by

\[(M_1 \cup M_2, N_{11} \cup N_{21}, N_{12} \cup N_{22}, ..., N_{1n} \cup N_{2n})\]

such that \(M_1 \cup M_2\) is defined by

\[
\begin{align*}
\mu_{(M_1 \cup M_2)}^P(x) &= (\mu_{M_1}^P \lor \mu_{M_2}^P)(x) = \mu_{M_1}^P(x) \lor \mu_{M_2}^P(x), \\
\mu_{(M_1 \cup M_2)}^N(x) &= (\mu_{N_1}^N \lor \mu_{N_2}^N)(x) = \mu_{N_1}^N(x) \land \mu_{N_2}^N(x) \forall x \in U_1 \cup U_2
\end{align*}
\]

(assuming \(\mu_{M_j}^P(x) = 0, \mu_{M_j}^N(x) = 0 \ i f \ x \not\in U_j, j = 1, 2\))

and \(N_{1i} \cup N_{2i}\) for \(i = 1, 2, ..., n\), is defined by

\[
\begin{align*}
\mu_{(N_{1i} \cup N_{2i})}^P(xy) &= (\mu_{N_{1i}}^P \lor \mu_{N_{2i}}^P)(xy) = \mu_{N_{1i}}^P(xy) \lor \mu_{N_{2i}}^P(xy), \\
\mu_{(N_{1i} \cup N_{2i})}^N(xy) &= (\mu_{N_{1i}}^N \lor \mu_{N_{2i}}^N)(xy) = \mu_{N_{1i}}^N(xy) \land \mu_{N_{2i}}^N(xy) \forall xy \in E_{1i} \cup E_{2i}
\end{align*}
\]

(assuming \(\mu_{N_{ji}}^P(xy) = 0, \mu_{N_{ji}}^N(xy) = 0 \ i f \ xy \not\in E_{ji}, j = 1, 2\)).

**Example 12.**

Let \(\tilde{G}_{b_1}\) and \(\tilde{G}_{b_2}\) be BFGSs as shown in Figure 2. Their union represented by \(\tilde{G}_{b_1} \cup \tilde{G}_{b_2} = (M_1 \cup M_2, N_{11} \cup N_{21}, N_{12} \cup N_{22})\) is shown in Figure 14.

![Figure 14: Union of Two BFGSs](image-url)
Example 13.

Let $G_{b1}$ and $G_{b2}$ be BFGSs as shown in Figure 4. Their union represented by $\bigcup G_{b1} = (M_1 \cup M_2, N_{11} \cup N_{21}, N_{12} \cup N_{22})$ is shown in Figure 15.

![Figure 15: Union of Two BFGSs](image)

**Theorem 6.**

Let $G^* = (U_1 \cup U_2, E_{11} \cup E_{21}, E_{12} \cup E_{22}, ..., E_{1n} \cup E_{2n})$ be the union of GSs $G_1^* = (U_1, E_{11}, E_{12}, ..., E_{1n})$ and $G_2^* = (U_2, E_{21}, E_{22}, ..., E_{2n})$. Let $G_{b1} = (M_1, N_{11}, N_{12}, ..., N_{1n})$ and $G_{b2} = (M_2, N_{21}, N_{22}, ..., N_{2n})$ be respective BFGSs of $G_1^*$ and $G_2^*$. Then $G_{b1} \cup G_{b2} = (M_1 \cup M_2, N_{11} \cup N_{21}, N_{12} \cup N_{22}, ..., N_{1n} \cup N_{2n})$ is a BFGS of $G^*$.

**Proof:**

Let $u_1 u_2 \in E_{1i} \cup E_{2i}$.

