Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Vol. 10, Issue 1 (June 2015), pp. 495 - 520

Applications and Applied Mathematics:
An International Journal
(AAM)

Characterization of Gamma Hemirings by Generalized Fuzzy Gamma Ideals

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Received: March 5, 2014; Accepted: October 30, 2014

Abstract

This paper has explored theoretical methods of evaluation in the identification of the boundedness of the generalized fuzzy gamma ideals. A functional approach was used to undertake a characterization of this structure leading to a determination of some interesting gamma hemirings theoretic properties of the generated structures. Gamma hemirings are the generalization of the classical agebraic structure of hemirings. Our aim is to extend this idea and, to introduce the concept of generalized fuzzy gamma ideals, generalized fuzzy prime (semiprime) gamma ideals, generalized fuzzy h-gamma ideals and generalized fuzzy k- gamma ideals of gamma hemirings and related properties are investigated. We have shown that intersection of any family of generalized fuzzy (left, right) h-gamma ideals $(k\text{-gamma ideals})$ of a hemiring is a generalized fuzzy (left, right) h-gamma ideal $(k\text{-gamma ideal})$ of H. Similarly we proved that the intersection of any family of generalized fuzzy prime (resp. semiprime) gamma ideals of H is a generalized fuzzy prime (resp. semiprime) gamma ideal of H. We have proved that a fuzzy subset μ of H is fuzzy h-gamma ideal $(k\text{-gamma ideal})$ if and only if μ is a generalized fuzzy h-gamma ideal $(k\text{-gamma ideal})$ of H. Further level cuts provide a useful linkage between the classical set theorey and the fuzzy set theorey. Here we use this linkage to investigate some useful aspects of gamma hemirings and characterize the gamma hemmirings through level cuts in terms of generalized fuzzy (left, right, prime, semiprime) gamma ideals of gamma hemirings. We have also used the concept of support of a fuzzy set in order to obtain some interesting results of gamma hemirings using the generalized fuzzy (left, right, prime, semiprime) gamma ideals of hemirings.

Keywords: Γ-Hemirings, (α,β)-fuzzy Γ-ideals; (α,β)-fuzzy prime Γ-ideals; (α,β) semi prime; (α,β)-fuzzy h-Γ-ideals; (α,β)-fuzzy k-Γ-ideals; $(\in,\in)$ *fuzzy* h-Γ-ideal; $(\in,\in)$ *fuzzy* k-Γ-ideal

MSC 2000 No.: 20M10 and 20N99

1. Introduction

Hemirings provide a common generalization of rings and distributive latices that arise in such diverse areas of mathematics such as combinatorics, functional analysis, graph theory, automata theory, formal language theory, mathematical modeling of quantum physics and parallel computation system Aho et al.(1979), Glazek (2002), Golan (1999) and Hebisch and Weinert (1998). The concept of fuzzy sets was first introduced by Zadeh (1965). Researchers used these notions in various algebraic structures; see Yaqoob (2013), Yaqoob et al. (2012), Yaqoob et al. (2012), Yaqoob et al. (2013) and Yaqoob et al. (2013). As a generalization of Γ-rings and of semirings the concept of Γ-semirings was introduced by Rao (1995). Ideals of semirings play an important role in structure theory and are useful for many purposes. However, they do not in general coincide with the usual ring ideals if a semiring is a ring and, for this reason, their use is somewhat limited in trying to obtain analogues of ring theorems for semirings. Many results in rings apparently have no analogues in semirings using only ideals. In this regard Henriksen (1958) defined a new class of ideals namely k-ideals in semirings. However a still more restricted class of ideals in hemirings has been given by Iizuka (1959). The definition of ideal in any commutative semiring coincides with Iizuka's definition provided that S is a hemiring, and it is called h-ideal. Abdullah (2013) and Abdullah et al. (2011) introduced the idea of (α,β)-Intuitionistic fuzzy ideals of hemirings and N-dimensional (α,β)-fuzzy H-ideals in hemirings, respectively. Recently the concept of prime (semiprime) fuzzy h-ideals in Γ-hemirings were introduced by Sardar and Mandal (2009) and Sardar and Mandal (2011). For more about (α,β)-fuzzy ideals, Gulistan et al. (in press) and Zadeh (1965).

In this paper, we characterize the different types of (α,β)-fuzzy Γ-ideals of Γ-hemirings.

2. Preliminaries

We will recall some useful definitions and results which will be helpful in further pursuit of this study. A semiring is an algebraic system $(R,+,\cdot)$ consisting of a non-empty set R together with two binary operations called addition and multiplication (denoted in the usual manner) such that $(R,+)$ and $(R,.)$ are semigroups connected by the following distributive laws: $a(b+c)=ab+ac$ and $(b+c)a=ba+ca,$ valid for all $a,b,c\in R$. An element $0\in R$ is called a zero of R if $a+0=0+a=a$ and $0a=a0=0,$ for all $a\in R$ and an element $1\in R$ is called the identity of R if $1a=a1=a,$ for all $a\in R$. A semiring with zero and a commutative addition is called a hemiring.

A non-empty subset A of a semiring R is called a subsemiring of R if it is closed with respect to the addition and multiplicationin R. A non-empty subset I of a semiring R is its left (right) ideal if it is closed with respect to the addition and $RI\subseteq I$ $(IR\subseteq I)$. If A is left and right ideal of R then it is an ideal of R. An ideal I of R is called a (left, right) k-ideal of R if for any $a,b\in I$ and all $x\in R$ from $x+a=b\Rightarrow x\in I.$ An ideal I of R is called a (left, right) h-ideal

of R if for any $a,b \in I$ and $x,y \in R$ from $x+a+y=b+y \Rightarrow x \in I$. Every (left, right) h-ideal of R is (left, right) k-ideal of R but converse is not true. An ideal I of R is called prime (semiprime) if $xy \in I$, $(x^2 \in I)$ implies $x \in I$ or $y \in I$ $(x \in I)$. A k-ideal $(h\text{-ideal})$ of R is called prime (semiprime) if it is prime (semiprime) as an ideal of R, i.e if $xy \in I$ $(x^2 \in I)$ implies $x \in I$ or $y \in I$ $(x \in I)$.

Definition 1. (Sardar and Mandal, 2009)

Let H and Γ be two additive commutative semigroups with zero. Then H is called a Γ-hemiring if there exists a mapping $H \times \Gamma \times H \to H$ given by $(x,\alpha,y) \mapsto x\alpha y$, satisfying the following conditions:

(i) $(a+b)\gamma c = a\gamma c + b\gamma c,$

(ii) $a\gamma(b+c) = a\gamma b + a\gamma c,$

(iii) $a(\gamma+\beta)b = a\gamma b + a\beta b,$

(iv) $a\gamma(b\beta c) = (a\gamma b)\beta c,$

(v) $0_H \gamma a = 0_H = a\gamma 0_H,$

(vi) $a0_\Gamma b = 0_\Gamma = b0_\Gamma a,$

for $a,b,c \in H$ and $\gamma,\beta \in \Gamma$.

Throughout this paper H will denote a Γ-hemiring. A nonempty subset A of H is called a Γ-subhemiring of H if $A\Gamma A \subseteq A$ and $A+A \subseteq A$.

