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A Note on the Qualitative Behavior of Some Second Order Nonlinear Equation

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Abstract

In this paper, we present two qualitative results concerning the solutions of some second order nonlinear equations, under suitable assumptions. The first result centers on the boundedness of the solutions while the second discusses the square integrability of the solutions. These results are obtained by extending and improving the current literature through sound mathematical analysis.

Keywords: Bounded, L^2-solutions, square-integrable, asymptotic behavior

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1. Introduction

We consider here the equation

$$(p(t)x')'+f(t,x,x')x'+a(t)g(x)=q(t,x,x'), \tag{1}$$

under the following conditions:

i) p and q are continuous functions on $I:=[0,+\infty)$ such that $0 < p_0 \leq p(t) < +\infty$ and $0 < a_0 \leq a(t) \leq a_1 < +\infty$.

ii) f is a continuous function on $\boldsymbol{IxR^2}$ satisfying $0 < f_0 \leq f(t,x,x')$.

iii) g is also a continuous function for all x such that $xg(x)>0$ for $x \neq 0$ and $\int_0^{+\infty} g(x)dx = \infty$.

iv) $|q(t,x,x')| \leq e(t)$, where $e(t)$ is a non-negative and continuous function of t satisfying

$$\int_0^{\infty} e(t)dt \leq M < +\infty$$

and M is a constant.

We shall determine sufficient conditions for the boundedness and the L^2 properties of the solutions of equation (1). Our approach differs from those of the earlier research as all they constructed energy or Liapunov Functions; so, our results differ significantly from those obtained previously, see some attempts in that sense in Kroopnick (2008) and Tunc (2011a), and references cited therein.

The solutions of equation (1) are bounded if there exists a constant $K>0$ such that $|x(t)| < K$ for all $t \geq T > 0$ for some T.

By an L^2-solution, we mean a solution of equation (1) such that $\int_0^{\infty} x^2(t)dt < \infty$.

In the last four decades, many authors have investigated the Liénard equation

$$x''+f(x)x'+g(x)=0. \tag{2}$$

They examined some qualitative properties of the solutions. The book of Sansone and Conti (1964) contains an almost complete list of papers dealing with these equations as well as a summary of the results published up to 1960. The book of Reissig, Sansone and Conti (1963) updates this list and summary up to 1962. The list of the papers which appeared between 1960 and 1970 is presented in the paper of Graef (1972). Among the papers published in the last years are Burton (1965); Burton and Townsend (1968); Hara and Yoneyama (1985); Hricisakova (1993); Kato (1988); Nápoles (2000); Omari, Villari and Zanolin (1987); Tunc (2011a); Villari (1983); and Villari (1987).

If in (1) we make $p(t)\equiv 1$, $q(t,x,x')\equiv 0$, $f(t,x)=f(x)$ and $a(t)\equiv 1$, then equation (1) becomes equation (2) so, every qualitative result for the equation (1) produces a qualitative result for (2).

The paper is organized as follows: in §2 we state and prove our results on boundedness and L^2 properties of solutions of (1); in the §3 we present some applications of our results to show the advantages over those reported in the literature and in §4 we reflect on a particular case of Theorem 1, an open problem, a simple example of the necessary criterion for positivity of the function f and some relations with results obtained recently.

2. Results

We now state and prove a general boundedness theorem. Without loss of generality, we shall assume $t \geq 0$.

Theorem 1.

Assuming that conditions i)-iv) above holds, then any solution $x(t)$ of (1), as well as its derivative, is bounded as $t \to \infty$ and $\int_0^\infty x'^2(t)dt < \infty$

Proof:

By standard existence theory, there is a solution of (1) which exists on $[0,T)$ for some $T>0$. Multiply the equation (1) by x' and integrate from 0 to t and apply the assumptions i) and iv) we obtain

$$p\frac{[x'(t)]^2}{2} + \int_0^t f(s,x(s),x'(s))[x'(s)]^2 ds + a_0 \int_{x(0)}^{x(s)} g(u)du \leq p\frac{[x'(0)]^2}{2} + \int_0^t |e(s)x'(s)| ds. \tag{3}$$