**Case 1.** When $u_1, u_2 \in U_1$, then by definition 11

$$\mu^P_{M_2}(u_1) = \mu^P_{M_1}(u_2) = \mu^P_{N_{21}}(u_1 u_2) = 0, \mu^N_{M_2}(u_1) = \mu^N_{M_1}(u_2) = \mu^N_{N_{21}}(u_1 u_2) = 0,$$

so we have

$$\mu^P_{(N_{11} \cup N_{21})}(u_1 u_2) = \mu^P_{N_{11}}(u_1 u_2) \vee \mu^P_{N_{21}}(u_1 u_2) = \mu^P_{N_{11}}(u_1 u_2) \vee 0 = [\mu^P_{M_1}(u_1) \vee 0] \wedge [\mu^P_{M_1}(u_2) \vee 0] = [\mu^P_{M_1}(u_1) \vee \mu^P_{M_2}(u_2)] \wedge [\mu^P_{M_2}(u_1) \vee \mu^P_{M_2}(u_2)]$$

for $u_1, u_2 \in U_1 \cup U_2$. 
Case 2. When $u_1, u_2 \in U_2$, then by definition 11

$$
\mu^P_{M_1}(u_1) = \mu^P_{M_1}(u_2) = \mu^P_{N_{1i}}(u_1u_2) = 0, \quad \mu^N_{M_1}(u_1) = \mu^N_{M_1}(u_2) = \mu^N_{N_{1i}}(u_1u_2) = 0,
$$
so we have

$$
\mu^P_{(N_{1i}\cup N_{2i})}(u_1u_2) = \mu^P_{N_{11}}(u_1u_2) \vee \mu^P_{N_{21}}(u_1u_2)
= 0 \vee \mu^P_{N_{21}}(u_1u_2)
\leq 0 \vee [\mu^P_{M_2}(u_1) \wedge \mu^P_{M_2}(u_2)]
= [0 \vee \mu^P_{M_2}(u_1)] \vee [0 \vee \mu^P_{M_2}(u_2)]
= [\mu^P_{M_1}(u_1) \vee \mu^P_{M_2}(u_1)] \wedge [\mu^P_{M_1}(u_2) \vee \mu^P_{M_2}(u_2)]
= \mu^P_{(M_1\cup M_2)}(u_1) \wedge \mu^P_{(M_1\cup M_2)}(u_2),
$$

$$
\mu^N_{(N_{1i}\cup N_{2i})}(u_1u_2) = \mu^N_{N_{11}}(u_1u_2) \wedge \mu^N_{N_{21}}(u_1u_2)
= 0 \wedge \mu^N_{N_{21}}(u_1u_2)
\geq 0 \wedge [\mu^N_{M_2}(u_1) \vee \mu^N_{M_2}(u_2)]
= [0 \wedge \mu^N_{M_2}(u_1)] \wedge [0 \wedge \mu^N_{M_2}(u_2)]
= [\mu^N_{M_1}(u_1) \wedge \mu^N_{M_2}(u_1)] \vee [\mu^N_{M_1}(u_2) \wedge \mu^N_{M_2}(u_2)]
= \mu^N_{(M_1\cup M_2)}(u_1) \vee \mu^N_{(M_1\cup M_2)}(u_2),
$$

for $u_1, u_2 \in U_1 \cup U_2$.

Both cases hold for $i = 1, 2, ..., n$. This completes the proof. □

**Theorem 7.**

If $G^* = (U_1\cup U_2, E_{11}\cup E_{21}, E_{12}\cup E_{22}, ..., E_{1n}\cup E_{2n})$ is the union of GSs $G^*_1 = (U_1, E_{11}, E_{12}, ..., E_{1n})$ and $G^*_2 = (U_2, E_{21}, E_{22}, ..., E_{2n})$. Then every BFGS $\tilde{G}_b = (M, N_{1i}, N_{2i}, ..., N_{ni})$ of $G^*$ is the union of a BFGS $\tilde{G}_{b1}$ of $G^*_1$ and a BFGS $\tilde{G}_{b2}$ of $G^*_2$.

