Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466
Vol. 9, Issue 1 (June 2014), pp. 435 – 448

Applications and Applied Mathematics:
An International Journal
(AAM)

Stability of an Inhomogeneous Damped Vibrating String

Siddhartha Misra and Ganesh C. Gorain

Chandernagore Sree Siksha Niketan(H.S),
Chandernagore, Hooghly, West Bengal, 712136, India
misra.siddhartha2@gmail.com
&
Department of Mathematics, J. K. College,
Purulia, West Bengal, 723101, India
goraing@gmail.com

Received: September 26, 2013; Accepted: January 9, 2014

Abstract

In this paper, we consider the vibrations of an inhomogeneous damped string under a distributed disturbing force which is clamped at both ends. The well-possedness of the system is studied. We prove that the amplitude of such vibrations is bounded under some restriction of the disturbing force. Finally, we establish the uniform exponential stabilization of the system when the disturbing force is insignificant. The results are established directly by means of an exponential energy decay estimate.

Keywords: Bounded-input bounded-output stability; C_0-semigroup; Distributed damping; Energy decay estimate; Exponential stability; Inhomogeneous string.

MSC 2010 No.: 35L05; 37C75; 93D05

1. Introduction

The mathematical theory of stabilization of a distributed parameter system is currently a topic of interest in application of vibrating control of various structures like strings, beams, plates, etc.

The study of the stabilization for these problems is significant in the sense that it is intended to suppress the vibrations to assure a good performance of the overall system. The vibrations of flexible structures are usually non-linear in practice. As the non-linear study of such structures is rather cumbersome for analytical treatment, linearized mathematical models are chosen for simplicity and concise result. The linearized vibrations of flexible structures are usually governed by a different type of partial differential equations. The question of energy decay estimates in the context of boundary stabilization of a wave equation has earlier been studied by several authors (see (Chen, 1979), (Gorain, 1997), (Lagnese, 1988), (Komornik and Zuazua, 1990) and the reference therein). There are several papers on the problem for the solution of wave equation in a bounded domain (see (Chen, 1979), (Chen, 1981), (Lagnese, 1988), (Lagnese, 1983), (Komornik and Zuazua, 1990) and the reference therein). (Chen, 1979) first established explicitly the exponential energy decay rate for the solution of wave equation by considering certain geometries of the domain. The theory of boundary stabilization of wave equation has been improved in (Lagnese, 1983), (Komornik, 1991), obtained faster energy decay rate for such problem by constructing a special type of feedback. There are different type of stability for the vibrations of flexible structures and the most important of all these is uniform stability. The question of uniform stabilization or point-wise stabilization of Euler-Bernoulli beams or serially connected beams has been studied by a number of authors (Lions, 1988). Recently, (Gorain, 2013) has established the uniform stabilization of longitudinal vibrations of inhomogeneous clamped beam.

2. Mathematical Formulation of the Problemn

In this paper, we consider a clamped inhomogeneous string of length L. The vibrations of the string can be described by the following partial differential equation

$$m(x)\frac{\partial^2 u}{\partial t^2} + 2\alpha(x)\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f \quad \text{on} \quad (0, L) \times \mathbb{R}^+, \tag{2.1}$$

where $\mathbb{R}^+ := (0, \infty)$. The variable parameters $m(x)$ and $\alpha(x)$ respectively denote mass per unit length and coefficient of damping at the point x which are assumed to be continuous up to second order partial derivatives over $[0, L]$. In fact, for a general inhomogeneous string they belong to $C^2[0, L]$. The distributed force $f : (0, L) \times \mathbb{R}^+ \to \mathbb{R}$ is the uncertain disturbance appearing in the model.

For a clamped string, the boundary conditions are

$$u(0, t) = 0, \quad u(L, t) = 0 \quad \text{on} \quad \mathbb{R}^+. \tag{2.2}$$

Let initially the string is set to vibrate with

$$u(x, 0) = u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = u_1(x) \quad \text{on} \quad (0, L). \tag{2.3}$$

The function $u_0(x)$ and $u_1(x)$ are assumed to be continuous up to second order partial derivatives over $[0, L]$ so that the solution is continuously differentiable on $(0, L) \times \mathbb{R}^+$.

Our aim in this work is to study the stability results of different types for the solutions of the mathematical problem (2.1) subject to the boundary and initial conditions (2.2) and (2.3). To achieve the results, we adopt a direct method by constructing a suitable Layapunov functional associated with the energy functional of the system.

3. Existence and Uniqueness of Solutions

In this section, we study the setting of the semigroup as in (Pazy, 1983) and we establish the well-possedness for the problem (2.1)-(2.3). We will use the following standard $L^2(0,L)$ space in which the scaler product and norm are denoted by

$$\Big\langle u, v\Big\rangle_{L^2(0,L)} = \int_0^L u\overline{v}dx, \qquad \| u \|^2_{L^2(0,L)} = \int_0^L | u |^2 dx.$$

We have the Poincaré inequality

$$\| u \|^2_{L^2(0,L)} \leq C_p \| u_x \|^2_{L^2(0,L)} \quad \text{for all } u \in H_0^1(0,L)$$

where C_p is the Poincaré constant.