Definition 2. (Rao, 1995)

By a fuzzy subset of H we mean any map $f : H \to [0,1]$.

Definition 3. (Sardar and Mandal, 2009)

A fuzzy subset μ of a Γ-hemiring H is called a fuzzy Γ-subhemiring of H if for all $x,y \in H$, we have

(i) $\mu(x+y) \geq \min\{\mu(x),\mu(y)\}$,

(ii) $\mu(x\alpha y) \geq \min\{\mu(x),\mu(y)\}$.

Definition 4. (Rao, 1995)

A fuzzy subset μ of a Γ-hemiring H is called a fuzzy left (right) Γ-ideal of H if for all $x, y \in H$ it satisfies:

(i) $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$,

(ii) $\mu(x\alpha y) \geq \mu(y)$ ($\mu(x\alpha y) \geq \mu(x)$).

Note that a fuzzy Γ-ideal of H can be defined as a fuzzy subset of H for which all $x, y \in H$, satisfies $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$ and $\mu(x\alpha y) \geq \max\{\mu(x), \mu(y)\}$. Every fuzzy (left, right) Γ-ideal is a fuzzy Γ-subhemiring.

Definition 5. (Rao, 1995)

A fuzzy (left, right) Γ-ideal μ of H is called a fuzzy (left, right) k-Γ-ideal if

$$x+y=z \Rightarrow \mu(x) \geq \min\{\mu(y), \mu(z)\},$$

and fuzzy (left, right) h-Γ-ideal of H if

$$x+a+z=b+z \Rightarrow \mu(x) \geq \min\{\mu(a), \mu(b)\},$$

for all $x, y, z, a, b \in H$.

Definition 6. (Rao, 1995)

A fuzzy Γ-ideal of H is called prime (semiprime) if

$$\mu(x\alpha y) = \mu(x) \text{ or } \mu(y) \ \left(\mu(x\alpha x) = \mu(x)\right), \text{ for all } x, y \in H \text{ and } \alpha \in \Gamma.$$

3. (α, β)-Fuzzy Γ-Ideals

Let H be a Γ-hemiring and $\alpha \in \{\in, q, \in \vee q\}$, $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$ unless otherwise specified.

Definition 7.

A fuzzy subset μ of a Γ-hemiring H is called (α, β)-fuzzy right (resp. left) Γ-ideal of H if

(i) $x_a \alpha \mu, y_b \alpha \mu$ implies $(x+y)_{\min(a,b)} \beta \mu$,

(ii) $x_a \alpha\mu, y_b \alpha\mu$ implies $(x\gamma y)_{\min(a,b)} \beta\mu$ (resp. $(y\gamma x)_{\min(a,b)} \beta\mu$) for all $x, y \in H$ and $a, b \in (0,1]$ and $\gamma, \beta \in \Gamma$.

A fuzzy subset μ of a Γ-hemiring H is called (α, β)-fuzzy Γ-ideal of H if it is both a fuzzy right and a fuzzy left Γ-ideal of H.

Theorem 1.

Let μ be a non-zero (α, β)-fuzzy (left, right) Γ-ideal of H. Then, the set

$$H_\mu = \{x \in H \mid \mu(x) > 0\},$$

is a (left, right) Γ-ideal of H.

Proof:

Let $x, y \in H_\mu$, then $\mu(x) > 0$ and $\mu(y) > 0$. Assume that $\mu(x+y) = 0$. If $\alpha \in \{\in, \in \vee q\}$, then $x_{\mu(x)} \alpha\mu, y_{\mu(y)} \alpha\mu$.

As

$$\mu(x+y) = 0 < \min\{\mu(x), \mu(y)\} \text{ and } \mu(x+y) + \min\{\mu(x), \mu(y)\} \leq 0 + 1 = 1,$$

so,

$$(x+y)_{\min\{\mu(x),\mu(y)\}} \overline{\beta}\mu, \text{ for every } \beta \in \{\in, q, \in \vee q, \in \wedge q\},$$

a contradiction. Hence,

$$\mu(x+y) > 0, \text{ that is, } x + y \in H_\mu.$$

Let $x \in H_\mu$, $y \in H$ and $\gamma \in \Gamma$. Then, $\mu(x) > 0$ and assume that $\mu(x\gamma y) = 0$. If $\alpha \in \{\in, \in \vee q\}$, then $x_{\mu(x)} \alpha\mu$, but $\mu(x\gamma y) = 0 < \min\{\mu(x), \mu(x)\}$ and $\mu(x\gamma y) + \min\{\mu(x), \mu(x)\} \leq 0 + 1 = 1$.

So,

$$(x\gamma y)_{\min\{\mu(x),\mu(x)\}} \overline{\beta} f \text{ and } (y\gamma x)_{\min\{\mu(x),\mu(x)\}} \overline{\beta} f, \text{ for every } \beta \in \{\in, q, \in \vee q, \in \wedge q\},$$

a contradiction. Hence, $\mu(x\gamma y) > 0$, that is, $x\gamma y \in H_\mu$ and $y\gamma x \in H_\mu$. Thus, the set

$$H_\mu = \{x \in H \mid \mu(x) > 0\}$$

is an (left, right) Γ-ideal of H.

Theorem 2.

Let I be a left (resp. right) Γ-ideal of H and let μ be a fuzzy subset in H such that,

$$\mu(x) = \begin{cases} 0, & \text{if } x \in H - I, \\ \geq 0.5, & \text{if } x \in I. \end{cases}$$

Then,

(i) μ is an $(\in, \in \vee q)$-fuzzy left (resp. right) Γ-ideal of H,
(ii) μ is a $(q, \in \vee q)$-fuzzy left (resp. right) Γ-ideal of H,
(iii) μ is a $(\in \vee q, \in \vee q)$-fuzzy left (resp. right) Γ-ideal of H.

Proof:

(i)

Let $x, y \in H$, $\gamma \in \Gamma$ and $t_1, t_2 \in (0,1]$ such that $x_t \in \mu$ and $y_t \in \mu$. Then $\mu(t) \geq t$ and $\mu(y) \geq t$. Thus, $x, y \in I$ and so $x + y \in I$. So $\mu(x+y) \geq 0.5$. Thus, if $\min\{t_1, t_2\} \leq 0.5$, then $\mu(x+y) \geq 0.5 \geq \min\{t_1, t_2\}$ and so $(x+y)_{\min\{t_1,t_2\}} \in \mu$. If $\min\{t_1, t_2\} > 0.5$, then $\mu(x+y) + \min\{t_1, t_2\} > 0.5 + 0.5 = 1$ and so $(x+y)_{\min\{t_1,t_2\}}\, q\mu$. Therefore, $(x+y)_{\min\{t_1,t_2\}} \in \vee qf$.

Let $x, y \in H$, $\gamma \in \Gamma$ and $t \in (0,1]$ such that $x_t \in \mu$. Then, $\mu(x) \geq t$. Thus, $x \in I$ and so $x\gamma y \in I$. Consequently, $\mu(x\gamma y) \geq 0.5$. Thus, if $t \leq 0.5$, then $\mu(x\gamma y) \geq 0.5 \geq t$ and so $(x\gamma y)_t \in \mu$. If $t > 0.5$, then $\mu(x\gamma y) + t > 0.5 + 0.5 = 1$ and so $(x\gamma y)_t\, q\mu$. Therefore, $(x\gamma y)_t \in \vee q\mu$. Thus, μ is an $(\in, \in \vee q)$-fuzzy left Γ-ideal of H.

