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A stage-structured two species competition mathematical model under the effect of disease

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Abstract

In this paper we study the dynamics of two competing species model; one of this competing species is stage structured and the disease spreads only in the other competing specie. In order to keep the model simple, we present it under the strong assumption that the disease can not cross the species barrier. Dynamical behaviors such as positivity, boundedness, stability, bifurcation and persistence of the model are studied analytically using the theory of differential equations. Computer simulations are carried out to substantiate the analytical findings. It is noted that c the loss rate of the population, τ the maturation time and f the intraspecific coefficient are the key parameters which we need to control or protect th to keep away the mature healthy population from extinction and also keep the infected individuals of the latter species from extinction respectively.

Keywords: Competing Species; Stage-Structured Disease; Stability; Permanence; Numerical Simulation

AMS 2010 Classification: 34D20, 34D23

1. Introduction

Population that compete for common resources are known among ecologists. They are classically modeled by observing their interactions that hinder the growth of both populations and are thus described by negative bilinear terms in all the relevant differential equations. In the natural world, there are many species whose individual members have a life history that takes them through the two stages-immature and mature. In Freedman et al. a stage structured model of population growth consisting of immature and mature individuals was analyzed, where the stage-structured was modeled by the introduction of a constant time delay. Other population growth and infectious disease models with time delays were considered in D'Onofrio (2002), Hethcote (2000) and Roberts et al. (2002).

Another major problem in today's modern society is the spread of infectious diseases. In general, the spread of infectious disease in a population depends upon various factors such as the number of infective and susceptible, modes of transmission; as well as socio-economic factors, environmental factors and ecological and geographic conditions Dutour (1982). A detailed account of modeling and the study of epidemic diseases can be found in the literature in the form of lecture notes, monographs etc. Bailey (1975), Hethcote (1976), Waltman (1974), Bailey (1982), Hethcote et al. (1981) and Agarwal et al. (2012). The population biology of infectious diseases has also been presented in Anderson et al. (1979). A recent trend on modeling population dynamics is to emphasize infectious diseases as regulators of population size Mena-Lorca et al. (1992). A system where one disease-free species competes with another host which is infected by the epidemics is also considered in Begon et al. (1995). This is most closely related to the present investigation. The classical paper Anderson et al. (1986) considered the two-competitor, one of which is affected by a disease, which is assumed to annihilate the reproductive rate of the infected individuals. The possibility that a superior competitor favors coexistence with another one, which would otherwise be wiped out, is inferred from the study.

The organization of this paper is as follows: In Sec.2, we introduce our mathematical model. In Sec.3 and 4, we present positivity and boundedness of solutions, respectively. In Sec.5, we analyze our model with regard to equilibria and their stabilities. In Sec.6, we establish bifurcation criterion to show the regulating impact of maturation delay. In Sec.7, we obtain conditions that influence the permanence of the solutions. Computer simulations are performed to illustrate the feasibility of our analytical findings in Sec.8. In the last Sec.9, we present the conclusion based on our analysis.

2. Mathematical Model

Here we consider a competition model with infection as studied in Venturino (2001) where two logistically growing populations $P(t)$ and $Q(t)$ which are competing for the same resource are analyzed. It is assumed that one of the competing species $P(t)$ is stage structured and the disease spreads only in the other competing species, $Q(t)$. We specify the healthy individuals of respectively as immature and mature $P_i(t)$ and $P_m(t)$, the healthy individuals $Q(t)$ and the infected individuals of the latter species denoted by $V(t)$. To study the effect of the disease in the competing species system we have proposed the following model:

$$\begin{aligned}
\dot{P}_i(t) &= \alpha P_m(t) - \gamma P_i(t) - \alpha e^{-\gamma\tau} P_m(t-\tau), \\
\dot{P}_m(t) &= \alpha e^{-\gamma\tau} P_m(t-\tau) - \beta P_m^2(t) - c P_m(t) Q(t) - \eta P_m(t) V(t), \qquad (2.1)\\
\dot{Q}(t) &= d Q(t) - e P_m(t) Q(t) - f(Q(t) + V(t)) Q(t) - \delta V(t) Q(t),
\end{aligned}$$

$$\dot{V}(t) = \delta V(t)Q(t) - gP_m(t)V(t) - f(Q(t)+V(t))V(t),$$

where

$$P_m(t) = \varphi_m(t) \geq 0, \ -\tau \leq t \leq 0 \text{ and } P_i(0) > 0, Q(0) > 0, V(0) > 0.$$

Now for continuity of initial conditions we require,

$$P_i(0) = \int_{-\tau}^{0} \alpha e^{-\gamma s} \varphi_m(s) ds \tag{2.2}$$

And with the help of (2.2) the solution of the first equation of system (2.1) can be written in terms of the solution for $P_m(t)$ as

$$P_i(t) = \int_{t-\tau}^{t} \alpha e^{-\gamma(t-s)} P_m(s) ds. \tag{2.3}$$

Equations (2.2) and (2.3) suggest that, mathematically no information of the past history of $P_i(t)$ is needed for the system (2.1) because the properties of $P_i(t)$ can be obtained from (2.2) and (2.3) if we know the properties of $P_m(t)$. Therefore, in the rest of this paper we need only to consider the following model:

$$\begin{aligned} \dot{P}_m(t) &= \alpha e^{-\gamma\tau} P_m(t-\tau) - \beta P_m^2(t) - cP_m(t)Q(t) - \eta P_m(t)V(t), \\ \dot{Q}(t) &= dQ(t) - eP_m(t)Q(t) - f(Q(t)+V(t))Q(t) - \delta V(t)Q(t), \\ \dot{V}(t) &= \delta V(t)Q(t) - gP_m(t)V(t) - f(Q(t)+V(t))V(t), \end{aligned} \tag{2.4}$$

where

$$P_m(t) = \varphi_m(t) \geq 0, \ -\tau \leq t \leq 0 \text{ and } Q(0) > 0, V(0) > 0.$$

At any time $t > 0$, birth into the immature healthy population is proportional to the existing mature healthy population with proportionality constant $\alpha > 0$. The immature healthy population will transfer to mature healthy class after its birth with a maturity period τ. The immature healthy population has the natural death rate $\gamma > 0$. The death rate of the mature healthy population is proportional to the square of the existing mature healthy population with proportionality constant $\beta > 0$. The term $\alpha e^{-\gamma\tau} P_m(t-\tau)$ that appears in the first and second equations of system (2.1) represents the immature healthy population born at time $(t-\tau)$ and surviving at the time t and therefore represents the transformation from immature healthy to mature healthy population. c is the loss rate in the population $P_m(t)$ due to the competitor $Q(t)$ and e is the loss rate in the population $Q(t)$ due to the competitor $P_m(t)$. η is the loss rate in the population $P_m(t)$ due to the competitor $V(t)$ and g is loss rate in the population $V(t)$ due to the competitor $P_m(t)$. β, f are intraspecific coefficients of the competition of $P_m(t)$, $Q(t)$ and $V(t)$. δ is the transmission rate of the infection.