Taking $u_t = v$, the initial boundary problem (2.1)-(2.3) can be reduced to the following abstract initial value problem

$$\frac{d}{dt}U(t) = \mathcal{A}U(t) + F(x,t), \qquad U(0) = U_0 \quad \text{for all} \quad t > 0, \tag{3.1}$$

with $U(t) = (u,v)^T$ and $U_0 = (u_0, v_0)^T$, where the linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is given by

$$\mathcal{A}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \frac{1}{m(x)}\Big(u_{xx} - 2\alpha(x)v\Big) \end{pmatrix}, F = \begin{pmatrix} 0 \\ f(x,t) \end{pmatrix} \tag{3.2}$$

Now, we introduce the energy space $\mathcal{H} = H_0^1(0,L) \times L^2(0,L)$ equipped with inner product given by

$$\Big\langle (u,v), (u',v')\Big\rangle_{\mathcal{H}} = \int_0^L m(x)v\overline{v'}dx + \int_0^L u_x\overline{u_x{'}}dx, \tag{3.3}$$

and norm by

$$||(u,v)||_{\mathcal{H}} = \int_0^L m(x)| v |^2 dx + \int_0^L | u_x |^2 dx \tag{3.4}$$

we can show that norm $||.||_{\mathcal{H}}$ is equivalent to the usual norm in $\mathcal{H}$. Instead of dealing with the system of equation (2.1)-(2.3) we conceder (3.1) in the Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(\mathcal{A})$ of the operator $\mathcal{A}$ given by

$$\mathcal{D}(\mathcal{A}) = \Big[(u,v) \in \mathcal{H} : \quad u \in H_0^1(0,L) \cap H^2(0,L)\Big].$$

We now show that the operator space $\mathcal{A}$ generates a C_0-semigroup of contractions on the space $\mathcal{H}$.

Theorem 1. *The operator $\mathcal{A}$ generates a C_0-semigroup $\mathcal{S}_{\mathcal{A}}(t)$ of contractions on the space $\mathcal{H}$.*

Proof: We will show that $\mathcal{A}$ is a dissipative operator and 0 belong to resolvent set of $\mathcal{A}$, denoted by $\rho(\mathcal{A})$. Then our conclusion will follow using the well known Lumer-Phillips theorem (see (Pazy, 1983)). We observe that if $U = (u, v) \in \mathcal{D}(\mathcal{A})$, then

$$\begin{aligned} \Big\langle \mathcal{A}U, U \Big\rangle_{\mathcal{H}} &= \int_0^L m(x)\frac{1}{m(x)}(u_{xx} - 2\alpha(x)v)\overline{v}dx + \int_0^L v_x\overline{u_x}dx \\ &= \int_0^L u_{xx}\overline{v}dx - 2\int_0^L \alpha(x)|\, v\,|^2 dx + \int_0^L v_x\overline{u_x}dx \\ &= 2iIm\int_0^L v_x\overline{u_x}dx - 2\int_0^L \alpha(x)|\, v\,|^2 dx. \end{aligned} \tag{3.5}$$

Hence,

$$Re\Big\langle \mathcal{A}U, U \Big\rangle_{\mathcal{H}} = -2\int_0^L \alpha(x)|\, v\,|^2 dx \leq 0. \tag{3.6}$$

Thus $\mathcal{A}$ is a dissipative operator. Now, we show that $0 \in \rho(\mathcal{A})$. In fact, given $\mathcal{F} = (f_1, g_1) \in \mathcal{H}$, we must show that there exits a unique $U = (u, v) \in \mathcal{D}(\mathcal{A})$ such that $\mathcal{A}U = \mathcal{F}$, that is,

$$v = f_1, \tag{3.7}$$

$$u_{xx} - 2\alpha(x)v = m(x)g_1. \tag{3.8}$$

Replacing (3.7) in (3.8), we have

$$u_{xx} = m(x)g_1 + 2\alpha(x)f_1. \tag{3.9}$$

It is known that there is a unique $u \in H^2(0, L)$ satisfying (3.9). It is easy to show that $||U||_{\mathcal{H}} \leq C||\mathcal{F}||_{\mathcal{H}}$ for a positive constant C. Therefore, we conclude that $0 \in \rho(\mathcal{A})$.

Now from theorem 1 and theorem 2.4 in (Pazy, 1983), we can state the following result.