Similarly $(y\gamma x)_t \in \vee q\mu$. Hence, μ is an $(\in, \in \vee q)$-fuzzy Γ-ideal of H.

(ii)

Let $x, y \in H$, $\gamma \in \Gamma$ and $t_1, t_2 \in (0,1]$ such that $x_t q\mu$ and $y_t q\mu$. Then, $\mu(t) + t > 1$ and $\mu(y) + t > 1$. As $x + y \in I$. So, $\mu(x+y) \geq 0.5$. If $\min\{t_1, t_2\} \leq 0.5$, then $\mu(x+y) \geq 0.5 \geq \min\{t_1, t_2\}$ and so $(x+y)_{\min\{t_1,t_2\}} \in \mu$. If $\min\{t_1, t_2\} > 0.5$, then

$\mu(x+y)+\min\{t_1,t_2\}>0.5+0.5=1$ and so $(x+y)_{\min\{t_1,t_2\}}\, q\mu$. Therefore, $(x+y)_{\min\{t_1,t_2\}}\in\vee q\mu$.

Let $x,y\in H$, $\gamma\in\Gamma$ and $t\in(0,1]$ such that $x_t q\mu$. Then, $\mu(x)+t\geq 1$. Thus, $x\in I$ and so $x\gamma y\in I$ and $y\gamma x\in I$. Consequently, $\mu(x\gamma y)\geq 0.5$, and $\mu(y\gamma x)\geq 0.5$. Thus, if $t\leq 0.5$, then $\mu(x\gamma y)\geq 0.5\geq t$ and so $(x\gamma y)_t\in\mu$. If $t>0.5$, then $\mu(x\gamma y)+t>0.5+0.5=1$ and so $(x\gamma y)_t\, q\mu$. Therefore, $(x\gamma y)_t\in\vee q\mu$. Thus, μ is a $(q,\in\vee q)$-fuzzy left Γ-ideal of H.

Similarly $(y\gamma x)_t\in\vee q\mu$. Hence μ is a $(q,\in\vee q)$-fuzzy Γ-ideal of H.

(iii)
Follows from (i) and (ii).

Definition 8.

An (α,β)-fuzzy Γ-ideal μ of H is called an (α,β)-fuzzy k-Γ-ideal of H if $x+a=b$, $a_r\alpha\mu$ and $b_s\alpha\mu$ implies $x_{\min\{r,s\}}\beta$ and if $x+a+y=b+y$, $a_r\alpha\mu$ and $b_s\alpha\mu$ implies $x_{\min\{r,s\}}\beta$ for all $a,b,x,y\in H,\alpha\in\Gamma$ and $r,s\in(0,1]$ then it is called (α,β)-fuzzy h-Γ-ideal of H.

Theorem 3.

Let μ be a non-zero (α,β)-fuzzy h-Γ-ideal $(k$-Γ-ideal$)$ of H. Then, the set

$$H_\mu=\{x\in H\mid \mu(x)>0\}$$

is a Γ-ideal of H.

Proof:

Straightforward.

Theorem 4.

Let μ be a non-zero (α,β)-fuzzy h-Γ-ideal $(k$-Γ-ideal$)$ of H. Then, the set

$$H_\mu=\{x\in H\mid \mu(x)>0\}$$

is an h-Γ-ideal (k-Γ-ideal) of H.

Proof:

By Theorem 3, the set $H_\mu = \{x \in H \mid \mu(x) > 0\}$ is an h-Γ-ideal of H. Let $a, b \in H_\mu$, so $\mu(a) > 0$, $\mu(b) > 0$, and $x, y \in H$ be such that $a + x + y = b + y$. Let $\mu(x) = 0$. If $\alpha \in \{\in, \in \vee q\}$, then $a_{\mu(x)} \alpha \mu$, $b_{\mu(x)} \alpha \mu$, but $x_{\min\{\mu(a), \mu(b)\}} \overline{\beta} f$ and for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Also $a_1 q \mu$ and $b_2 q \mu$, but $x_1 \overline{\beta} \mu$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Hence, $\mu(x) > 0$, that is $x \in H_\mu$. Thus, the set $H_\mu = \{x \in H \mid \mu(x) > 0\}$ is an h-Γ-ideal of H.

For the proof of (α, β)-fuzzy k-Γ-ideal put $y = 0$ in the above result.

Theorem 5.

Let I be an h-Γ-ideal $(k\text{-}\Gamma\text{-ideal})$ of H and let μ be a fuzzy subset in H such that,

$$\mu(x) = \begin{cases} 0, & \text{if } x \in H - I, \\ \geq 0.5, & \text{if } x \in I. \end{cases}$$

Then,

(i) μ is an $(\in, \in \vee q)$-fuzzy h-Γ-ideal $(k\text{-}\Gamma\text{-ideal})$ of H,

(ii) μ is a $(q, \in \vee q)$-fuzzy h-Γ-ideal $(k\text{-}\Gamma\text{-ideal})$ of H,

(iii) μ is an $(\in \vee q, \in \vee q)$-fuzzy h-Γ-ideal $(k\text{-}\Gamma\text{-ideal})$ of H.

Proof:

(i)

By Theorem 2, μ is an $(\in, \in \vee q)$-fuzzy Γ-ideal of H. Let us assume that I is an h-Γ-ideal of H and let $a, b, x, y \in H$ such that $x + a + y = b + y$. Let $a_{t_1}, b_{t_2} \in \mu$ for some $t_1, t_2 \in (0,1]$. So $\mu(a) \geq t_1$ and $\mu(b) \geq t_2$. So by hypothesis $a, b \in I$. As I is an h-Γ-ideal, also $x \in I$. Hence, $\mu(x) \geq 0.5$. If $\min\{t_1, t_2\} \leq 0.5$, then clearly $\mu(x) \geq \min\{t_1, t_2\}$ so $x_{\min\{t_1, t_2\}} \in \mu$ and if $\min\{t_1, t_2\} > 0.5$. Then, $\mu(x) + \min\{t_1, t_2\} > 1$, and so $x_{\min\{t_1, t_2\}} q \mu$. Hence, $x_{\min\{t_1, t_2\}} \in \vee q \mu$. Thus, μ is a $(\in, \in \vee q)$-fuzzy h-Γ-ideal of H.

(ii) and (iii) Easy to prove.

If we put $y=0$ we get the required result for $(\in,\in\vee q)$-fuzzy k-Γ-ideal of H.

Theorem 6.

A fuzzy subset μ of H is an $(\in,\in)$-fuzzy Γ-ideal of H if and only if it is a fuzzy (left, right) Γ-ideal of H.

Proof:

Straightforward.

Corollary 1.

A fuzzy subset μ of H is fuzzy h-Γ-ideal $(k\text{-}\Gamma\text{-ideal})$ if and only if μ is an $(\in,\in)$ fuzzy h-Γ-ideal $(k\text{-}\Gamma\text{-ideal})$ of H.