Now if $x(t)$ becomes unbounded then it follows that all terms on the left hand side of (1) are positive from our hypotheses. By the Cauchy-Schwarz inequality for integrals on the right hand side of (3), we get

$$p\frac{[x'(t)]^2}{2} + \int_0^t f(s,x(s),x'(s))[x'(s)]^2 ds + a_0 \int_{x(0)}^{x(s)} g(u)du \leq p\frac{[x'(0)]^2}{2} + \left(\int_0^t e^2(s)ds\right)^{\frac{1}{2}} \left(\int_0^t x'^2(s)ds\right)^{\frac{1}{2}}.$$

Now, let $X(t) = \left(\int_0^t x'^2(s)ds\right)^{\frac{1}{2}}$. Dividing both sides by $X(t)$ yields

$$\begin{aligned} X^{-1}(t)\left[p\frac{[x'(t)]^2}{2} + \int_0^t f(s,x(s),x'(s))[x'(s)]^2 ds + a_0 \int_{x(0)}^{x(s)} g(u)du\right] \\ \leq X^{-1}(t)p\frac{[x'(0)]^2}{2} + \left(\int_0^t e^2(s)ds\right)^{\frac{1}{2}}. \end{aligned} \tag{4}$$

Taking into account the positivity of the left hand side of (4) if $x(t)$ increase without bound and the term

$$X^{-1}(t)f_0\int_0^t x'^2(s)ds = f_0\left(\int_0^t x'^2(s)ds\right)^{\frac{1}{2}}$$

is bounded by the right hand side of equation (4) we obtain that *x'* is square integrable and is also bounded after we examine the first term of the left hand side of (4). However, the above implies that *x(t)* must be bounded. Otherwise, the left hand side of (4) becomes infinite which is impossible. A standard argument now permits the solution to be extended to all t of $\boldsymbol{I}$, see for example Boudonov (nodated); Reissig, Sansone and Conti (1963) and Sansone and Conti (1964). The proof is thus complete.

By imposing more stringent conditions on $g(t,x)$ and $p(t)$, all solutions become L^2-solutions. This case is covered by the following result.

Theorem 2.

Under hypotheses of Theorem 1, we suppose that $g(x)x > g_0 x^2$ for some positive constant $g_0>0$, and $0< p < p(t) < P <+\infty$, then all the solutions of equation (1) are L^2-solutions.

Proof:

In order to see that $x \in L^2[0,\infty)$ we must first multiply equation (1) by x, the integration from 0 to t yields

$$x(p(t)x') - \int_0^t p(s)x'^2(s)ds + \int_0^t f(t,x,x')x(s)x'(s)ds + \int_0^t x(s)a(s)g(x(s))ds$$
$$= x(0)p(0)x'(0) + \int_0^t q((s,x(s,x'(s))x(s)ds.$$

Next, let $\int_{x(0)}^{x(t)} zf(x^{-1}(z),z,z')dz = F(x)$. So, the above equation may be rewritten as

$$px(t)x'(t) - P\int_0^t x'^2(s)ds + F(x) + a_0 g_0 \int_0^t x^2(s)ds \le K, \tag{5}$$

where

$$K = P|x(0)x'(0)| + \left|\int_0^t e(s)x(s)ds\right|.$$

Notice that the last term is bounded by

$$\left(\int_0^t e^2(s)ds\right)^{\frac{1}{2}}\left(\int_0^t x^2(s)ds\right)^{\frac{1}{2}}$$

by using the Cauchy-Scwharz inequality. Dividing the left hand side of (9) by

$$M(t) = \int_0^t e(s)x(s)ds$$

and using the hypotheses of Theorem 2 we obtain

$$M^{-1}(t)\left[px(t)x'(t) - P\int_0^t x'^2(s)ds + F(x) \right] + a_0 g_0 \left(\int_0^t x^2(s)ds \right)^{\frac{1}{2}} \leq \frac{p\left|x(0)x'(0)\right|}{M(t)} + \left(\int_0^t e^2(s)ds \right)^{\frac{1}{2}}. \qquad (6)$$

Since the right hand side of (10) is bounded and all the terms of the left hand side are either bounded or positive, the result follows because the left hand side cannot be unbounded. Here, we need that x is square integrable.

3. Some Applications

In this section our results are applied to some reported in the literature.

If in (1) the functions involved satisfies $p(t) \equiv 1$, $e(t) \equiv 0$, $f(t,x,x') \equiv f_0$ and $g(x)=g_0x$, from assumptions ii) and iii) of Theorem 1 we obtain

ii') $f_0>0$,

iii') $g_0>0$.