3. Positivity of Solutions

Theorem 3.1.

All solutions of the system (2.4) are positive for all $t \leq 0$.

Proof:

Clearly $Q(t) > 0$ and $V(t) > 0$ for $Q(0) > 0, V(0) > 0, \mathrm{t} > 0$. Now $P_m(0) > 0$ hence if there exist at such that $P_m(t_0) = 0,$ then $t_0 > 0$. Assume that t_0 is the first time such that $P_m(t) = 0,$ that is, $t_0 = \inf\{t.0 : P_m(t) = 0\}$, then

$$\dot{P}_m(t_0) = \begin{cases} \alpha e^{-\gamma\tau}\phi_m(t_0 - \tau), & 0 \leq t_0 \leq \tau, \\ \alpha e^{-\gamma\tau}P_m(t_0 - \tau), & t_0 > \tau. \end{cases}$$

so that $\dot{P}_m(t_0) > 0$. Hence, for sufficiently small $\varepsilon > 0$, $\dot{P}_m(t_0 - \varepsilon) > 0$. But by definition of t_0, $\dot{P}_m(t_0 - \varepsilon) \leq 0$, a contradiction. Hence, $P_m(t) > 0$ for all $t_0 \geq 0$.

4. Bondedness of Solutions

To prove the boundedness of the solutions, we shall need the following result, which is a direct application of Theorem (4.9.1) in Kuang.Y (1993) (p159).

Lemma 4.1.

Consider the equation,

$$\dot{P}_m(t) = \alpha e^{-\gamma\tau}P_m(t-\tau) - \beta P_m^2(t) - cP_m(t), \big(\alpha, c, \beta, \tau > 0,\ P_m(t) > 0,\ for - \tau \leq t \leq 0\ \big).$$

Then, we have

(i) If $\alpha e^{-\gamma\tau} > c$ then $\lim\limits_{t\to\infty} P_m(t) = \dfrac{(\alpha e^{-\gamma\tau} - c)}{\beta}$,

(ii) If $\alpha e^{-\gamma\tau} < c$ then $\lim\limits_{t\to\infty} P_m(t) = 0$.

Theorem 4.1.

All solutions of model (2.4) will lie in the region,

$$\Omega = \left\{(P_m, Q, V) \in R_+^3 :\ 0 \leq P_m \leq P_{\max}, 0 \leq Q \leq Q_{\max}, 0 \leq V \leq V_{\max}\right\} \text{ as } t \to \infty,$$
for all positive initial values $(P_0, Q_0, V_0) \in R_+^3$.

Proof:

First from equation (2.4) we get,

$$\dot{P}_m(t) \le \alpha e^{-\gamma\tau} P_m(t-\tau) - \beta P_m^2(t).$$

According to lemma (4.1) and the comparison theorem D'Onofrio (2002), there is a $T > 0$ and $\varepsilon > 0$ such that

$$P_m(t) \le \frac{\alpha e^{-\gamma\tau}}{\beta} + \varepsilon \quad \text{for} \quad t > T + \tau.$$

This implies that $\lim\limits_{t\to\infty} Sup\, P_m(t) \le \frac{\alpha e^{-\gamma\tau}}{\beta} = P_{\max}$.

Similarly from the second and third equation in model (2.4) we get as $t \to \infty$,

$$\lim_{t\to\infty} Sup\, Q(t) \le \frac{d}{f} = Q_{\max} \quad \text{and} \quad \lim_{t\to\infty} Sup\, V(t) \le \frac{\delta d}{f^2} = V_{\max}.$$

This completes the proof of the theorem.

5. Boundary Equlibria and their Stability

Setting $\dot{P}_m(t) = \dot{Q}(t) = \dot{V}(t) = 0$ in model (2.4) and solving the resulting equations,

$$\alpha e^{-\gamma\tau} P_m(t-\tau) - \beta P_m^2(t) - cP_m(t)Q(t) - \eta P_m(t)V(t) = 0$$
$$dQ(t) - eP_m(t)Q(t) - f(Q(t)+V(t))Q(t) - \delta V(t)Q(t) = 0$$
$$\delta V(t)Q(t) - gP_m(t)V(t) - f(Q(t)+V(t))V(t) = 0$$

We see that model (2.4) has six non negative equilibria:

$$E_0(0,0,0),\ E_1(P_{m1},0,0),\ E_2(0,Q_{2,}0)\ E_3(P_{m3},Q_3,0),\ E_4(0,Q_4,V_4) \text{ and } \hat{E}(\hat{P}_m,\hat{Q},\hat{V}),$$

where

$$P_{m1} = \frac{\alpha e^{-\gamma\tau}}{\beta},\ Q_2 = \frac{d}{f},\ P_{m3} = \frac{(\alpha f e^{-\gamma\tau} - cd)}{(\beta f - ec)},\ Q_3 = \frac{(\beta d - \alpha e e^{-\gamma\tau})}{(\beta f - ec)},\ Q_4 = \frac{df}{\delta^2},$$
$$V_4 = \frac{(\delta - f)d}{\delta^2}.$$