Proof:

Straightforward.

Lemma 1.

For any fuzzy subset μ of H the following conditions are equivalent:

(i) $x_{t_1}\in\mu,\ y_{t_2}\in\mu$ implies $(x+y)_{\min(t_1,t_2)}\in\vee q\mu,$

(ii) $\mu(x+y)\geq\min\{\mu(x),\mu(y),0.5\},$ for all $x,y\in H.$

Proof:

$(i)\Rightarrow(ii)$

Let $x,y\in H$ and let us consider the case when $\min\{\mu(x),\mu(y)\}\leq 0.5.$ For

$$\mu(x+y)\geq\min\{\mu(x),\mu(y)\}$$

equation (ii) is valid. Now if

$$\mu(x+y)<\min\{\mu(x),\mu(y)\},$$

there exist t such that

$$\mu(x+y) < t < \min\{\mu(x), \mu(y)\},$$

which implies that $x_t, y_t \in \mu,$ but $(x+y)_t \overline{\in \vee q\mu}.$ But, this is a contradiction.

Now consider the case when $\min\{\mu(x), \mu(y)\} \geq 0.5.$ As $\mu(x+y) < 0.5,$ so we have $x_{0.5}, y_{0.5} \in \mu,$ but $(x+y)_{0.5} \overline{\in \vee q\mu},$ which is again a contradiction. We have $\mu(x+y) \geq 0.5.$ Thus,

$$\mu(x+y) \geq 0.5 \geq \min\{\mu(x), \mu(y), 0.5\}, \text{ for all } x, y \in H.$$

$(ii) \Rightarrow (i)$

Let $x_{t_1}, y_{t_2} \in \mu,$ then

$$\mu(x+y) \geq \min\{\mu(x), \mu(y), 0.5\} \geq \min\{t_1, t_2, 0.5\}.$$

Thus,

$$\mu(x+y) \geq \min\{t_1, t_2\} \text{ for } \min\{t_1, t_2\} \leq 0.5,$$

and

$$\mu(x+y) \geq \min\{t_1, t_2\} \text{ for } \min\{t_1, t_2\} > 0.5.$$

This means that $(x+y)_{\min(t_1, t_2)} \in \vee q\mu.$

Lemma 2.

For any fuzzy subset μ of H the following conditions are equivalent.

(i) $x_t \in \mu,\ y \in H$ Imply $(y\alpha x)_t \in \vee q\mu,$

(ii) $\mu(y\alpha x) \geq \min\{\mu(x), 0.5\},$ For all $x, y \in H$ and $\alpha \in \Gamma.$

Proof:

Straightforward.

Lemma 3.

For any fuzzy subset μ of H the following conditions are equivalent:

(i) $x_t \in \mu,\ y \in H$ Imply $(x\alpha y)_t \in \vee q\mu,$

(ii) $\mu(x\alpha y) \geq \min\{\mu(x), 0.5\},$ For all $x, y \in H$ and $\alpha \in \Gamma.$

Proof:

Straightforward.

Lemma 4.

For any fuzzy subset μ of H the following conditions are equivalent:

(i) $x_t \in \mu,\ y \in H$ Imply $(y\alpha x)_t \in \vee q\mu,$ and $(x\alpha y)_t \in \vee q\mu$ for all $x, y \in H$ and $\alpha \in \Gamma,$

(ii) $\mu(x\alpha y) \geq \min\{\max\{\mu(x), \mu(y)\}, 0.5\},$ For all $x, y \in H$ and $\alpha \in \Gamma.$

Proof:

Straightforward.

Lemma 5.

Let μ be a fuzzy subset of H and $x, y, a, b \in H$ be such that $x + a + y = b + y.$ Then, the following conditions are equivalent:

(i) $a_{t_1}, b_{t_2} \in \mu \Rightarrow x_{\min\{t_1, t_2\}} \in \vee q\mu,$

(ii) $\mu(x) \geq \min\{\mu(a), \mu(b), 0.5\}.$

Proof:

$(i) \Rightarrow (ii)$

Let $x, y, a, b \in H$ be such that $x + a + y = b + y.$ Let $\mu(x) < \min\{\mu(a), \mu(b), 0.5\}.$ Since for $\mu(x) < \min\{\mu(a), \mu(b)\}$ and there exist t such that $\mu(x) < t < \min\{\mu(a), \mu(b)\}.$ This implies $a_{t_1}, b_{t_2} \in \mu$ and $x_t \overline{\in \vee q}\mu,$ which is contradiction to hypothesis. So $\min\{\mu(a), \mu(b)\} \geq 0.5.$ Also for $\mu(x) < 0.5,$ we have $a_{0.5}, b_{0.5} \in \mu$ and $x_{0.5} \overline{\in \vee q}\mu,$ which is contradiction to hypothesis. So

$\mu(x) \geq \min\{\mu(a), \mu(b), 0.5\}$.

$(ii) \Rightarrow (i)$

Let $x+a+y=b+y$ and $a_{t_1}, b_{t_2} \in \mu$ for some $x, y, a, b \in H$ and $t_1, t_2 \in (0,1]$. Then, $\mu(x) \geq \min\{\mu(a), \mu(b), 0.5\} \geq \min\{t_1, t_2, 0.5\}$. Thus, $\mu(x) \geq \min\{t_1, t_2\}$ for $\min\{t_1, t_2\} \leq 0.5$, and $\mu(x) \geq 0.5$, for $\min\{t_1, t_2\} > 0.5$. Therefore, $x_{\min\{t_1, t_2\}} \in \vee q\mu$.

Theorem 7.

A fuzzy subset μ of H is an $(\in, \in \vee q)$-fuzzy left (resp. right) Γ-ideal of H if and only if it satisfies $\mu(x+y) \geq \min\{\mu(x), \mu(y), 0.5\}$ and $\mu(y\alpha x) \geq \min\{\mu(x), 0.5\}$, for all $x, y \in H$ and $\alpha \in \Gamma$.

Proof:

Straightforward.

Theorem 8.

A fuzzy subset μ of H is an $(\in, \in \vee q)$-fuzzy Γ-ideal of H if and only if

$$U(\mu; t) = \{x \in H \ : \ \mu(x) \geq t\} \neq \varphi$$

is a Γ-ideal of H for all $0 < t \leq 0.5$.

Proof:

Let μ be an $(\in, \in \vee q)$-fuzzy Γ-ideal of H. Assume that $x, y \in U(\mu; t)$ for some $0 < t \leq 0.5$. Then.

$$\mu(x+y) \geq \min\{\mu(x), \mu(y), 0.5\} \geq \min\{t, 0.5\} = t,$$

which implies $(x+y) \in U(\mu; t)$. Moreover, for all $x \in U(\mu; t)$ and $y \in H$ we have

$$\mu(x\alpha y) \geq \min\{\mu(x), 0.5\} \geq \min\{t, 0.5\} = t,$$

and similarly $\mu(y\alpha x) \geq t$, which implies that $x\alpha y, y\alpha x \in U(\mu; t)$. Hence,

$$U(\mu; t) = \{x \in H \ : \ \mu(x) \geq t\} \neq \varphi$$

is a Γ-ideal of H for all $0 < t \leq 0.5$.