**Proof:**

We define $M_1, M_2, N_{1i}$ and $N_{2i}$ for $i = 1, 2, ..., n$ as

$$
\begin{align*}
\mu^P_{M_1}(u) &= \mu^P_{M}(u), \quad \mu^N_{M_1}(u) = \mu^N_{M}(u), \quad & \text{if } u \in U_1 \\
\mu^P_{M_2}(u) &= \mu^P_{M}(u), \quad \mu^N_{M_2}(u) = \mu^N_{M}(u), \quad & \text{if } u \in U_2 \\
\mu^P_{N_{11}}(u_1u_2) &= \mu^P_{N_1}(u_1u_2), \quad \mu^N_{N_{11}}(u_1u_2) = \mu^N_{N_1}(u_1u_2), \quad & \text{if } u_1u_2 \in E_{1i} \\
\mu^P_{N_{21}}(u_1u_2) &= \mu^P_{N_2}(u_1u_2), \quad \mu^N_{N_{21}}(u_1u_2) = \mu^N_{N_2}(u_1u_2), \quad & \text{if } u_1u_2 \in E_{2i}.
\end{align*}
$$

Then, $M = M_1 \cup M_2$ and $N_i = N_{1i} \cup N_{2i}, i = 1, 2, ..., n$.

Now for $u_1u_2 \in E_{ji}, j = 1, 2$ and $i = 1, 2, ..., n$

$$
\begin{align*}
\mu^P_{N_{ji}}(u_1u_2) &= \mu^P_{N_j}(u_1u_2) \leq \mu^P_{M}(u_1) \wedge \mu^P_{M}(u_2) = \mu^P_{M}(u_1) \wedge \mu^P_{M}(u_2) \\
\mu^N_{N_{ji}}(u_1u_2) &= \mu^N_{N_j}(u_1u_2) \geq \mu^N_{M}(u_1) \vee \mu^N_{M}(u_2) = \mu^N_{M}(u_1) \vee \mu^N_{M}(u_2),
\end{align*}
$$

i.e.,

$$
\tilde{G}_{bj} = (M_j, N_{j1}, N_{j2}, ..., N_{jn}) \text{ is a BFGS of } G^*_j, j = 1, 2.
$$
Thus, $\tilde{G}_b = (M, N_1, N_2, \ldots, N_n)$, a BFGS of $G^* = G_1 \cup G_2$, is the union of a BFGS of $G_1^*$ and a BFGS of $G_2^*$. □

**Definition 12.**

Let $\tilde{G}_{b1} = (M_1, N_{11}, N_{12}, \ldots, N_{1n})$ and $\tilde{G}_{b2} = (M_2, N_{21}, N_{22}, \ldots, N_{2n})$ be respective BFGSs of GSs $G_1^* = (U_1, E_{11}, E_{12}, \ldots, E_{1n})$ and $G_2^* = (U_2, E_{21}, E_{22}, \ldots, E_{2n})$ and let $U_1 \cap U_2 = \emptyset$. The join $\tilde{G}_{b1} + \tilde{G}_{b2}$ of $\tilde{G}_{b1}$ and $\tilde{G}_{b2}$, is then a BFGS of $G_1^* + G_2^* = (U_1 + U_2, E_{11} + E_{21}, E_{12} + E_{22}, \ldots, E_{1n} + E_{2n})$ is given by

$$(M_1 + M_2, N_{11} + N_{21}, N_{12} + N_{22}, \ldots, N_{1n} + N_{2n})$$

such that $M_1 + M_2$ is defined by

\[
\begin{align*}
\mu^P_{(M_1+M_2)}(x) &= \mu^P_{(M_1 \cup M_2)}(x), \\
\mu^N_{(M_1+M_2)}(x) &= \mu^N_{(M_1 \cup M_2)}(x) \quad \forall x \in U_1 \cup U_2,
\end{align*}
\]

$N_{1i} + N_{2i}$ for $i = 1, 2, \ldots, n$ is defined by

\[
\begin{align*}
\mu^P_{(N_{1i}+N_{2i})}(xy) &= \mu^P_{(N_{1i} \cup N_{2i})}(xy), \\
\mu^N_{(N_{1i}+N_{2i})}(x) &= \mu^N_{(N_{1i} \cup N_{2i})}(x) \quad \forall x \in E_{1i} \cup E_{2i}
\end{align*}
\]

and

\[
\begin{align*}
\mu^P_{(N_{1i}+N_{2i})}(xy) &= (\mu^P_{N_{1i}} + \mu^P_{N_{2i}})(xy) = \mu^P_{M_1}(x) \land \mu^P_{M_2}(y), \\
\mu^N_{(N_{1i}+N_{2i})}(x) &= (\mu^N_{N_{1i}} + \mu^N_{N_{2i}})(xy) = \mu^N_{M_1}(x) \lor \mu^N_{M_2}(y) \quad \forall x \in U_1, \ y \in U_2.
\end{align*}
\]

**Example 14.**

Let $\tilde{G}_{b1}$ and $\tilde{G}_{b2}$ be BFGSs as shown in Figure 2. Their join represented by $\tilde{G}_{b1} + \tilde{G}_{b2} = (M_1 + M_2, N_{11} + N_{21}, N_{12} + N_{22})$ is shown in Figure 16.