Equilibrium $\hat{E}$ exists if the system of following equations;

$$\begin{aligned} &\alpha e^{-\gamma\tau} - \beta P_m(t) - cQ(t) - \eta V(t) = 0 \\ &d - eP_m(t) - f(Q(t)+V(t)) - \delta V(t) = 0 \\ &\delta Q(t) - gP_m(t) - f(Q(t)+V(t)) = 0 \end{aligned} \tag{5.1}$$

has a positive solution $(\hat{P}_m, \hat{Q}, \hat{V})$. From second and third equation of system (5.1) we get,

$$V = \frac{(d - eP_m - fQ)}{(f + \delta)} \tag{5.2}$$

and

$$V = \frac{(\delta - f)Q - gP_m}{f}. \tag{5.3}$$

From equation (5.2) and (5.3) we get,

$$\frac{(d - eP_m - fQ)}{(f + \delta)} = \frac{(\delta - f)Q - gP_m}{f}.$$

Now solving above equation we get,

$$Q = \frac{fd - (ef + gf + \delta g)P_m}{\delta^2}. \tag{5.4}$$

Using equation (5.4) in equation (5.3) we get,

$$\hat{V} = \left[\frac{(\delta - f)(fg + g\delta - ef)}{\delta^2 f} - \frac{g}{f} \right] \hat{P}_m + \frac{(\delta - f)d}{\delta^2} \tag{5.5}$$

$$\hat{V} = a_1 \hat{P}_m + a_2,$$

where

$$a_1 = \frac{(\delta - f)(fg + g\delta - ef)}{\delta^2 f} - \frac{g}{f} \text{ and } a_2 = \frac{(\delta - f)d}{\delta^2}.$$

Also using equation (5.5) in equation (5.3) we get,

$$\hat{Q} = \frac{fd}{\delta^2} + \frac{(gf + g\delta - ef)}{\delta^2} \hat{P}_m , \tag{5.6}$$

$\hat{Q} = a_3 + a_4 \hat{P}_m$, where $a_3 = \dfrac{fd}{\delta^2}$ and $a_4 = \dfrac{(gf + g\delta - ef)}{\delta^2}$.

Now putting values of $\hat{Q}$ and $\hat{V}$ in first equation of system (5.1) we get,

$$\hat{P}_m = \frac{(\alpha e^{-\gamma\tau} - ca_3 - \eta a_2)}{(\beta + ca_4 + \eta a_1)}.$$

The interior equilibrium $\hat{E}$ is feasible when

$$\delta > f,\ g(f+\delta) > ef,\ \alpha e^{-\gamma\tau} > (ca_3+\eta a_2) \text{ and } (\delta-f)(gf+g\delta-ef) > g\delta^2.$$

The characteristic equation of the equilibrium E_0 is

$$(\lambda-\alpha e^{-(\lambda+\gamma)\tau})\ (\lambda-d)\ (\lambda-0)=0.$$

Clearly, $\lambda=\alpha e^{-(\lambda+\gamma)\tau}$, $\lambda=d$ and $\lambda=0$ are all positive equilibrium therefore equilibrium E_0 is completely unstable. The characteristic equation of equilibria E_1 is

$$(\lambda-\alpha e^{-(\lambda+\gamma)\tau}+2\beta P_{m1})\ (\lambda+eP_{m1}-d)\ (\lambda+gP_{m1})=0.$$

Since above characteristic equation has one negative eigenvalue corresponding to the $V-$direction and all other eigenvalues, i.e., eigenvalues in the P_m and $Q-$directions are given by solution of $\lambda=\alpha e^{-(\lambda+\lambda)\tau}-2\beta P_{m1}$ and $\lambda=d-eP_{m1}$ which always has a positive solution provided that E_1 is unstable. The characteristic equation of equilibria E_2 is

$$(\lambda-\alpha e^{-(\lambda+\gamma)\tau}+cQ_2)\ (\lambda+2fQ_2-d)\ \left(\lambda-(\delta-f)Q_2\right)=0.$$

From this $\lambda=d-2fQ_2$, $\lambda=(\delta-f)Q_2$ and $\lambda=\alpha e^{-(\lambda+\gamma)\tau}-cQ_2$.

Since one eigenvalue $\lambda=(\delta-f)Q_2$ is always positive therefore equilibrium E_2 is unstable. The characteristic equation of equilibrium E_3 is

$$\left(\lambda+gP_{m3}-(\delta-f)Q_3\right)\left(\lambda^2+B_1\lambda+B_2-(B_3\lambda+B_4)e^{-\gamma\tau}\right)=0,$$

where

$$B_1=(2f+c)Q_3+(e+2\beta P_{m3})-d,$$

$$B_2=(2fQ_3-d)(2\beta P_{m3}+cQ_3)+2\beta eP_{m3}^2,$$

$$B_3=\alpha e^{-\gamma\tau}=(\beta P_{m3}+cQ_3),$$

$$B_4=(2f-Q_3+eP_{m3})\alpha e^{-\gamma\tau}=(2f-Q_3+eP_{m3})(\beta P_{m3}+cQ_3).$$

Clearly, $\lambda=(\delta-f)Q_3-gP_{m3}>0.$ Therefore, equilibrium E_3 is unstable.

The characteristic equation of equilibrium E_4 is

$$\left(\lambda+cQ_4+\eta V_4-\alpha e^{-(\lambda+\gamma)\tau}\right)(\lambda^2+\lambda H_1+H_2)=0,$$

where

$$H_1 = (3fQ + 3fV + \delta V - \delta Q - d),$$

$$H_2 = \left(d(\delta - f)Q_4 - 2fdV_4 - 2f(\delta - f)Q_4^2 + 4f^2Q_4V_4 + 2f(\delta + f)V_4^2\right).$$

The characteristic equation of equilibrium $\hat{E}$ is

$$\lambda^3 + \lambda^2 A_1 + \lambda A_2 + A_3 - (A_4\lambda^2 + A_5\lambda + A_6)e^{-\lambda\tau} = 0, \tag{5.7}$$

where

$$A_1 = \left((2\beta\hat{P}_m + c\hat{Q} + \eta\hat{V}) - (d - e\hat{P}_m - 2f\hat{Q} - (f + \delta)\hat{V} + (\delta - f)\hat{Q} - g\hat{P}_m - 2f\hat{V})\right),$$

$$A_2 = \begin{bmatrix} \left(d - e\hat{P}_m - 2f\hat{Q} - (f + \delta)\hat{V}\right)\left((\delta - f)\hat{Q} - g\hat{P}_m - 2f\hat{V}\right) \\ + (\delta^2 - f^2)\hat{Q}\hat{V} - ce\hat{P}\hat{Q} - g\eta P_m\hat{V} - (2\beta\hat{P}_m + c\hat{Q} + \eta\hat{V}) \\ \left(d - e\hat{P}_m - 2f\hat{Q} - (f + \delta)\hat{V} + (\delta - f)\hat{Q} - g\hat{P}_m - 2f\hat{V}\right) \end{bmatrix},$$