Conversely, assume that for every $0 < t \leq 0.5$,

$$U(\mu;t) = \{x \in H \ : \ \mu(x) \geq t\} \neq \varphi$$

is a Γ-ideal of H. Then, $\mu(x+y) \geq \min\{\mu(x), \mu(y), 0.5\}$, for all $x, y \in H$. Let

$$\mu(x+y) < \min\{\mu(x), \mu(y), 0.5\}$$

and there exist $t \in (0,1]$ such that

$$\mu(x+y) < t < \min\{\mu(x), \mu(y), 0.5\}.$$

This implies that $x, y \in U(\mu;t)$, but $x+y \notin U(\mu;t)$, which is contradiction to hypothesis. So

$$\mu(x+y) \geq \min\{\mu(x), \mu(y), 0.5\}$$

and in a similar way $\mu(x\alpha y) \geq \min\{\mu(x), 0.5\}$ and $\mu(y\alpha x) \geq \min\{\mu(y), 0.5\}$ for all $x, y \in H$. Hence, μ is an $(\in, \in \vee q)$-fuzzy Γ-ideal of H.

Corollary 2.

A fuzzy subset μ of H is an $(\in, \in \vee q)$-fuzzy (left, right) h-Γ-ideal $(k$-Γ-ideal$)$ of H if and only if

$$U(\mu;t) = \{x \in H \ : \ \mu(x) \geq t\} \neq \varphi$$

is a (left, right) h-Γ-ideal $(k$-Γ-ideal$)$ of H for all $0 < t \leq 0.5$.

Proof:

Let μ be an $(\in, \in \vee q)$-fuzzy (left, right) h-Γ-ideal $(k$-Γ-ideal$)$ of H, then $U(\mu;t) \neq \varphi$ is a Γ-ideal of H. If $x+a+y = b+y$ for some $a, b \in U(\mu;t)$ and $x, y \in H$. Then,

$$\mu(x) \geq \min\{\mu(a), \mu(b), 0.5\} \geq t \Rightarrow x \in U(\mu;t).$$

Thus,

$$U(\mu;t)=\{x\in H\ :\ \mu(x)\geq t\}\neq\varphi$$

is a (left, right) h-Γ-ideal $(k$-Γ-ideal$)$ of H for all $0<t\leq 0.5$.

Conversely, assume that

$$U(\mu;t)=\{x\in H\ :\ \mu(x)\geq t\}\neq\varphi$$

is a (left, right) h-Γ-ideal $(k$-Γ-ideal$)$ of H for all $0<t\leq 0.5$. Then, by above theorem μ is an $(\in,\in\vee q)$ fuzzy Γ-ideal of H. Let us assume that $x+a+y=b+y$ for some a,b , $x,y\in H$. If $\mu(x)<\min\{\mu(a),\mu(b),0.5\}$, then there exist some $t\in(0,1]$ such that

$$\mu(x)<t<\min\{\mu(a),\mu(b),0.5\},$$

which implies $a,b\in U(\mu;t)$ but $x\notin U(\mu;t)$, which is contradiction to hypothesis. Hence,

$$\mu(x)\geq\min\{\mu(a),\mu(b),0.5\}.$$

Thus, μ is an $(\in,\in\vee q)$-fuzzy (left, right) h-Γ-ideal $(k$-Γ-ideal$)$ of H.

Theorem 9.

The intersection of any family of $(\in,\in\vee q)$-fuzzy (left, right) Γ-ideals of H is an $(\in,\in\vee q)$-fuzzy (left, right) Γ-ideals of H.

Proof:

Let $\{\mu_i\ :\ i\in\wedge\}$ be a fixed family of $(\in,\in\vee q)$-fuzzy (left, right) Γ-ideals of H and μ be the intersection of this family. We have to show that

$$(x+y)_{\min\{t_1,t_2\}}\in\vee q\mu \text{ and } (x\alpha y)(y\alpha x)\in\vee q\mu \text{ for all } x_{t_1},y_{t_2}\in\mu \text{ and } y\in H.$$

Let us assume that

$$(x+y)_{\min\{t_1,t_2\}}\overline{\in\vee q}\mu \text{ for some } x,y\in H \text{ and } t_1,t_2\in(0,1].$$

Then,

$$\mu(x+y) < \min\{t_1, t_2\} \text{ and } \mu(x+y) + \min\{t_1, t_2\} \leq 1.$$

Thus, $\mu(x+y) < 0.5$. As each $\{\mu_i \ : \ i \in \wedge\}$ is an $(\in, \in \vee q)$-fuzzy Γ-ideal of H, so the family can be divided into two disjoint parts:

$$\wedge_1 = \{\mu_i \ : \ \mu_i(x+y) \geq \min\{t_1, t_2\}\}$$

and

$$\wedge_2 = \{\mu_i \ : \ \mu_i(x+y) < \min\{t_1, t_2\} \text{ and } \mu_i(x+y) + \min\{t_1, t_2\} > 1\}.$$

If $\mu_i(x+y) \geq \min\{t_1, t_2\}$ for all μ_i, then also $\mu(x+y) \geq \min\{t_1, t_2\}$, which is a contradiction. So for some μ_i, we have $\mu_i(x+y) < \min\{t_1, t_2\}$ and $\mu_i(x+y) + \min\{t_1, t_2\} > 1$. Thus, $\min\{t_1, t_2\} > 0.5$, whence $\mu_i(x) \geq \mu(x) \geq t \geq \min\{t_1, t_2\} > 0.5$, for all $\mu_i \in \wedge_2$.

In the similar way $\mu_i(y) > 0.5$ for all $\mu_i \in \wedge$. Now let us assume that $t = \mu_i(x+y) < 0.5$ for some μ_i. Let $t^{\sim} \in (0, 0.5)$ such that $t > t^{\sim}$, then $\mu_i(x) > 0.5 > t^{\sim}$ and $\mu_i(y) > 0.5 > t^{\sim}$, that is $x_{t^{\sim}} \in \mu_i$ and $y_{t^{\sim}} \in \mu_i$ but $\mu_i(x+y) = t > t^{\sim}$ and $\mu_i(x+y) + t^{\sim} < 1$. So $(x+y)_{t^{\sim}} \overline{\in \vee q} \mu_i$, which is not possible because for all $x, y \in H$ we have $\mu(x+y) < 0.5$. Therefore, $(x+y)_{\min\{t_1, t_2\}} \in \vee q \mu$.

In the similar way, we can easily prove that

$$(x \alpha y)(y \alpha x) \in \vee q \mu \text{ for all } x_{t_1}, y_{t_2} \in \mu \text{ and } y \in H.$$

Hence, intersection of any family of $(\in, \in \vee q)$-fuzzy (left, right) Γ-ideals of H is an $(\in, \in \vee q)$-fuzzy (left, right) Γ-ideals of H.

Corollary 3.

The intersection of any family of $(\in, \in \vee q)$-fuzzy (left, right) h-Γ-ideals $(k$-Γ-ideals$)$ of H is an $(\in, \in \vee q)$-fuzzy (left, right) h-Γ-ideals $(k$-Γ-ideals$)$ of H.