Then, these assumptions amount to the usual Routh-Hurwitz criterion (see Boudonov).

In Nápoles (1999) the author proved for the generalized Liénard equation (2) with restoring term $h(t)$, the following result:

Theorem 1. We assume that $g\in C(\boldsymbol{R})$, with limit at infinity and

$g(-\infty)<g(x)<g(+\infty), \ \forall x\in \boldsymbol{R}$.

In addition, either $p\in V$, $g(-\infty)<p(t)<g(+\infty)$, or $p\in L^{\infty}(I)$, $g(-\infty)=-\infty$, $g(+\infty)=g(+\infty)$, where

$$V = \{h \in L^{\infty}(I) : h_m = \underset{T\to\infty}{Lim} \int_{\alpha}^{\alpha+T} h(t)dt \quad uniformly \quad in \quad \alpha\};$$

denoting by h_m the medium value of h, then (2) has a solution in $W^{2,\infty}(\boldsymbol{R})$. Also, $\forall\gamma > 0$, $\exists\Gamma>0$ such that for any solution $x(t)$ of (2) with $|x(t_0)| + |x'(t_0)| \leq \gamma$, for some $t_0\in I$, then $|x(t)| + |x'(t)| \leq \Gamma$, $t \geq t_0$.

This result is easily obtained from our Theorem 1.

In Repilado and Ruiz (1985) and Repilado and Ruiz (1986) the authors studied the asymptotic behaviour of the solutions of the equation

$$x''+f(x)x'+a(t)g(x)=0 \tag{7}$$

under the following conditions:

a) f is a continuous and nonnegative function for all $x\in\boldsymbol{R}$,

b) g is also a continuous function with $xg(x)>0$, for $x\neq 0$, and

c) $a(t)>0$, for all $t\in I$ and $a\in C^1$.

In particular in Repilado and Ruiz (1986), the following result is proved:

Theorem 2. Under conditions

1. $\int_0^{+\infty} a(t)dt=+\infty.$
2. $\int_0^{+\infty} \frac{a'(t)_-}{a(t)}dt=+\infty$, $a'(t)_-=\max\{-a'(t),0\}$.
3. There exists a positive constant N such that $|G(x)|\leq N$ for $x\in(-\infty,\infty)$, where $G(x)=\int_0^x g(s)ds$, all solutions of equation (6) are bounded if and only if

$$\int_0^{+\infty} a(t)f[\pm k(t-t_0)]dt=\pm\infty, \tag{8}$$

for all $k>0$ and some $t_0\geq 0$.

The first result of this nature was obtained by Burton and Grimmer (1971) when they showed that all continuable solutions of equations $x''+a(t)f(x)=0$ under condition b) and c) are oscillatory (and bounded) if and only if the condition (8) is fulfilled, with f instead of g.
It is easy to obtain the sufficiency of the above result from our Theorem 1.

If in (1) we take $f(t,x)\equiv 0$, $e(t)\equiv 0$ and $g(t,x)=g(t)x$, our result becomes Theorem 1 of Nápoles and Negrón (1996), referent to boundedness of $x(t)$ and $p(t)x'(t)$, for all $t\geq a$ with a some positive constant.

Castro and Alonso (1987) considered the special case

$$x''+h(t)x'+x=0, \tag{9}$$

of equation (1) under condition $h \in C^1(I)$ and $h(t) \geq b > 0$. Further, they required that the condition $ah'(t)+2h(t) \leq 4a$ be fulfilled, and obtained various results on the stability of the trivial solution of (8). It is clear that all assumptions of Theorem 1 are satisfied. Thus, we obtain a consistent result under milder conditions.

This result completes those of Ignatiev referred to in equation (5), see Ignatiev (1997), with restoring term

$$x''+f(t)x'+g(t)x=h(t), \tag{10}$$

Taking $h(t)$ continuous on $\boldsymbol{I}$ (in Ignatiev's results $h \equiv 0$) such that $\int_0^\infty h^2(t)dt < \infty$ and $f(t)>f_0>0$, $g(t)>g_0>0$ with continuous nonpositive derivatives we have that all the solutions of (10), as well as their derivatives, are bounded and in $L^2(I)$.

Our results contain and improve those of Ruiz (1988), obtained with $h \equiv 0$, referring to the boundedness of the solutions of equation

$$x''+f(t)x'+a(t)g(x)=h(t),$$

because the author used regularity assumptions on function $a(t)$, which are not used here.