![Figure 16: Join of Two BFGSs](image-url)
Example 15.
Let \( \tilde{G}_{b_1} \) and \( \tilde{G}_{b_2} \) be BFGSs as shown in Figure 4. Their join represented by \( \tilde{G}_{b_1} + \tilde{G}_{b_2} = (M_1 + M_2, N_{11} + N_{21}, N_{12} + N_{22}) \) is shown in Figure 17.

![Figure 17: Join of Two BFGSs](image)

**Theorem 8.**

Let \( G^* = (U_1 + U_2, E_{11} + E_{21}, E_{12} + E_{22}, \ldots, E_{1n} + E_{2n}) \) be the join of GSs \( G_1^* = (U_1, E_{11}, E_{12}, \ldots, E_{1n}) \) and \( G_2^* = (U_2, E_{21}, E_{22}, \ldots, E_{2n}) \). Let \( G_{b_1} = (M_1, N_{11}, N_{12}, \ldots, N_{1n}) \) and \( G_{b_2} = (M_2, N_{21}, N_{22}, \ldots, N_{2n}) \) be respective BFGSs of \( G_1^* \) and \( G_2^* \). Then \( G_{b_1} + G_{b_2} = (M_1 + M_2, N_{11} + N_{21}, N_{12} + N_{22}, \ldots, N_{1n} + N_{2n}) \) is a BFGS of \( G^* \).

**Proof:**

Let \( u_1, u_2 \in E_{1i} + E_{2i} \).

Case 1. When \( u_1, u_2 \in U_1 \), then by definition 12

\[
\mu^P_{M_2}(u_1) = \mu^P_{M_2}(u_2) = \mu^P_{N_{21}}(u_1 u_2) = 0, \quad \mu^N_{M_2}(u_1) = \mu^N_{M_2}(u_2) = \mu^N_{N_{21}}(u_1 u_2) = 0,
\]

so we have

\[
\mu^P_{(N_{11} + N_{21})}(u_1 u_2) = \mu^P_{N_{11}}(u_1 u_2) \lor \mu^P_{N_{21}}(u_1 u_2) = \mu^P_{N_{11}}(u_1 u_2) \lor 0 = 0, \quad \mu^N_{(M_1 + M_2)}(u_1 u_2) = \mu^N_{M_1}(u_1) \lor \mu^N_{M_2}(u_2) = \mu^N_{M_1}(u_1) \lor \mu^N_{M_2}(u_2) = 0.
\]
$\mu^N_{(N_{11}+N_{21})}(u_1u_2) = \mu^N_{N_{11}}(u_1u_2) \land \mu^N_{N_{21}}(u_1u_2)$
$= \mu^N_{N_{11}}(u_1u_2) \land 0$
$\geq [\mu^N_{M_1}(u_1) \lor \mu^N_{M_1}(u_2)] \land 0$
$= [\mu^N_{M_1}(u_1) \land 0] \lor [\mu^N_{M_1}(u_2) \land 0]$
$= [\mu^N_{M_1}(u_1) \land \mu^N_{M_2}(u_1)] \lor [\mu^N_{M_1}(u_2) \land \mu^N_{M_2}(u_2)]$
$= \mu^N_{(M_1+M_2)}(u_1) \lor \mu^N_{(M_1+M_2)}(u_2),$

for $u_1, \ u_2 \in U_1 + U_2$.  