Proof:

Straightforward.

Definition 9.

For any fuzzy subset μ of H we define $Q(\mu, t) = \{x \in H \ : \ x_t q \mu\}$ and

$$[\mu]_t = \{x \in H \ : \ x_t \in \vee q\mu\} \text{ for any } t \in (0,1].$$

Theorem 10.

A fuzzy subset μ of H is an $(\in, \in \vee q)$-fuzzy Γ-ideal of H if and only if

$$[\mu]_t = \{x \in H \ : \ x_t \in \vee q\mu\} \neq \varphi \text{ is a } \Gamma\text{-ideal of } H \text{ for all } 0 < t \leq 0.5.$$

Proof:

Let μ be an $(\in, \in \vee q)$-fuzzy Γ-ideal of H. Assume that $x, y \in [\mu]_t$ for some $t \in (0,1]$. Then,

$$\mu(x) \geq 1 \text{ or } \mu(x) + t > 1 \text{ and } \mu(y) \geq 1 \text{ or } \mu(y) + t > 1.$$

As μ of H is an $(\in, \in \vee q)$-fuzzy Γ-ideal, so we have

$$\mu(x+y) \geq \min\{\mu(x), \mu(y), 0.5\} \geq \min\{t, 0.5\} = t.$$

So that $(x+y) \in [\mu]_t$. Also

$$\mu(x\alpha y) \geq \min\{\mu(x), \mu(y), 0.5\} \geq \min\{t, 0.5\} = t,$$

for any $\alpha \in \Gamma$. Thus, $(x\alpha y) \in [\mu]_t$ and in the similar way $(y\alpha x) \in [\mu]_t$, for all $x, y \in H$ and $\alpha \in \Gamma$. Hence,

$$[\mu]_t = \{x \in H \ : \ x_t \in \vee q\mu\} \neq \varphi$$

is a Γ-ideal of H for all $0 < t \leq 0.5$.

Conversely, let

$$[\mu]_t = \{x \in H \ : \ x_t \in \vee q\mu\} \neq \varphi$$

be a Γ-ideal of H for all $0 < t \leq 0.5$. If there exist some $t \in (0,1]$ such that

$$\mu(x+y) < t < \min\{\mu(x), \mu(y), 0.5\}.$$

Then. $x, y \in [\mu]_t$, but $(x+y) \notin [\mu]_t$, which is a contradiction. Hence,

$$\mu(x+y) \geq \min\{\mu(x), \mu(y), 0.5\}$$

and in the similar way

$$\mu(x\alpha y) \geq \min\{\mu(x), \mu(y), 0.5\}, \ \ \mu(y\alpha x) \geq \min\{\mu(x), \mu(y), 0.5\} \text{ for all } x, y \in H$$

and $\alpha \in \Gamma$. Hence, μ is an $(\in, \in \vee q)$-fuzzy Γ-ideal of H.

Theorem 11.

A fuzzy subset μ of H is an $(\in, \in \vee q)$-fuzzy (left, right) h-Γ-ideal $(k\text{-}\Gamma\text{-ideal})$ of H if and only if

$$[\mu]_t \neq \varphi$$

is a (left, right) h-Γ-ideal $(K\text{-}\Gamma\text{-ideal})$ of H for all $0 < t \leq 0.5$.

Proof:

Let

$$\mu \text{ be an } (\in, \in \vee q)\text{-fuzzy } (\text{left, right}) \ h\text{-}\Gamma\text{-ideal } (k\text{-}\Gamma\text{-ideal}) \text{ of } H.$$

Then,

$$[\mu]_t = \{x \in H \ : \ x_t \in \vee q\mu\} \neq \varphi$$

is a Γ-ideal of H for all $0 < t \leq 0.5$. Assume that $x + a + y = b + y$ for some $x, y \in H$ and $a, b \in [\mu]_t$. Then, $\mu(a) \geq 1$ or $\mu(a) + t > 1$ and $\mu(b) \geq 1$ or $\mu(b) + t > 1$. Since μ is an

$$(\in, \in \vee q)\text{-fuzzy } (\text{left, right}) \ h\text{-}\Gamma\text{-ideal } (k\text{-}\Gamma\text{-ideal}) \text{ of } H,$$

we have

$$\mu(x) \geq \min\{\mu(x), \mu(y), 0.5\}.$$

This implies $\mu(x) \geq \min\{t, 0.5\} = t$. Indeed for $\mu(x) < t$, we obtain $a_t \overline{\in \vee q} \mu$ or $b_t \overline{\in \vee q} \mu$, which is a contradiction. So $\mu(x) \geq t$. Now if $t \leq 0.5$, then $\mu(x) \geq \{t, 0.5\} = t$, that is $x_t \in \mu$. Hence

$x \in [\mu]_t$. If $t > 0.5$, then $\mu(x) \geq \{t, 0.5\} = 0.5$. Thus if $\mu(x) + t > 0.5 + 0.5 = 1$ and $x_t q \mu$. Therefore, $x \in [\mu]_t$. This shows that

$$[\mu]_t = \{x \in H \ : \ x_t \in \vee q\mu\} \neq \varphi$$

is a (left, right) h-Γ-ideal $(k\text{-}\Gamma\text{-ideal})$ of H for all $0 < t \leq 0.5$.

Conversely, if

$$[\mu]_t = \{x \in H \ : \ x_t \in \vee q\mu\} \neq \varphi$$

is a (left, right) h-Γ-ideal $(k\text{-}\Gamma\text{-ideal})$ of H for all $0 < t \leq 0.5$. Then, it is an $(\in, \in \vee q)$-fuzzy Γ-ideal of H. Let $a, b, x, y \in H$ be such that $x + a + y = b + y$. There exist some t such that $\mu(x) < t < \min\{\mu(x), \mu(y), 0.5\}$. This implies $t < 0.5$, $\mu(x) + t > 1$, $\mu(a) > t$, $\mu(b) > t$. Thus, $x \notin Q(\mu, t)$, $x \notin U(\mu, t)$ and $a, b \in U(\mu, t) \subset [\mu]_t$, which contradicts to the given hypothesis. Hence, $\mu(x) \geq \min\{\mu(x), \mu(y), 0.5\}$, which shows that μ is an $(\in, \in \vee q)$-fuzzy (left, right) h-Γ-ideal $(k\text{-}\Gamma\text{-ideal})$ of H.

Lemma 6.

Let μ be an arbitrary set defined on H and $x \in H$. Then $\mu(x) = t$ if and only if $x \in U(\mu, t)$, $x \notin U(\mu, s)$ for all $s > t$.

Proof:

Straightforward.

Theorem 12.

Let $\{A_t\}_{t \in F}$, where $F \subseteq (0, 0.5)$ be the collection of left (right) h-Γ-ideals of H such that $H = \cup_{t \in F} A_t$ and for $s, t \in F$, $s < t$ if and only if $A_t \subset A_s$. Then, a fuzzy set μ defined by $\mu(x) = \sup\{t \in F \mid x \in A_t\}$ is an $(\in, \in \vee q)$-fuzzy (left, right) h-Γ-ideal of H.