$$A_3 = \begin{bmatrix} (2\beta\hat{P}_m + c\hat{Q} + \eta\hat{V})\left(d - e\hat{P}_m - 2f\hat{Q} - (f + \delta)\hat{V}\right)\left((\delta - f)\hat{Q} - g\hat{P}_m - 2f\hat{V}\right) \\ + (2\beta\hat{P}_m + c\hat{Q} + \eta\hat{V})(\delta^2 - f^2)\hat{Q}\hat{V} + \left((\delta - f)\hat{Q} - g\hat{P}_m - 2f\hat{V}\right)ce\hat{P}_m\hat{Q} \\ + \left(c(f + \delta)g + \eta(f - \delta)e\right)\hat{P}_m\hat{Q}\hat{V} + g\eta\hat{P}_m\hat{V}\left(d - e\hat{P}_m - 2f\hat{Q} - (f + \delta)\hat{V}\right) \end{bmatrix},$$

$$A_4 = \alpha e^{\gamma\tau} = \beta\hat{P}_m + c\hat{Q} + \eta\hat{V},$$

$$A_5 = \left((e + g)\hat{P}_m + 2f(\hat{Q} + \hat{V}) + (\delta + f)\hat{V} - (\delta - f)\hat{Q} - d\right)\alpha e^{\gamma\tau}, \tag{5.8}$$

$$= \left((e + g)\hat{P}_m + 2f(\hat{Q} + \hat{V}) + (\delta + f)\hat{V} - (\delta - f)\hat{Q} - d\right)\left(\beta\hat{P}_m + c\hat{Q} + \eta\hat{V}\right),$$

$$A_6 = (\delta^2 - f^2)\hat{Q}\hat{V}\alpha e^{-\gamma\tau} = (\delta^2 - f^2)(\beta\hat{P}_m + c\hat{P}_m + \eta\hat{V})\hat{Q}\hat{V},$$

Let

$$\varphi(\lambda, \tau) = \lambda^3 + \lambda^2 A_1 + \lambda A_2 + A_3 - (A_4\lambda^2 + A_5\lambda + A_6)e^{-\lambda\tau} = 0. \tag{5.9}$$

To show the positive equilibria $\hat{E}(\hat{P}_m, \hat{Q}, \hat{V})$ is locally asymptotically stable for all $\tau > 0$, we use the following Theorem 5.1 Begon et al. (1995).

Theorem 5.1.

A set of necessary and sufficient conditions for $\hat{E}(\hat{P}_m, \hat{Q}, \hat{V})$ to be asymptotically stable for all $\tau \geq 0$ is

(i) The real part of all roots of $\varphi(\lambda, 0) = 0$ are negative.

(ii) For all real ω_0 and $\tau \geq 0$, $(i\omega_0, \tau) \neq 0$ where $i = \sqrt{-1}$.

Theorem 5.2.

Assume that

$$\delta > f,\ g(f+\delta) > ef,\ \alpha e^{-\gamma\tau} > (ca_3 + \eta a_2) \text{ and} (\delta - f)(gf + g\delta - ef) > g\delta^2.$$

Then, the positive equilibrium of system (2.4) is asymptotically stable.

Proof:

We now apply Theorem (5.1) to prove Theorem (5.2). We prove this theorem in two steps.

Step 1. Substituting $\tau = 0$ in equation (5.7), we get

$$\varphi(\lambda,0) = \lambda^3 + \lambda^2 A_1 + \lambda A_2 + A_3 - (A_4\lambda^2 + A_5\lambda + A_6) = 0,$$

$$\varphi(\lambda,0) = \lambda^3 + S\lambda^2 + T\lambda + U = 0, \tag{5.10}$$

where

$$S = (A_1 - A_4) > 0,\ T = (A_2 - A_5) > 0,\ U = (A_3 - A_6) > 0 \text{ and } ST - U > 0.$$

Therefore, by Routh-Hurwitz criterion, all roots of equation (5.10) have negative real parts. Hence, condition (i) of Theorem (5.1) is satisfied and $\hat{E}$ is a locally asymptotically stable equilibrium in the absence of delay.

Step 2. Suppose that $\phi(i\omega_0, \tau) = 0$ holds for some real ω_0.

Firstly, when $\omega_0 = 0$,

$$\varphi(0,\tau) = A_3 - A_6 \neq 0.$$

Secondly, suppose $\omega_0 \neq 0$,

$$\varphi(i\omega_0,\tau) = -i\omega_0^3 - A_1\omega_0^2 + iA_2\omega_0 + A_3 - (-A_4\omega_0^2 + iA_5\omega_0 + A_6)e^{i\omega_0\tau}. \tag{5.11}$$

Equating real and imaginary parts of equation (5.11), we obtain

$$-A_1\omega_0^2 + A_3 = (A_4\omega_0^2 - A_6)\cos\omega_0\tau - A_5\omega_0\sin\omega_0\tau, \tag{5.12}$$

$$-\omega_0^3 + A_2\omega_0 = -(A_4\omega_0^2 - A_6)\sin\omega_0\tau - A_5\omega_0\cos\omega_0\tau \tag{5.13}$$

Squaring and adding equation (5.12) and (5.13), we get

$$\omega_0^6 + (A_1^2 - 2A_2 - A_4^2)\omega_0^4 + (A_2^2 - 2A_1A_3 + 2A_4A_6 - A_5^2)\omega_0^2 + (A_3^2 - A_6^2) = 0, \qquad (5.14)$$

where

$$(A_1^2 - 2A_2 - A_4^2) > 0,\ (A_2^2 - 2A_1A_3 + 2A_4A_6 - A_5^2) > 0 \text{ and } (A_3^2 - A_6^2) > 0.$$

It follows that

$$\omega_0^6 + (A_1^2 - 2A_2 - A_4^2)\omega_0^4 + (A_2^2 - 2A_1A_3 + 2A_4A_6 - A_5^2)\omega_0^2 + (A_3^2 - A_6^2) > 0.$$

This contradicts (5.14). Hence, $\varphi(i\omega_0, \tau) \neq 0.$ For any real ω_0, it satisfies condition (ii) of Theorem (5.1). Therefore, the unique positive equilibrium $\hat{E}(\hat{P}_m, \hat{Q}, \hat{V})$ is locally asymptotically stable for all $\tau \geq 0$ and the delay is harmless in this case.