Theorem 2. *For any $U_0 \in \mathcal{H}$, there exists a unique solution $U(t) = (u, u_t)$ of the system* (2.1)-(2.3) *satisfying*

$$u \in C([0, \infty[: H_0^1(0, L)) \cap C^1([0, \infty[: L^2(0, L)).$$

However, if $U_0 \in \mathcal{D}(\mathcal{A})$ then

$$u \in C([0, L[: H_0^1(0, L)) \cap H^2(0, L)) \cap C^1([0, \infty[: L^2(0, L)).$$

4. Energy of the System

Now we proceed as in (Gorain, 2007), (Komornik, 1991), (Shahruz, 1996) by defining energy functional for every solution $u(x, t)$ of the system (2.1)-(2.3). We define energy of u at any instant t by the functional

$$E(u(t)) := \frac{1}{2}\int_0^L \Big[m(x)\Big(\frac{\partial u}{\partial t}\Big)^2 + \Big(\frac{\partial u}{\partial x}\Big)^2\Big]dx \quad \text{for all} \quad t \geq 0. \tag{4.1}$$

Differentiate (4.1) with respect to t to obtain

$$\frac{dE}{dt} = \int_0^L \Big[m(x)\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}\Big]dx. \tag{4.2}$$

Using (2.1) in (4.2) and applying the conditions in (2.2), we get

$$\frac{dE}{dt} = -2\int_0^L \alpha(x)\Big(\frac{\partial u}{\partial t}\Big)^2 dx + \int_0^L \frac{\partial u}{\partial t} f\,dx. \tag{4.3}$$

Since $\dfrac{dE}{dt} \neq 0$, it follows from (4.3) that the system is not energy conserving. Now, when the uncertain disturbing force is not present that is $f(x,t) \equiv 0$ for all $(x,t) \in (0,L)\times(0,\infty)$, the system (2.1)-(2.3) is energy dissipating and hence on integration (4.3) with respect to t over $[0,t]$, the solution u satisfy energy estimate

$$E(u(t)) - E(u(0)) = -2\int_0^t \int_0^L \Big[\alpha(x)\Big(\frac{\partial u}{\partial \tau}(x,\tau)\Big)^2\Big]dx d\tau \quad \text{for} \quad t \geq 0, \tag{4.4}$$

where

$$E(u(0)) = \frac{1}{2}\int_0^L \Big[m(x)(u_1)^2 + ({u_0}')^2\Big]dx. \tag{4.5}$$

In view of (4.4) and (4.5), we may conclude that $u_0 \in H_0^1(0,L)$ and $u_1 \in L^2(0,L)$, where

$$H_0^1(0,L) = \Big[\varphi : \varphi \in H^1(0,L) \quad \text{and} \quad \varphi(0) = \varphi(L) = 0\Big]$$

is the subspace of the classical Sobolev space

$$H^1(0,L) = \Big[\varphi : \varphi \in L^2(0,L), \quad \frac{d\varphi}{dx} \in L^2(0,L)\Big]$$

of real valued function of order one. Then obviously

$$E(u(t)) \leq E(u(0)) < \infty \quad \text{for} \quad t \geq 0. \tag{4.6}$$

Now, we have to study bounded-input and bounded-output stability of the system in presence of uncertain input disturbance $f(x,t)$. We introduce two function spaces as specified in (Gorain, 2007)

$$X := \Big[\varphi(x,t) : (0,L)\times\mathbb{R}^+ \to \mathbb{R} : \sup_{t\in\mathbb{R}^+}\Big[\int_0^L \varphi^2 dx\Big]^{\frac{1}{2}} < \infty\Big], \tag{4.7}$$

$$Y := \Big[\varphi(x,t) : (0,L)\times\mathbb{R}^+ \to \mathbb{R}^+ : \sup_{t\in\mathbb{R}^+}\sup_{x\in(0,L)} |\varphi| < \infty\Big], \tag{4.8}$$

with $\|\varphi\|_X = \sup\limits_{t\in\mathbb{R}^+}\Big[\int_0^L \varphi^2 dx\Big]^{\frac{1}{2}} < \infty$ and $\|\varphi\|_Y^2 = \sup\limits_{t\in\mathbb{R}^+}\sup\limits_{x\in(0,L)} |\varphi| < \infty$. From (4.7) and (4.8), it follows that $Y \subset X$ as $L^\infty(0,L) \subset L^2(0,L)$.

5. Stability Results

On account of uncertain disturbance force $f(x,t)$ as an input disturbance, the system evolves from its initial state (u_0, u_1) to $\Big(u, \dfrac{\partial u}{\partial t}\Big)$ at any instant t. The result of the bounded-output solution for the restriction of $f(x,t)$ is in the following theorems.

Theorem 3. *If $u(x,t)$ be the solution of the system (2.1)-(2.3) with $f \in X$ then $u \in Y$ for every set of initial values $(u_0, u_1) \in H_0^1(0,L) \times L^2(0,L)$.*

Theorem 4. *Let $u(x,t)$ be the solution of the system (2.1)-(2.3) corresponding to the initial value $(u_0, u_1) \in H_0^1(0,L) \times L^2(0,L)$ then for every $T > 0$*

$$\int_0^T E(u(t))dt \leq \kappa E(u(0)) + \sigma \int_0^T \|f\|^2_{L^2(0,L)} dt \tag{5.1}$$

where κ and σ are positive constant given by (5.38).