Case 2. When $u_1, \ u_2 \in U_2$, then by definition 12 

$\mu^P_{M_1}(u_1) = \mu^P_{M_1}(u_2) = \mu^P_{N_{11}}(u_1u_2) = 0, \ \mu^N_{M_1}(u_1) = \mu^N_{M_1}(u_2) = \mu^N_{N_{11}}(u_1u_2) = 0,$

so we have

$\mu^P_{(N_{11}+N_{21})}(u_1u_2) = \mu^P_{N_{11}}(u_1u_2) \lor \mu^P_{N_{21}}(u_1u_2)$
$= 0 \lor \mu^P_{N_{21}}(u_1u_2)$
$\leq 0 \lor [\mu^P_{M_2}(u_1) \land \mu^P_{M_2}(u_2)]$
$= [0 \lor \mu^P_{M_2}(u_1)] \lor [0 \lor \mu^P_{M_1}(u_2)]$
$= [\mu^P_{M_1}(u_1) \lor \mu^P_{M_2}(u_1)] \land [\mu^P_{M_1}(u_2) \lor \mu^P_{M_2}(u_2)]$
$= \mu^P_{(M_1+M_2)}(u_1) \lor \mu^P_{(M_1+M_2)}(u_2),$

$\mu^N_{(N_{11}+N_{21})}(u_1u_2) = \mu^N_{N_{11}}(u_1u_2) \land \mu^N_{N_{21}}(u_1u_2)$
$= 0 \land \mu^N_{N_{21}}(u_1u_2)$
$\geq 0 \land [\mu^N_{M_2}(u_1) \lor \mu^N_{M_2}(u_2)]$
$= [0 \land \mu^N_{M_2}(u_1)] \lor [0 \land \mu^N_{M_1}(u_2)]$
$= [\mu^N_{M_1}(u_1) \land \mu^N_{M_2}(u_1)] \lor [\mu^N_{M_1}(u_2) \land \mu^N_{M_2}(u_2)]$
$= \mu^N_{(M_1+M_2)}(u_1) \lor \mu^N_{(M_1+M_2)}(u_2),$

for $u_1, \ u_2 \in U_1 + U_2$.  

Case 3. When $u_1 \in U_1, \ u_2 \in U_2$ then by definition 12 

$\mu^P_{M_1}(u_2) = \mu^P_{M_2}(u_1) = \mu^P_{M_2}(u_2) = \mu^N_{M_2}(u_1) = 0,$

and we have

$\mu^P_{(N_{11}+N_{21})}(u_1u_2) = \mu^P_{M_1}(u_1) \land \mu^P_{M_2}(u_2)$
$= [\mu^P_{M_1}(u_1) \lor 0] \land [0 \lor \mu^P_{M_2}(u_2)]$
$= [\mu^P_{M_1}(u_1) \lor \mu^P_{M_2}(u_1)] \land [\mu^P_{M_1}(u_2) \lor \mu^P_{M_2}(u_2)]$
$= \mu^P_{(M_1+M_2)}(u_1) \land \mu^P_{(M_1+M_2)}(u_2),$
\[
\begin{align*}
\mu_{(N_{1i}+N_{2i})}(u_1u_2) &= \mu_{M_1}^N(u_1) \lor \mu_{M_2}^N(u_2) \\
&= [\mu_{M_1}^N(u_1) \land 0] \lor [0 \land \mu_{M_2}^N(u_2)] \\
&= [\mu_{M_1}(u_1) \land \mu_{M_2}(u_1)] \lor [\mu_{M_1}(u_2) \land \mu_{M_2}(u_2)] \\
&= \mu_{(M_1+M_2)}(u_1) \lor \mu_{(M_1+M_2)}(u_2),
\end{align*}
\]

for \( u_1, u_2 \in U_1 + U_2 \).

All three cases hold for \( i = 1, 2, \ldots, n \). This completes the proof. \( \Box \)

**Theorem 9.**

If \( G^* = (U_1 + U_2, E_{11} + E_{21}, E_{12} + E_{22}, \ldots, E_{1n} + E_{2n}) \) is the join of GSs \( G_1^* = (U_1, E_{11}, E_{12}, \ldots, E_{1n}) \) and \( G_2^* = (U_2, E_{21}, E_{22}, \ldots, E_{2n}) \) and \( G_b = (M, N_1, N_2, \ldots, N_n) \) is a strong BFGS of \( G^* \) Then \( G_b \) is the join of \( G_{b1} \), a strong BFGS of \( G_1^* \), and \( G_{b2} \), a strong BFGS of \( G_2^* \).