In Kroopnick (2008) the author discussed the boundedness and L^2 character of equation (1) with $f(t,x,x')=c(t)f(x)$ and $p(t)\equiv 1$. Thus, our results contain those of Kroopnick.

Tunc studied the boundedness of the solutions of equation (1) under derivability assumptions on $p(t)$ and $a(t)$, see Tunc (2011a). Taking into account the results obtained above, these complete and improve Tunc's work.

The results obtained in this paper are consistent with those of Shao and Song (2011) where the authors study the sublinear equation $(a(t)y')'+b(t)y^{,k}=0$, with regularity assumptions on $r(t)$.

4. Conclusions

At the end of this paper and to compare our results with several of the references, we give in this section a particular case of Theorem 1, an open problem, a simple example of the necessary character of positivity of function f and some concluding remarks.

A particular case.

Ignatiev (1997) considered an oscillator described by the following equation

$$x''+f(t)x'+g(t)x=0, \tag{11}$$

where the damping and rigidity coefficients $f(t)$ and $g(t)$ are continuous and bounded functions. If in equation (1) we put $p(t)\equiv 1$, $e(t)\equiv 0$, $f(t,x)=f(t)$ and $g(t,x)=g(t)x$, then we improve the Theorem 1 of Ignatiev, since the assumption

$$\frac{1}{2}\frac{g'(t)}{g(t)}+f(t)>\alpha_2>0,$$

is not necessary, and

$$|f(t)|<M_1,\ |g(t)|<M_2,\ |g'(t)|<M_3,$$

is dropped. Under the above remarks, the Ignatiev's Corollary 1 is obvious.

An open problem.

Taking into account the Application related to Theorem 2 of Repilado and Ruiz (1986), and Theorem 1 of same reference, raises the following open problem

Under which additional hypotheses, the assumption $\int_0^\infty g(x)dx=\infty$ *is a necessary and sufficient condition for boundedness of the solutions of equation (1)?*

This is not a trivial problem. The resolution implies obtaining a necessary and sufficient condition for completing the study of asymptotic nature of solutions of (1).

On the positivity of *f*.

Under assumptions $f(t,x,x')\geq f_0>0$ for some positive constant f_0, the class of equation (1) is not very large, but if this condition is not fulfilled, we can exhibit equations that have unbounded solutions. For example

$$\left(e^{-\left(\frac{t^3}{3}+3t\right)}x'\right)'+2(t^2+1)e^{-\left(\frac{t^3}{3}+3t\right)}=0,$$

has the unbounded solution $x(t)=e^{2t}$ and $f(t,x,x')\equiv 0$.

Final Remarks.

Tunc (2010) established some new sufficient conditions which guarantee the boundedness of solutions of non-linear differential equation:

$$x''+f(t,x,x')g(t,x,x')+b(t)h(x)=e(t,x,x'), \tag{12}$$

where b, f, g, h and e are continuous functions in their respective domains and $b'(t)$ exists and its continuous. Under these assumptions he proved the following result.

Theorem 2.

In addition to the basic assumptions imposed on the functions b, f, g, h and e that appear in equation (12), we assume that there exists a positive constant α such that the following assumptions hold:

i) $f(t,x,x')g(t,x,x')\geq 0$, for all $t\in \mathbf{R}_+ := [0,+\infty)$ and

$$x,y\in \mathbf{R},\ b(t)\geq 1,\ b'(t)\leq 0 \text{ , for all } t\in \mathbf{R}_+,\ \frac{h(x)}{x}\geq \alpha \text{ , for all } x\neq 0.$$

ii) $|e(t,x,y)|\leq |p(t)|$, with $\int_0^t |p(s)|ds<\infty$.

Then all solutions of equation (12) are bounded.

Clearly this theorem and our results are consistent and complement each other and they have different applications. Equation (1) contains a general function $p(t)$ while in equation (12) $p(t)\equiv 1$, the forcing term of both are similar, the function $a(t)$ in equation (1) is wider than that $p(t)$ in (12), the consideration of the function $h(x)$ in (12) is more restrictive than ours on $g(x)$, although the term restorer of equation (12) is more general than ours.

The same author extends and completes several known results on boundedness and stability, to the case of functional differential equations of various types, which illustrates a work address very promising (see Tunc (2011b); Tunc (2012a) and Tunc (2012b)).

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