Proof:

To prove the theorem it will be sufficient to show that each non-empty $U(\mu, t)$ is a (left, right) h-Γ-ideal of H. Let us consider the two cases:

$(i)\quad t=\sup\{s\in F \mid s<t\}$,

$(ii)\quad t\neq\sup\{s\in F \mid s\geq t\}$.

If we consider the first case

$$x\in U(\mu,t)\Leftrightarrow(x\in A_s,\ \text{for all } s<t)\Leftrightarrow x\in\cap_{s<t}A_s.$$

So,

$$U(\mu,t)=\cap_{s<t}A_s,$$

which is a (left, right) h-Γ-ideal of H. Now considering the second case we have to show that $U(\mu,t)=\cup_{s\geq t}A_s$. For it let $x\in\cap_{s\geq t}A_s\Rightarrow x\in A_s$, for some $s\geq t$, which implies that

$$\mu(x)\geq s\geq t\Rightarrow x\in\cup_{s\geq t}A_s\subset U(\mu,t).$$

Now for the inverse inclusion let us consider $x\notin\cup_{s\geq t}A_s\Rightarrow x\notin A_s$ for all $t\neq\sup\{s\in F \mid s\geq t\}$, there exist $\varepsilon>0$ such that $(t-\varepsilon,t)\cap F=\varphi$. Hence, $x\notin A_s$ $s>t-\varepsilon$, which shows that if $x\in A_s$, then $s\leq t-\varepsilon$. Thus $\mu(x)\leq t-\varepsilon<t$ and so $x\notin U(\mu,t)$. Hence, $U(\mu,t)=\cup_{s\geq t}A_s$. Also it is easy to show that $\cup_{s\geq t}A_s$ is a left (right) h-Γ-ideal of H. Hence, $U(\mu,t)$ is a left (right) h-Γ-ideal of H.

3. Prime (α,β)-fuzzy Γ-ideals

Definition 10.

An (α,β)-fuzzy Γ-ideal μ of H is called semiprime if for all $x\in H$,

$$\alpha\in\Gamma \text{ and } t\in(0,1],\ (x\alpha x)_t\,\alpha\mu\Rightarrow x_t\beta\mu.$$

An (α,β)-fuzzy Γ-ideal μ of H is called prime if for all $x,y\in H$,

$$\alpha\in\Gamma \text{ and } t\in(0,1],\ (x\alpha y)_t\,\alpha\mu\Rightarrow x_t\beta\mu \text{ or } y_t\beta\mu.$$

An (α,β)-fuzzy h-Γ-ideal $(k\text{-}\Gamma\text{-ideal})$ μ of H is called prime (semiprime) if it is prime (semiprime) as an (α,β)-fuzzy Γ-ideal.

Proposition 1.

An $(\in,\in\vee q)$-fuzzy Γ-ideal μ of H is prime if and only if

$$\max\{\mu(x),\mu(y)\}\geq\min\{\mu(x\alpha y),0.5\} \text{ for all } x,y\in H \text{ and } \alpha\in\Gamma.$$

Proof:

Let μ be an $(\in,\in\vee q)$-fuzzy prime Γ-ideal of H. Let for any $x,y\in H$ and there exist $t\in(0,1]$ such that

$$\max\{\mu(x),\mu(y)\}<t<\min\{\mu(x\alpha y),0.5\}.$$

This means that $(x\alpha y)_t\in H$ but $x_t\overline{\in\vee q}\mu$ and $y_t\overline{\in\vee q}\mu$, which is a contradiction to given hypothesis. Hence,

$$\max\{\mu(x),\mu(y)\}\geq\min\{\mu(x\alpha y),0.5\}, \text{ for all } x,y\in H \text{ and } \alpha\in\Gamma.$$

Conversely, assume that

$$\max\{\mu(x),\mu(y)\}\geq\min\{\mu(x\alpha y),0.5\}, \text{ for all } x,y\in H \text{ and } \alpha\in\Gamma.$$

Then,

$$(x\alpha y)_t\in\mu\Rightarrow\max\{\mu(x),\mu(y),0.5\}\geq\min\{t,0.5\}.$$

For $t\leq 0.5$ we have that either $\mu(x)\geq t$ or $\mu(y)\geq t$. It means that either $x_t\in\mu$ or $y_t\in\mu$. For $t>0.5$, we have $\max\{\mu(x),\mu(y)\}\geq 0.5$, i.e, either $\mu(x)+t>0.5+0.5=1$ or $\mu(y)+t>1$. Thus, either $x_t\in\vee q\mu$ or $y_t\in\vee q\mu$. Therefore, μ is a fuzzy prime.

Corollary 4.

An $(\in,\in\vee q)$-fuzzy h-Γ-ideal $(k$-Γ-ideal$)$ μ of H is prime if and only if

$$\max\{\mu(x),\mu(y)\}\geq\min\{\mu(x\alpha y),0.5\}, \text{for all } x,y\in H \text{ and } \alpha\in\Gamma.$$

Proof:

Straightforward.

Theorem 13.

An $(\in,\in\vee q)$-fuzzy Γ-ideal μ of H is prime if and only if for all $0<t\leq 0.5$ each non-empty $U(\mu,t)$ is a prime Γ-ideal of H.

Proof:

Let us suppose that μ is an $(\in,\in\vee q)$-fuzzy Γ-ideal of H and let $t\in(0,0.5]$. Then, each non-empty $U(\mu,t)$ is a Γ-ideal of H. By Proposition 1, for each $x\alpha y\in U(\mu,t)$ we have

$$\max\{\mu(x),\mu(y)\}\geq\min\{\mu(x\alpha y),0.5\}\geq\min\{t,0.5\}=t.$$

So, $\mu(x)\geq t$ or $\mu(y)\geq t$. Thus, $x\in U(\mu,t)$ or $y\in U(\mu,t)$. Hence, $U(\mu,t)$ is a prime Γ-ideal of H.

Conversely let us assume that $U(\mu,t)$ is a prime Γ-ideal of H for each $t\in(0,0.5]$, then it is obviously μ is an $(\in,\in\vee q)$-fuzzy Γ-ideal of H. Let $(x\alpha y)_t\in\mu$ and $t\leq 0.5$. Then, $(x\alpha y)\in U(\mu,t)$. So either $x\in U(\mu,t)$ or $y\in U(\mu,t)$. This implies that $x_t\in\mu$ or $y_t\in\mu$ or $t>0.5$ and $(x\alpha y)_t\in\mu$, we have $\mu(x\alpha y)\geq t>0.5$. Thus, $x\alpha y\in U(\mu,0.5)$, when $x\in U(\mu,0.5)$ or $y\in U(\mu,0.5)$. Therefore, $x_t q\mu$ or $y_t q\mu$. Hence, μ is an $(\in,\in\vee q)$-fuzzy Γ-ideal of H.

Corollary 5.

An $(\in,\in\vee q)$-fuzzy h-Γ-ideal $(k$-Γ-ideal$)$ μ of H is prime (resp. semiprime) if and only if for all $0<t\leq 0.5$ each non-empty $U(\mu,t)$ is a prime (resp. semiprime) h-Γ-ideal $(k$-Γ-ideal$)$ of H.

Proof:

Straightforward.