In an ideal case, when the uncertain disturbance force is not present in the system (2.1)-(2.3), then the energy function given by (4.1) is a dissipative function of time. So naturally a question arises as to whether this decay is exponentially or not, and the affirmative answer of this question is found in the following theorem.

Theorem 5. *If $u(x,t)$ be the solution of the system (2.1)-(2.3) with $f(x,t) \equiv 0$ and $(u_0, u_1) \in H_0^1(0,L) \times L^2(0,L)$ then the solution $\to 0$ exponentially as time $t \to +\infty$, that is, the energy function given by (4.1) satisfy*

$$E(u(t)) \leq Ae^{-\nu t} E(u(0)) \quad \text{for all} \quad t \in \mathbb{R}^+ \tag{5.2}$$

for some reals $\nu > 0$ and $A > 1$.

To prove the above theorems, we need the following inequalities and few lemmas.

I. For any real number $\alpha > 0$, we have Young's Inequality (see (Mitrinović et al., 1991))

$$\Big|f.g\Big| \leq \frac{1}{2}\Big(\alpha |f|^2 + \frac{|g|^2}{\alpha}\Big). \tag{5.3}$$

II. Poincaré type Scheeffer's inequality (see (Mitrinović et al., 1991))

$$\int_0^L u^2 dx \leq \frac{L^2}{\pi^2} \int_0^L \Big(\frac{\partial u}{\partial x}\Big)^2 dx, \tag{5.4}$$

as $u(x,t)$ satisfy boundary condition (2.2).

III. CauchySchwartz inequality for integral calculus (see (Mitrinović et al., 1991))

$$\Big|\int_0^L f.g dx\Big| \leq \Big[\int_0^L f^2 dx \int_0^L g^2 dx\Big]^{\frac{1}{2}}. \tag{5.5}$$

Thus by Mean value theorem of integral calculus, there are reals $\xi_1, \xi_2, \eta_1, \eta_2, \zeta \in [0, L]$ satisfying

$$\int_0^L m(x)u^2\,dx = m(\xi_1)\int_0^L u^2\,dx, \tag{5.6}$$

$$\int_0^L m(x)\Big(\frac{\partial u}{\partial t}\Big)^2 dx = m(\xi_2)\int_0^L \Big(\frac{\partial u}{\partial t}\Big)^2 dx, \tag{5.7}$$

$$\int_0^L \alpha(x)u^2\,dx = \alpha(\eta_1)\int_0^L u^2\,dx, \tag{5.8}$$

$$\int_0^L \alpha(x)\Big(\frac{\partial u}{\partial t}\Big)^2 dx = \alpha(\eta_2)\int_0^L \Big(\frac{\partial u}{\partial t}\Big)^2 dx. \tag{5.9}$$

Now we define

$$\mu_0 = \frac{L}{\pi}\sqrt{m(\xi_1)}, \quad \mu_1 = 2\alpha(\eta_1)\frac{L^2}{\pi^2}, \quad \mu_2 = \frac{\alpha(\eta_2)}{m(\xi_2)}. \tag{5.10}$$

It is obvious that $m(\xi_1), m(\xi_2), \alpha(\eta_1), \alpha(\eta_2)$ are all positive and they are bounded above by their corresponding upper bound over $[0, L]$.

Now, we need the following lemmas.

Lemma 1. *For every solution* $u = u(x,t)$ *of the system* (2.1)-(2.3), *the time derivative of the functional* G *defined by*

$$G(u(t)) := \int_0^L \Big[m(x)u\frac{\partial u}{\partial t} + \alpha(x)u^2\Big]dx, \tag{5.11}$$

satisfies

$$\frac{dG}{dt} = 2\int_0^L m(x)\Big(\frac{\partial u}{\partial t}\Big)^2 dx + \int_0^L u\,f\,dx - 2E(u(t)). \tag{5.12}$$

Proof: Differentiate (5.11) with respect to t, we get

$$\frac{dG}{dt} = \int_0^L \Big[m(x)u\frac{\partial^2 u}{\partial t^2} + m(x)\Big(\frac{\partial u}{\partial t}\Big)^2 + 2\alpha(x)u\frac{\partial u}{\partial t}\Big]dx. \tag{5.13}$$

Using (2.1) in (5.13) and applying condition (2.2), we get

$$\frac{dG}{dt} = \int_0^L uf dx + \int_0^L m(x)\Big(\frac{\partial u}{\partial t}\Big)^2 dx + \int_0^L u\frac{\partial^2 u}{\partial x^2}dx. \tag{5.14}$$

Using energy equation (4.1) we get from (5.14)

$$\frac{dG}{dt} = 2\int_0^L m(x)\Big(\frac{\partial u}{\partial t}\Big)^2 dx + \int_0^L uf dx - 2E((u(t)). \tag{5.15}$$

Hence the lemma is proved.