6. Bifurcation Analysis

Substituting $\lambda = a(\tau) + ib(\tau)$ in (5.9) and separating real and imaginary parts, we obtain the following transcendental equations

$$\begin{aligned} &a^3 - 3ab^2 + A_1(a^2 - b^2) + A_2b + A_3 - e^{-a\tau}\left[A_4(a^2 - b^2) + aA_5 + A_6\right]\cos b\tau \\ &\qquad\qquad - e^{-a\tau}(2abA_4 + bA_5)\sin b\tau = 0, \qquad (6.1) \\ &-b^3 + 3a^2b + 2A_1ab + A_2b - e^{-a\tau}(2abA_4 + bA_5)\cos b\tau \\ &\qquad\qquad + e^{-a\tau}\left[A_4(a^2 - b^2) + aA_5 + A_6\right]\sin b\tau = 0, \qquad (6.2) \end{aligned}$$

where a and b are functions of τ. We are interested in the change of stability of $\hat{E}$ which will occur at the values of τ for which $a = 0$ and $b \neq 0$.

Let $\hat{\tau}$ be such that $a(\hat{\tau}) = 0$ and $b(\hat{\tau}) = \hat{b} \neq 0.$ then equation (6.1) and (6.2) become

$$-A_1\hat{b}^2 + A_3 - (-A_4\hat{b}^2 + A_6)\cos\hat{b}\hat{\tau} - A_5\hat{b}\sin\hat{b}\hat{\tau} = 0, \qquad (6.3)$$

$$-\hat{b}^3 + A_2\hat{b} - A_5\hat{b}\cos\hat{b}\hat{\tau} + (-A_4\hat{b}^2 + A_6)\sin\hat{b}\hat{\tau} = 0. \qquad (6.4)$$

Now eliminating $\hat{\tau}$ from (6.3) and (6.4), we get

$$\hat{b}^6 + (A_1^2 - 2A_2 - A_4^2)\hat{b}^4 + (A_2^2 - 2A_1A_3 + 2A_4A_6 - A_5^2)\hat{b}^2 + (A_3^2 - A_6^2) = 0. \qquad (6.5)$$

To analyze the change in the behavior of stability of $\hat{E}$ with respect to τ, we examine the sign of $\dfrac{da}{d\tau}$ as a crosses zero. If this derivative is positive (negative), then clearly a stabilization (destabilization) cannot take place at that value of τ. Differentiating equations (6.1) and (6.2) with respect to τ, then setting $\tau = \hat{\tau}$, $a = 0$ and $b = \hat{b}$, we get

$$\theta_1 \frac{da}{d\tau}(\hat{\tau}) + \theta_2 \frac{db}{d\tau}(\hat{\tau}) = k, \tag{6.6}$$

$$-\theta_2 \frac{da}{d\tau}(\hat{\tau}) + \theta_1 \frac{db}{d\tau}(\hat{\tau}) = l, \tag{6.7}$$

where,

$$\theta_1 = -3\hat{b}^2 + A_2 + (-A_4\hat{b}^2 + A_6)\hat{\tau}\cos\hat{b}\hat{\tau} - A_5\cos\hat{b}\hat{\tau} + A_5\hat{b}\hat{\tau}\sin\hat{b}\hat{\tau} - 2A_4\hat{b}\sin\hat{b}\hat{\tau},$$

$$\theta_2 = -2A_1\hat{b} + (-A_4\hat{b}^2 + A_6)\hat{\tau}\sin\hat{b}\hat{\tau} - A_5\sin\hat{b}\hat{\tau} - A_5\hat{b}\hat{\tau}\cos\hat{b}\hat{\tau} + 2A_4\hat{b}\cos\hat{b}\hat{\tau},$$

$$k = A_5\hat{b}^2\cos\hat{b}\hat{\tau} - (-A_4\hat{b}^2 + A_6)\hat{b}\sin\hat{b}\hat{\tau}, \tag{6.8}$$

$$l = -A_5\hat{b}^2\sin\hat{b}\hat{\tau} - (-A_4\hat{b}^2 + A_6)\hat{b}\cos\hat{b}\hat{\tau}.$$

Solving (6.6) and (6.7), we get

$$\frac{da}{d\tau}(\hat{\tau}) = \frac{k\theta_1 - l\theta_2}{\theta_1^2 + \theta_2^2}. \tag{6.9}$$

From (6.9), it is clear that $\frac{da}{d\tau}(\hat{\tau})$ has the same sign as $k\theta_1 - l\theta_2$.

From (6.8) after simplification and solving (6.3) and (6.4), we get

$$k\theta_1 - l\theta_2 = \hat{b}^2\left[3\hat{b}^4 + 2(A_1^2 - 2A_2 - A_4^2)\hat{b}^2 + (A_2^2 - 2A_1A_3 + 2A_4A_6 - A_5^2)\right]. \tag{6.10}$$

Let

$$G(u) = u^3 + S_1u^2 + S_2u + S_3, \tag{6.11}$$

where

$$S_1 = A_1^2 - 2A_2 - A_4^2,\ S_2 = A_2^2 - 2A_1A_3 + 2A_4A_6 - A_5^2,\ S_3 = A_3^2 - A_6^2.$$

From (6.11), we note that $G(u)$ is the left hand side of equation (6.5) with $\hat{b}^2 = u$. Therefore,

$$G(\hat{b}^2) = 0. \tag{6.12}$$

Now

$$\frac{dG(\hat{b}^2)}{du} = 3\hat{b}^4 + 2S_1\hat{b}^2 + S_2$$

$$= 3\hat{b}^4 + 2(A_1^2 - 2A_2 - A_4^2)\hat{b}^2 + (A_2^2 - 2A_1A_3 + 2A_4A_6 - A_5^2)$$

$$= \frac{k\theta_1 - l\theta_2}{\hat{b}^2} = \frac{\theta_1^2 + \theta_2^2}{\hat{b}^2}\frac{da}{d\tau}(\hat{\tau}).$$

This implies that,

$$\frac{da}{d\tau}(\hat{\tau}) = \frac{\hat{b}^2}{\theta_1^2 + \theta_2^2}\frac{dG(\hat{b}^2)}{du}. \tag{6.13}$$

Hence, the criterion for instability (stability) of $\hat{E}$ are

(1) If the polynomial $G(u)$ has no positive root, there can be no change of stability.