**Proof:**

Let define \( M_j \) and \( N_{ji} \) for \( i = 1, 2, \ldots, n \) and \( j = 1, 2 \) as

\[
\begin{align*}
\mu_{M_j}^P(u) &= \mu_{M}^P(u), \quad \mu_{M_j}^N(u) = \mu_{M}^N(u), \quad \text{if} \ u \in U_j \\
\mu_{N_{ji}}^P(u_1u_2) &= \mu_{N_{j}}^P(u_1u_2), \quad \mu_{N_{ji}}^N(u_1u_2) = \mu_{N_{j}}^N(u_1u_2), \quad \text{if} \ u_1u_2 \in E_{ji}.
\end{align*}
\]

By similar way as in the proof of Theorem 7, for \( u_1u_2 \in E_{ji}, \ j = 1, 2 \) and \( i = 1, 2, \ldots, n \)

\[
\begin{align*}
\mu_{N_{ji}}^P(u_1u_2) &= \mu_{N_{j}}^P(u_1u_2) = \mu_{M}^P(u_1) \land \mu_{M}^P(u_2) = \mu_{M_j}^P(u_1) \land \mu_{M_j}^P(u_2) \\
\mu_{N_{ji}}^N(u_1u_2) &= \mu_{N_{j}}^N(u_1u_2) = \mu_{M}^N(u_1) \lor \mu_{M}^N(u_2) = \mu_{M_j}^N(u_1) \lor \mu_{M_j}^N(u_2).
\end{align*}
\]

So \( G_{bj} = (M_j, N_{j1}, N_{j2}, \ldots, N_{jn}) \) is a strong BFGS of \( G_j^*, \ j = 1, 2 \).

Moreover, \( G_b \) is the join of \( G_{b1} \) and \( G_{b2} \) as shown in the following.

Using definitions 11 and 12, \( M = M_1 \cup M_2 = M_1 + M_2 \) and

\[
N_i = N_{1i} \cup N_{2i} = N_{1i} + N_{2i}, \quad \forall \ u_1u_2 \in E_{1i} \cup E_{2i}.
\]

When \( u_1u_2 \in E_{11} + E_{21} \setminus (E_{1i} \cup E_{2i}), \ \text{i.e.}, \ u_1 \in U_1 \text{ and } u_2 \in U_2 \)

\[
\begin{align*}
\mu_{N_{1i}}^P(u_1u_2) &= \mu_{M}^P(u_1) \land \mu_{M}^P(u_2) = \mu_{M_1}^P(u_1) \land \mu_{M_2}^P(u_2) = \mu_{N_{1i} + N_{2i}}^P(u_1u_2) \\
\mu_{N_{1i}}^P(u_1u_2) &= \mu_{N_{j}}^P(u_1) \lor \mu_{N_{j}}^P(u_2) = \mu_{N_{j}}^P(u_1) \lor \mu_{N_{j}}^P(u_2) = \mu_{N_{1i} + N_{2i}}^P(u_1u_2).
\end{align*}
\]

There are similar calculations when \( u_1 \in U_2 \) and \( u_2 \in U_1 \). This is true for \( i = 1, 2, \ldots, n \). This ends the proof. \( \Box \)

4. **Conclusions**

Graph theoretical concepts are widely used to study and model various applications in different areas. However, in many cases, some aspects of a graph-theoretic problem may be vague or uncertain. It is natural to deal with the vagueness and uncertainty using the methods of fuzzy sets or bipolar fuzzy sets which have shown advantages in handling vagueness and uncertainty.
than fuzzy sets. So we have applied the concept of bipolar fuzzy sets to graph structures. We have discussed some operations on bipolar fuzzy graph structures. We are extending our work to: (1) Bipolar fuzzy soft graph structures, (2) Soft graph structures, (3) Rough fuzzy soft graph structures, and (4) Roughness in fuzzy graph structures.

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