Lemma 2. *The functional* $G(u(t))$ *given by* (5.11) *satisfies the inequality*

$$-\mu_0 E(u(t)) \le G(u(t)) \le (\mu_0 + \mu_1)E(u(t)) \quad \text{for} \quad t \ge 0. \tag{5.16}$$

Proof: By using (5.4), (5.8) and (5.9), we get

$$\begin{aligned}\int_0^L \alpha(x)u^2dx &= \alpha(\eta_1)\int_0^L u^2dx \\ &\leq \alpha(\eta_1)\frac{L^2}{\pi^2}\int_0^L \Big(\frac{\partial u}{\partial x}\Big)^2dx \\ &\leq 2\alpha(\eta_1)\frac{L^2}{\pi^2}E(u(t)) \\ &= \mu_1 E(u(t)) \quad for \quad t\geq 0. \end{aligned} \tag{5.17}$$

By using (5.3), (5.4) and (5.6), we get

$$\begin{aligned}\Big|\int_0^L m(x)u\frac{\partial u}{\partial t}dx\Big| &= \Big|\int_0^L \sqrt{m(x)}u \quad \sqrt{m(x)}\frac{\partial u}{\partial t}dx\Big| \\ &\leq \frac{1}{2}\Big[\frac{\pi}{L\sqrt{m(\eta_1)}}\int_0^L m(x)u^2(x)dx \\ &\quad +\frac{L\sqrt{m(\eta_1)}}{\pi}\int_0^L m(x)\Big(\frac{\partial u}{\partial t}\Big)^2dx\Big] \\ &\leq \frac{1}{2}\Big[\frac{L\sqrt{m(\eta_1)}}{\pi}\int_0^L \Big(\frac{\partial u}{\partial x}\Big)^2+\frac{L\sqrt{m(\eta_1)}}{\pi}\int_0^L m(x)\Big(\frac{\partial u}{\partial t}\Big)^2dx\Big] \\ &= \frac{1}{2}\frac{L\sqrt{m(\eta_1)}}{\pi}\int_0^L\Big[m(x)\Big(\frac{\partial u}{\partial t}\Big)^2+\Big(\frac{\partial u}{\partial x}\Big)^2\Big]dx \\ &= \frac{L\sqrt{m(\eta_1)}}{\pi}E(u(t)) \\ &= \mu_0 E(u(t)). \end{aligned} \tag{5.18}$$

Now by (5.11), (5.17) and (5.18), we get

$$-\mu_0E(u(t)) \leq G(u(t)) \leq (\mu_0+\mu_1)E(u(t)) \quad \text{for} \quad t\geq 0.$$

Hence the lemma is proved.

To prove the above theorems, we proceed like (Komornik, 1994), (Gorain, 2007), (Gorain, 1997). Let us introduce an energy like Layapunov functional V defined by

$$V(u(t)) := E(u(t)) + \varepsilon G(u(t)) \quad \text{for} \quad t\geq 0, \tag{5.19}$$

where ε is a small positive real number given by (5.25). The lemma 2 yields for $V(u(t))$ that estimates

$$(1-\mu_0\varepsilon)E(u(t)) \leq V(u(t)) \leq [1+(\mu_0+\mu_1)\varepsilon]E(u(t)) \quad \text{for} \quad t\geq 0, \tag{5.20}$$

where we choose $\quad \varepsilon < \dfrac{1}{\mu_0}, \quad$ so that $V(u(t))\geq 0 \quad$ for $\quad t\geq 0.$

Also by (5.3) and (5.4), we can estimate

$$\begin{aligned}\int_0^L uf dx &\leq \frac{1}{2}\Big[\int_0^L \frac{2p\pi^2}{L^2}\int_0^L u^2 dx + \frac{L^2}{2p\pi^2}\int_0^L f^2 dx\Big] \\ &\leq \frac{1}{2}\Big[2p\int_0^L \Big(\frac{\partial u}{\partial x}\Big)^2 dx + \frac{L^2}{2p\pi^2}\int_0^L f^2 dx\Big] \\ &= p\int_0^L \Big(\frac{\partial u}{\partial x}\Big)^2 dx + \frac{L^2}{4p\pi^2}\int_0^L f^2 dx \end{aligned} \quad (5.21)$$

and also by (5.3) and (5.4), we can estimate

$$\begin{aligned}\int_0^L \frac{\partial u}{\partial t} f dx &\leq \frac{1}{2}\Big[2p\varepsilon m(\eta_2)\int_0^L \Big(\frac{\partial u}{\partial t}\Big)^2 dx + \frac{1}{2p\varepsilon m(\eta_2)}\int_0^L f^2 dx\Big] \\ &= p\varepsilon\int_0^L m(x)\Big(\frac{\partial u}{\partial t}\Big)^2 dx + \frac{1}{4p\varepsilon m(\eta_2)}\int_0^L f^2 dx, \end{aligned} \quad (5.22)$$

where p is a real number satisfying $0 < p < 1$.