(2) If $G(u)$ is increasing (decreasing) at all of its positive roots, instability (stability) is preserved. Now in this case, if

  (i) $S_3 < 0,$ $G(u)$ has unique positive real root then it must increase at that point (since $G(u)$ is a cubic in $u,$ $\lim_{u\to\infty} it\, G(u) = \infty$).

  (ii) $S_3 > 0,$ then (1) is satisfied, i.e. there can be no change of stability.

Therefore, we have the following theorems.

Theorem 6.1.

If $S_3 < 0$ and $\hat{E}$ is unstable for $\tau = 0,$ it will remain unstable for $\tau > 0.$

Theorem 6.2.

If $S_3 < 0$ and $\hat{E}$ is asymptotically stable for $\tau = 0,$ it is impossible that it will remain stable for $\tau > 0.$ Hence there exist a $\hat{\tau} > 0,$ such that for $\tau < \hat{\tau},$ $\hat{E}$ is asymptotically stable and for $\tau > \hat{\tau},$ $\hat{E}$ is unstable and as τ increases together with $\hat{\tau},$ $\hat{E}$ bifurcates into small amplitude periodic solutions of the Hopf type Begon et al. (1995). The value of $\hat{\tau}$ is given by the following equation;

$$\hat{\tau} = \frac{1}{\hat{b}}\sin^{-1}\left[\frac{(\hat{b}^3 - A_2\hat{b})(-A_4\hat{b}^2 + A_6) - (A_1\hat{b}^2 - A_3)A_5\hat{b}}{(-A_4\hat{b}^2 + A_6)^2 + A_5^2\hat{b}^2}\right]$$

7. Persistence

Theorem 7.1.

Assume that

$$\alpha e^{-\gamma\tau} > \left(c + \frac{\eta\delta}{f}\right)\left(\frac{d}{f}\right),\ d > \frac{e\alpha e^{-\gamma\tau}}{\beta} + \frac{(f+\delta)\delta d}{f^2}$$

and

$$\delta q^* > \frac{g\alpha e^{-\gamma\tau}}{\beta}\left(\frac{\delta d}{f^2}\right) + d\,,$$

where

$$q^* = \left[\frac{d - \frac{e\alpha e^{-\gamma\tau}}{\beta} - \frac{(f+\delta)\delta d}{f^2}}{f}\right].$$

Then system (2.4) is permanent.

Proof:

From the first equation of system (2.4), we have

$$\dot{P}_m(t) \geq \alpha e^{-\gamma\tau} P_m(t-\tau) - \beta P_m^2(t) - \left(cQ_{\max} + \eta V_{\max}\right)P_m(t).$$

According to Lemma (4.1) and comparing principal, it follows that

$$\liminf_{t\to\infty} P_m(t) \geq \left[\frac{\alpha e^{-\gamma\tau} - \left(c + \frac{\eta\delta}{f}\right)\left(\frac{d}{f}\right)}{\beta}\right] (> 0).$$

From the second equation of system (2.4), we have

$$\dot{Q}(t) \geq Q(t)\left(d - \frac{e\alpha e^{-\gamma\tau}}{\beta} - (f+\delta)\frac{\delta d}{f^2} - fQ(t)\right).$$

This yields that for $d > \frac{e\alpha e^{-\gamma\tau}}{\beta} + \frac{(f+\delta)\delta d}{f^2}$,

$$\liminf_{t\to\infty} Q(t) \geq \left[\frac{d - \dfrac{e\alpha e^{-\gamma\tau}}{\beta} - \dfrac{(f+\delta)\delta d}{f^2}}{f}\right](>0) = q^*\ (say).$$

From third equation of system (2.4), we have

$$\dot{V}(t) \geq V(t)\left[\delta\, q^* - \frac{g\alpha e^{-\gamma\tau}}{\beta}\left(\frac{\delta d}{f^2}\right) - fV(t) - f\left(\frac{d}{f}\right)\right].$$

This yields that for, $\delta q^* > \dfrac{g\alpha e^{-\gamma\tau}}{\beta}\left(\dfrac{\delta d}{f^2}\right) + d,$

$$\liminf_{t\to\infty} V(t) \geq \left[\frac{\delta q^* - \dfrac{g\alpha e^{-\gamma\tau}}{\beta}\left(\dfrac{\delta d}{f^2}\right) - d}{f}\right] (>0).$$

According to the above arguments and Theorem (4.1), we have

$$\left[\frac{\alpha e^{-\gamma\tau} - \left(c + \dfrac{\eta\delta}{f}\right)\left(\dfrac{d}{f}\right)}{\beta}\right] \leq \liminf_{t\to+\infty} P_m(t) \leq \lim_{t\to+\infty} Sup P_m(t) \leq \frac{\alpha e^{-\gamma\tau}}{\beta},$$

$$\left[\frac{d - \dfrac{e\alpha e^{-\gamma\tau}}{\beta} - \dfrac{(f+\delta)\delta d}{f^2}}{f}\right] \leq \liminf_{t\to+\infty} Q(t) \leq \lim_{t\to+\infty} Sup Q(t) \leq \frac{d}{f},$$

$$\left[\frac{\delta q^* - \dfrac{g\alpha e^{-\gamma\tau}}{\beta}\left(\dfrac{\delta d}{f^2}\right) - d}{f}\right] \leq \liminf_{t\to+\infty} V(t) \leq \lim_{t\to+\infty} Sup V(t) \leq \frac{\delta d}{f^2}.$$

This completes the proof of theorem (7.1).

8. Numerical Simulation

In this section, we present numerical simulation to explain the applicability of the result discussed above. We choose the following parameters in model (2.1) are,

$$\alpha = 1,\ \beta = 1,\ c = 0.1,\ d = 2,\ e = 0.1,\ \ f = 1,\ g = 0.2,\ \delta = 1.5,\ \eta = 0.1,\ \gamma = 0.1,\ \tau = 10.$$

For the above set of parameter values the equilibrium $\hat{E}$ is given by, $\hat{P}_m = 0.2330$, $\hat{Q} = 0.9304$, $\hat{V} = 0.4186$.

Here, we note that all conditions of local stability and permanence are satisfied. From the existence, stability and persistence criteria τ, c and f are recognized to be the important parameters. Using MATLAB software package, graphs are plotted for different values of τ, c and f in order to conclude and confirm some important points.

(i) Figure 1 shows that $P_m(t)$ decreases with τ , and becomes extinct if $\tau \geq 20.15$.