Theorem 14.

A non-empty subset I of H is a prime Γ-ideal if and only if a fuzzy subset μ of H such that

$$\mu(x)=\begin{cases}0, & \text{if } x\in H-I,\\ \geq 0.5, & \text{if } x\in I,\end{cases}$$

is an $(\in,\in\vee q)$ -fuzzy prime Γ-ideal of H.

Proof:

Let I of H is a prime Γ-ideal. Then, μ is an $(\in,\in\vee q)$-fuzzy Γ-ideal of H. If $(x\alpha y)_t \in \mu$, then $\mu(x\alpha y)=t\geq 0.5$, and consequently $x\alpha y\in I$. So $x\in I$ or $y\in I$. Thus, $\mu(x)=t$ or $\mu(y)=t$. Hence, $x_t\in\vee q\mu$ or $y_t\in\vee q\mu$. Therefore, μ is an $(\in,\in\vee q)$-fuzzy prime Γ-ideal of H.

Conversely, assume that μ is an $(\in,\in\vee q)$-fuzzy prime Γ-ideal of H and $x\alpha y\in I$. Then, $\mu(x\alpha y)=t\geq 0.5$. Thus, $(x\alpha y)_t\in\mu$, when $x_t\in\vee q\mu$ or $y_t\in\vee q\mu$. In the case $x_t\in\vee q\mu$ we get $\mu(x)\geq t$ or $\mu(x)+t>1$, and whence we get $\mu(x)=t\geq 0.5$. This means that $x\in I$. Similarly, from $y_t\in\vee q\mu$ we get $y\in I$. So $x\alpha y\in I\Rightarrow x\in I$ or $y\in I$.

Theorem 15.

The intersection of any family of $(\in,\in\vee q)$-fuzzy prime (resp. semiprime) Γ-ideals of H is an $(\in,\in\vee q)$-fuzzy prime (resp. semiprime) Γ-ideal of H.

Proof:

Straightforward.

Theorem 16.

A fuzzy subset μ of H is an $(\in,\in\vee q)$-fuzzy prime Γ-ideal of H if and only if each

$$[\mu]_t=\{x\in H\ :\ x_t\in\vee q\mu\}\neq\varphi$$

is a prime Γ-ideal of H for all $0<t\leq 0.5$.

Proof:

Let μ be an $(\in,\in\vee q)$-fuzzy prime Γ-ideal of H. Then, for $t\in(0,0.5]$ each non-empty $[\mu]_t$ is a Γ-ideal of H. Now let $(x\alpha y)\in[\mu]_t$. But $[\mu]_t=Q(\mu,t)\cup U(\mu,t)\Rightarrow(x\alpha y)\in Q(\mu,t)$ or $(x\alpha y)\in U(\mu,t)$. First consider the case when $(x\alpha y)\in Q(\mu,t)\setminus U(\mu,t)$. Then, $\mu(x\alpha y)+t>1$ and $\mu(x\alpha y)<t$. Hence, for $\mu(x\alpha y)\leq 0.5$, we get

$$\max\{\mu(x),\mu(y)\}+t \geq \min\{\mu(x\alpha y),0.5\}+t=\mu(x\alpha y)+t>1.$$

Hence,

$$x\in Q(\mu,t)\subseteq[\mu]_t \text{ or } y\in Q(\mu,t)\subseteq[\mu]_t.$$

Now if $\mu(x\alpha y)>0.5$ we have $0.5<\mu(x\alpha y)<t.$ Hence,

$$\max\{\mu(x),\mu(y)\}+t \geq \min\{\mu(x\alpha y),0.5\}+t>1.$$

Thus,

$$x\in Q(\mu,t)\subseteq[\mu]_t \text{ or } y\in Q(\mu,t)\subseteq[\mu]_t.$$

So

$$x\alpha y\in Q(\mu,t)\setminus U(\mu,t)\Rightarrow x\in[\mu]_t \text{ or } y\in[\mu]_t.$$

Now let $x\alpha y\in U(\mu,t).$ In this case $\mu(x\alpha y)\geq t.$ Hence, for $t\leq 0.5$ we have

$$\max\{\mu(x),\mu(y)\}\geq \min\{\mu(x\alpha y),0.5\}\geq 1.$$

Thus,

$$x\in U(\mu,t)\subseteq[\mu]_t \text{ or } y\in U(\mu,t)\subseteq[\mu]_t.$$

If $t>0.5,$ then

$$\max\{\mu(x),\mu(y)\}\geq t>0.5$$

and consequently

$$\max\{\mu(x),\mu(y)\}+t>1.$$

Therefore,

$$x\in Q(\mu,t)\subseteq[\mu]_t \text{ or } y\in Q(\mu,t)\subseteq[\mu]_t.$$

So in any case

$$x\alpha y\in[\mu]_t \Rightarrow \text{ either } x\in[\mu]_t \text{ or } y\in[\mu]_t.$$

Hence,

$$[\mu]_t = \{x \in H \ : \ x_t \in \vee q\mu\} \neq \varphi \text{ is a prime } \Gamma\text{-ideal of } H \text{ for all } 0 < t \leq 0.5.$$

Conversely, assume that

$$[\mu]_t = \{x \in H \ : \ x_t \in \vee q\mu\} \neq \varphi$$

is a prime Γ-ideal of H for all $0 < t \leq 0.5$. Then, it is obvious that μ is an $(\in, \in \vee q)$-fuzzy Γ-ideal of H. As

$$[\mu]_t = \{x \in H \ : \ x_t \in \vee q\mu\} \neq \varphi$$

is a prime Γ-ideal of H so from $x\alpha y \in U(\mu, t) \subseteq [\mu]_t$, we get either $x \in [\mu]_t$ or $y \in [\mu]_t$. This implies that $x_t \in \vee q\mu$ or $y \in \vee q\mu$. Hence, μ is an $(\in, \in \vee q)$-fuzzy prime Γ-ideal of H.

Corollary 6.

A fuzzy subset μ of H is an $(\in, \in \vee q)$-fuzzy prime h-Γ-ideal (resp. prime k-Γ-ideal) of H if and only if each $[\mu]_t = \{x \in H \ : \ x_t \in \vee q\mu\} \neq \varphi$ is a prime h-Γ-ideal (resp. prime k-Γ-ideal) of H for all $0 < t \leq 0.5$.

Proof:

Straightforward.

4. Conclusion

"Γ-hemirings" is ageneralization of the classical agebraic structure of hemirings. Our aim has been to extend this idea and, to introduce the concept of fuzzy Γ-ideals, (α, β)-fuzzy prime (semiprime) Γ-ideals, (α, β)-fuzzy h-Γ-ideals and (α, β)-fuzzy k-Γ-ideals of Γ-hemirings, and related properties have been investigated. In future we will focus on the following.

(i) A possible extension of this idea in order to introduce new types of Γ-ideals such as cubic Γ-ideals, Interval valued Γ-ideals and $(\in, \in \vee q)$-cubic-Γ-ideals.

(ii) Also to develop a new theory related to all types (Γ-cubic left ideals, Γ-cubic right ideals, Γ-cubic bi-ideals, Γ-cubic interior ideals) Γ-hemirings.

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