Now taking time derivative of (5.19) and applying the results (4.3), (5.15), (5.21) and (5.22), we get

$$\begin{aligned}\frac{dV}{dt} &= \frac{dE}{dt} + \varepsilon\frac{dG}{dt} \\ &= -2\int_0^L \alpha(x)\Big(\frac{\partial u}{\partial t}\Big)^2 dx + \int_0^L \frac{\partial u}{\partial t} f dx \\ &\quad +\varepsilon\int_0^L uf dx + 2\varepsilon\int_0^L m(x)\Big(\frac{\partial u}{\partial t}\Big)^2 dx - 2\varepsilon E(u(t)) \\ &\leq -2\int_0^L \alpha(x)\Big(\frac{\partial u}{\partial t}\Big)^2 dx - 2\varepsilon E(u(t)) \\ &\quad +2\varepsilon\int_0^L m(x)\Big(\frac{\partial u}{\partial t}\Big)^2 dx + p\varepsilon\int_0^L m(x)\Big(\frac{\partial u}{\partial t}\Big)^2 dx \\ &\quad +p\varepsilon\int_0^L \Big(\frac{\partial u}{\partial x}\Big)^2 dx + \frac{1}{4p\varepsilon m(\eta_2)}\int_0^L f^2 dx + \frac{L^2\varepsilon}{4p\pi^2}\int_0^L f^2 dx \\ &= -2\varepsilon E(u(t)) + p\varepsilon\int_0^L \Big[m(x)\Big(\frac{\partial u}{\partial t}\Big)^2 + \Big(\frac{\partial u}{\partial x}\Big)^2\Big] dx \\ &\quad -2(\mu_2 - \varepsilon)\int_0^L m(x)\Big(\frac{\partial u}{\partial t}\Big)^2 dx + \Big(\frac{1}{4p\varepsilon m(\eta_2)} + \frac{L^2\varepsilon}{4p\pi^2}\Big)\int_0^L f^2 dx \\ &\leq -2\varepsilon(1-p)E(u(t)) - 2(\mu_2 - \varepsilon)\int_0^L m(x)\Big(\frac{\partial u}{\partial t}\Big)^2 dx \\ &\quad +C\int_0^L f^2 dx, \end{aligned} \quad (5.23)$$

where

$$C = \frac{1}{4p}\Big(\frac{1}{\varepsilon m(\xi_2)} + \frac{L^2\varepsilon}{\pi^2}\Big). \quad (5.24)$$

Since ε is small, we assume that

$$\varepsilon < \varepsilon_0 = \min\Big\{\frac{1}{\mu_0}, \mu_2\Big\}. \tag{5.25}$$

Hence from (5.23), we get the differential inequality

$$\begin{aligned}\frac{dV}{dt} &\leq -2(1-p)\varepsilon E(u(t)) + C \parallel f \parallel^2_{L^2(0,L)} \\ &\leq \frac{-2(1-p)\varepsilon}{[1+(\mu_0+\mu_1)\varepsilon]} V(u(t)) + C \parallel f \parallel^2_{L^2(0,L)}\end{aligned}$$

in view of (5.20). Thus we have

$$\frac{dV}{dt} + \lambda V \leq C \parallel f \parallel^2_{L^2(0,L)}, \tag{5.26}$$

where

$$\lambda = \frac{2(1-p)\varepsilon}{[1+(\mu_0+\mu_1)\varepsilon]} > 0. \tag{5.27}$$

Multiplying (5.26) by $e^{\lambda t}$ and integrating over $[0,t]$ for any $t \geq 0$ we get

$$e^{\lambda t} V(u(t)) - V(u(0)) \leq \int_0^t C \parallel f \parallel^2_{L^2(0,L)} e^{\lambda\tau} d\tau.$$

Thus we have

$$V(u(t)) \leq e^{-\lambda t}\left[V(u(0)) + \int_0^t C \parallel f \parallel^2_{L^2(0,L)} e^{\lambda\tau} d\tau\right]. \tag{5.28}$$

By using (5.20) in (5.28), we get

$$E(u(t)) \leq \frac{1}{(1-\mu_0\varepsilon)}\left[(1+(\mu_0+\mu_1)\varepsilon)E(u(0))e^{-\lambda t} + C\int_0^t \parallel f \parallel^2_{L^2(0,L)} e^{-(t-\tau)\lambda} d\tau\right], \tag{5.29}$$

where $E(u(0))$ is given by (4.5).