(ii) Figure 2 shows that $Q(t)$ decreases with τ increases.

(iii) Figure 3 shows the behavior of $V(t)$ with time for different values of τ . From this figure, we can infer that τ increases with an increase in time and the maturity time, and finally attains its equilibrium level.

(iv) Figure 4 shows that the value of c at which the mature healthy population $P_m(t)$ tends to extinction is $c = 0.364$.

(v) Figure 5 shows that the value of c increases, as the population $Q(t)$ decreases.

(vi) Figure 6 shows that the value of c increases, as the infected population $V(t)$ increases.

(vii) Figure 7 shows that the value of intraspecific coefficient f increases, as the mature healthy population $P_m(t)$ increases.

(viii) Figure 8 shows the behavior of $Q(t)$ with time for different values of f. This figure shows that initially $Q(t)$ increases for some time, reaches the peak, then starts decreasing and finally attains its equilibrium level. From this figure, we also note that $Q(t)$ remains constant at its equilibrium level as f increases but the amplitude and timing of the peak decreases with an increase in f.

(ix) Figure 9 shows that the value of f at which the infected population $V(t)$ tends to extinction is $f = 1.4655$.

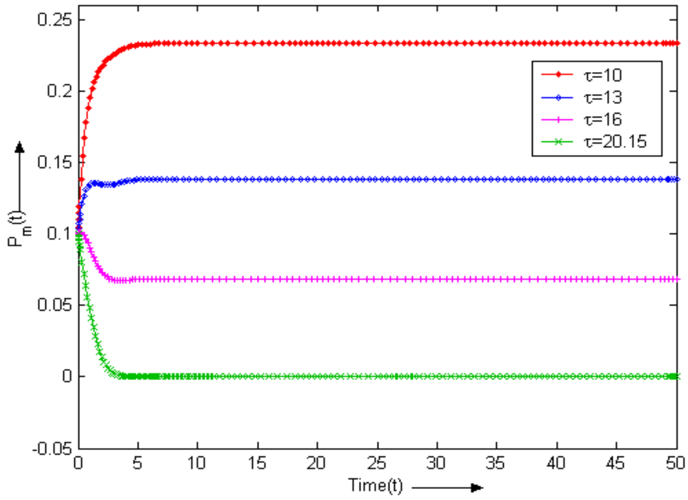

Figure 1. Variation of the mature healthy population with time for different τ and other values of the parameters are the same.

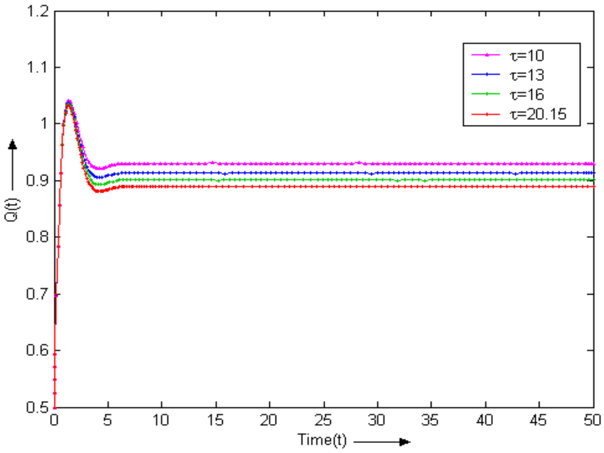

Figure 2. Variation of $Q(t)$ with time for different τ and other values of the parameters are the same.

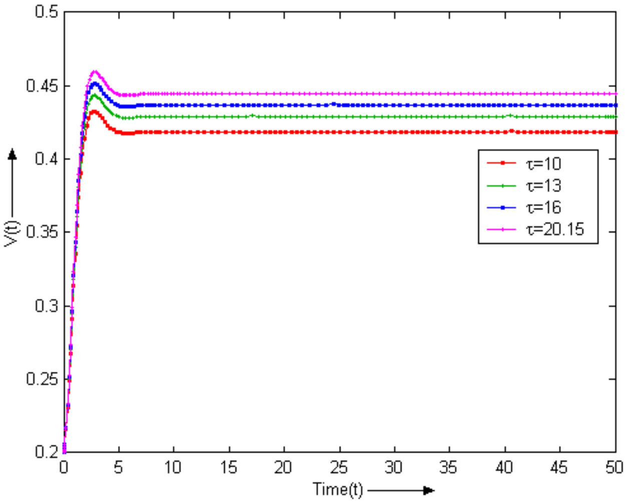

Figure 3. Variation of $V(t)$ with time for different τ and other values of the parameters are the same.

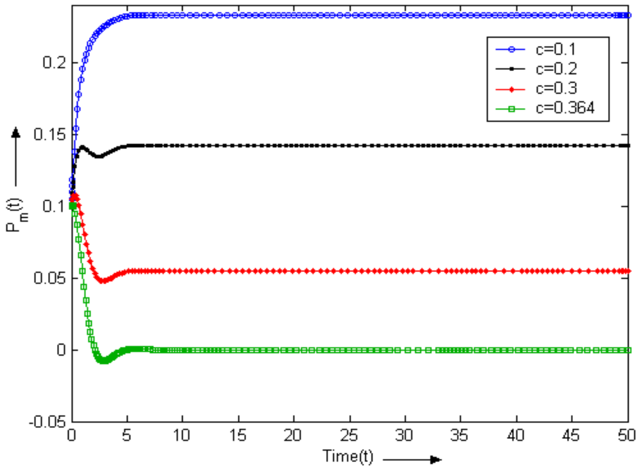

Figure 4. Variation of the mature healthy population $P_m(t)$ with time for different c and other values of the parameters are the same.

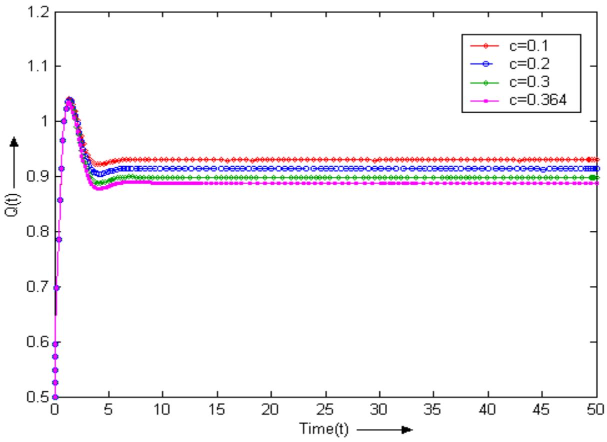

Figure 5. Variation of $Q(t)$ with time for different c and other values of parameters are the same.