Proof of theorem 3: Let $f \in X$ such that $||f||_X = \sup_{t\in\mathbb{R}^+} ||f||_{L^2(0,L)} < \infty$. Putting $t - \tau = \theta$ in (5.29), we get

$$\begin{aligned}E(u(t)) &\leq \frac{1}{(1-\mu_0\varepsilon)}\Big[(1+(\mu_0+\mu_1)\varepsilon)E(u(0))e^{-\lambda t} + C\int_0^t \parallel f \parallel^2_X e^{-\lambda\theta} d\theta\Big] \\ &\leq \frac{1}{(1-\mu_0\varepsilon)}\Big[(1+(\mu_0+\mu_1)\varepsilon)E(u(0))e^{-\lambda t} + C \parallel f \parallel^2_X \int_0^\infty e^{-\lambda\theta} d\theta\Big] \\ &\leq \frac{1}{\lambda(1-\mu_0\varepsilon)}\Big[2(1-p)\varepsilon E(u(0)) + C \parallel f \parallel^2_X\Big] \quad \text{for} \quad t \in \mathbb{R}^+. \end{aligned} \tag{5.30}$$

Hence,

$$\sup_{t\in\mathbb{R}^+} E(u(t)) < \infty \tag{5.31}$$

for every set of initial value $(u_0, u_1) \in H^1_0(0,L) \times L^2(0,L)$ and for every $f \in X$. Thus the energy of the system (2.1)-(2.3) is uniformly bounded function of time.

Again from (2.2), we have $u(0,t)=0$ so we can write

$$\begin{aligned} \Big|u(x,t)\Big| = \Big|\int_0^x \frac{\partial u}{\partial x}dx\Big| &\leq \Big(\int_0^L 1^2 dx\Big)^{\frac{1}{2}}\Big(\int_0^L \Big|\frac{\partial u}{\partial x}\Big|^2 dx\Big)^{\frac{1}{2}} \\ &\leq L^{\frac{1}{2}}\Big(\int_0^L \Big(\frac{\partial u}{\partial x}\Big)^2 dx\Big)^{\frac{1}{2}} \end{aligned} \tag{5.32}$$

using the inequality (5.5), for all $(x,t)\in(0,L)\times\mathbb{R}^+$.

Thus, in view of (4.1),

$$\Big|u(x,t)\Big|^2 \leq L\int_0^L \Big(\frac{\partial u}{\partial x}\Big)^2 dx \leq 2LE(u(t)) < \infty \tag{5.33}$$

for every $(x,t)\in(0,L)\times\mathbb{R}^+$.

Hence,

$$u \in Y \tag{5.34}$$

for every set of initial values $(u_0,u_1)\in H_0^1(0,L)\times L^2(0,L)$ and for every $f\in X$.

Hence the theorem is proved.

Remark 1. *This result shows that the output solution u is Y-bounded for every X-bounded input disturbance f. Thus the system is bounded-input bounded-output stable.*

Proof of Theorem 4: Integrating (5.29) over $[0,T]$ for $T>0$, we get

$$\begin{aligned} \int_0^T E(u(t))dt &\leq \frac{1}{(1-\mu_0\varepsilon)}\Big[\Big(1+(\mu_0+\mu_1)\varepsilon\Big)E(u(0))\int_0^T e^{-\lambda t}dt \\ &\quad +C\int_0^T e^{-\lambda t}F(t)dt\Big], \end{aligned} \tag{5.35}$$

where

$$F(t)=\int_0^t \| f \|^2_{L^2(0,L)} e^{\lambda\tau}d\tau. \tag{5.36}$$

Integrating (5.35) by parts, we have

$$\begin{aligned} \int_0^T E(u(t))dt &\leq \frac{1}{(1-\mu_0\varepsilon)}\Big[\frac{2(1-p)\varepsilon}{\lambda^2}E(u(0))(1-e^{-\lambda T}) \\ &\quad +\frac{C}{\lambda}\Big[F(0)-e^{-\lambda T}F(t)+\int_0^T e^{-\lambda t}\frac{dF}{dt}dt\Big]\Big] \\ &= \kappa E(u(0))(1-e^{-\lambda T}) \\ &\quad +\sigma\Big[F(0)-e^{-\lambda T}F(t)+\int_0^T e^{-\lambda t}\frac{dF}{dt}dt\Big] \\ &\leq \kappa E(u(0))+\sigma\int_0^T \|f\|^2_{L^2(0,L)}dt, \end{aligned} \tag{5.37}$$

where

$$\kappa = \frac{2(1-p)\varepsilon}{\lambda^2(1-\mu_0\varepsilon)} \quad \text{and} \quad \sigma = \frac{C}{\lambda(1-\mu_0\varepsilon)}, \tag{5.38}$$

since $F(0) = 0$ and $\dfrac{dF}{dt} = e^{\lambda t}||f||^2_{L^2(0,L)}$.

Hence, the theorem is proved.

Remark 2. *Thus the above result shows that if* $u(x,t)$ *is a solution of the system* (2.1)-(2.3) *with* $f \in L^2(0,T,H^1(0,L))$ *then the solution* $u \in L^2(0,T,\ H^1_0(0,L))$. *The factor* σ *in* (5.37) *may be defined as the tolerance factor of this disturbing force* f *on the total energy over* $[0,T]$.

Proof of Theorem 5: When the disturbing force $f(x,t)$ is not taking into our account in the equation (2.1), the result (4.3) shows that the energy $E(u(t))$ of the system (2.1)-(2.3) is a non-increasing function of time. Consequently, the terms in (5.21) and (5.22) are insignificant following (4.3) and (5.15). Thus we can get rid off the terms involving p in (5.23) and hence differential inequality (5.26) becomes

$$\frac{dV}{dt} + \nu V \leq 0 \quad \text{for} \quad t \geq 0, \tag{5.39}$$

where

$$\nu = \frac{2\varepsilon}{1+(\mu_0+\mu_1)\varepsilon}. \tag{5.40}$$

Multiplying (5.39) by $e^{\nu t}$ and integrating over $[0,t]$ for any $t \in \mathbb{R}^+$, we get

$$V(u(t)) \leq e^{-\nu t}V(u(0)). \tag{5.41}$$

Applying (5.20) in (5.41), we get

$$E(u(t)) \leq \frac{1+(\mu_0+\mu_1)\varepsilon}{1-\mu_0\varepsilon}e^{-\nu t}E(u(0)).$$

Thus

$$E(u(t)) \leq Ae^{-\nu t}E(u(0)),$$

where

$$A = \frac{1+(\mu_0+\mu_1)\varepsilon}{1-\mu_0\varepsilon} > 1. \tag{5.42}$$

Hence the theorem is proved.

Remark 3. *Thus the above result shows that the solution of the system* (2.1)-(2.3) *decay exponentially with time and* $u(x,t) \to 0$ *as* $t \to +\infty$ *for every* $(u_0,u_1) \in H^1_0(0,L) \times L^2(0,L)$.

Remark 4. *The exponential stability result* (5.2) *can be obtain directly by setting* $f \equiv 0$ *in* (5.29). *In that case, the exponential decay rate of energy would be* λ *which is less than* ν. *Thus the exponential energy decay rate* ν *given by* (5.40) *is a stronger one. Since*

$$\nu = \frac{2\varepsilon}{1+(\mu_0+\mu_1)\varepsilon},$$

we have

$$\frac{d\nu}{d\varepsilon} = \frac{2}{1 + (\mu_0 + \mu_1)\varepsilon} > 0.$$

The exponential decay rate ν *of energy as a function of* ε *will be maximum for the largest value* ε_0 *of* ε *given by* (5.25)*. In view of* (5.25)*, an upper bound of which is given by* ε_0 *that depends explicitly on* μ_0 *and* μ_2*, as defined by* (5.10)*. It signifies that the decay of energy will be slower for a longer string.*

6. Conclusion

In this study, we deal with the mathematical stability results of a vibrating clamped string modeled by linear differential equation (2.1) and well-possedness of the system. We have established the boundedness of output solution under boundedness of input disturbances. We also estimate the total energy of the system over any time interval with a tolerance level of the input disturbance. Finally, we prove that the energy of the system decay exponentially with time whenever the input disturbance is not so significant or important.

References

Chen, G. (1979). Energy decay estimate and exact boundary-value controllability for the wave equation in a bounded domain. *Journal de Mathematiques Pures et Appliques*, 58:249–273.

Chen, G. (1981). A note on the boundary stabilization of the wave equation. *SIAM Journal of control and optimization*, 19:106–113.

Gorain, G. (1997). Exponential energy decay estimate for the solution of internally damped wave equation in a bounded domain. *J. Math. Anal. Appl.*, 216:510–520.

Gorain, G. (2007). Stabilization of a quasi-linear vibrations of an inhomogeneous beam. *IEEE Trans. Automat. Contr.*, 52:1690–1695.

Gorain, G. (2013). Exponential stabilization of longitudinal vibrations of an inhomogeneous beam. *Non-linear Oscillation*, 16:157–164.

Komornik, V. (1991). Rapid boundary stabilization of wave equations. *SIAM Journal of control and optimization*, 29:197–208.

Komornik, V. (1994). *Exact Controllability and Stabilization, The Multiplier Method*. John Wiley-Masson, Paris.

Komornik, V. and Zuazua, E. (1990). A direct method for the boundary stabilization of the wave equation. *Journal de Mathematiques Pures etAppliquees*, 69:33–54.

Lagnese, J. (1983). Decay of solution of wave equations in a bounded region with boundary dissipation. *Journal of Differential Equations*, 50:163–182.

Lagnese, J. (1988). Note on boundary stabilization of wave equations. *SIMA Journal of control and optimization*, 26:1250–1256.

Lions, J. (1988). Exact controllability, stabilization and perturbations for distributed systems. *SIAM Review*, 30:1–68.

Mitrinović, D., Pečarić, J., and Fink, A. (1991). *Inequalities Involving Functions and Their Integrals and Derivatives*. Dordrecht, The Netherlands, Kluwer.

Pazy, A. (1983). *Semigroup of Linear Operators and Applications to Partial Differential Equations*. Springer-Verglas, New York.

Shahruz, S. (1996). Bounded-input bounded-output stability of a damped non-linear string. *IEEE. Trans. Automat. Contr.*, 41:1179–1182.