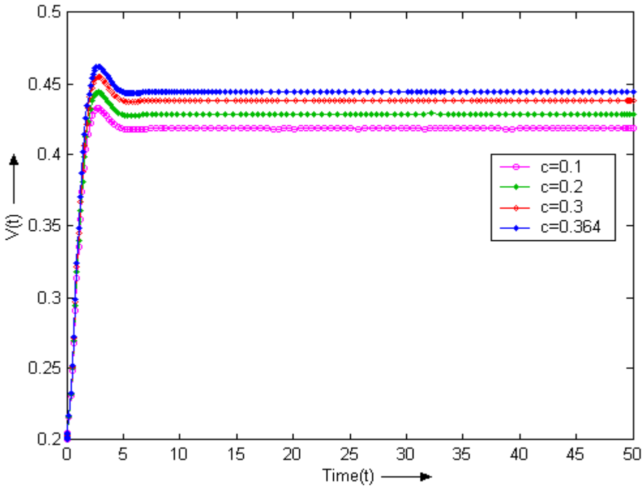

Figure 6. Variation of $V(t)$ with time for different c and other values of parameters are the same.

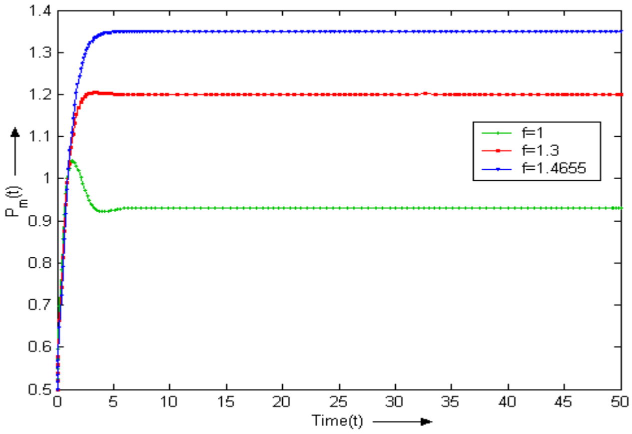

Figure 7. Variation of the mature healthy population $P_m(t)$ with time for different f and other values of the parameters arethe same.

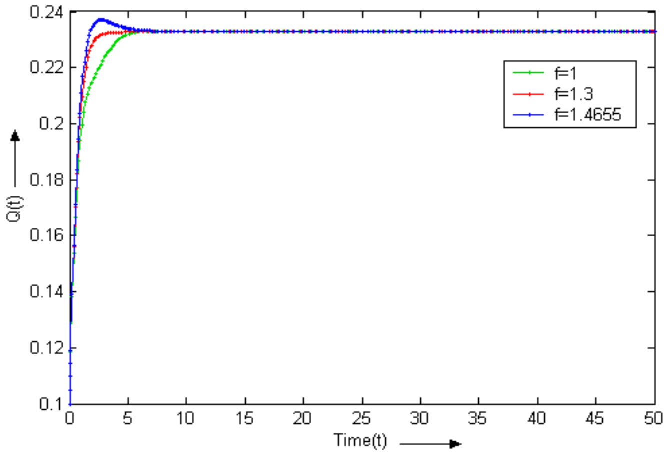

Figure 8. Variation of $Q(t)$ with time for different f and other values of the parameters are the same.

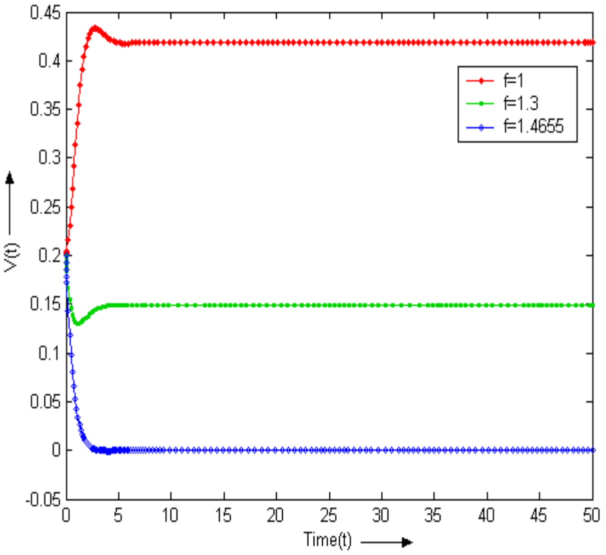

Figure 9. Variation of $V(t)$ with time for different f and other values of the parameters are the same.

9. Conclusion

In this paper we have studied a competition model with infection populations competing for the same resource are analyzed. Where the one competing species is divided into two stages, immature and mature by a constant time delay and the disease spreads only in the other competing species. This system is also analyzed for positivity and boundedness of solutions, equilibria and their stabilities. Conditions that influence the permanence of all populations are given. By Theorem 7.1, the population is permanent provided that

$$\alpha e^{-\gamma\tau} > \left(c+\frac{\eta\delta}{f}\right)\left(\frac{d}{f}\right),\ d > \frac{e\alpha e^{-\gamma\tau}}{\beta}+\frac{(f+\delta)\delta d}{f^2},\ \delta q^* > \frac{g\alpha e^{-\gamma\tau}}{\beta}\left(\frac{\delta d}{f^2}\right)+d.$$

These results indicate that the loss rate, intraspecific coefficient, death rate, and transmission rate of the infection of populations plays an important role for the permanence of the solutions. With the help of computer simulations, it is concluded that if the maturation time increases, then the system is not permanent and mature healthy population tends to extinction. It is also noted that if the value of maturation time increases, the healthy population $Q(t)$ and the infected individuals of the latter species $V(t)$ decreases and increases respectively. Also when the value of parameter c (loss rate) increases mature healthy population tends to extinction. It is also noted that if the value of loss rate increases, healthy population $Q(t)$ and the infected individuals of the latter species $V(t)$ decreases and increases respectively. When the value of the parameter f (intraspecific coefficient) increases, the mature healthy population $P_m(t)$ increases and the infected individuals of the latter species $V(t)$ tends to extinction respectively.

It is also noted that healthy population $Q(t)$ remains constant at its equilibrium level as f increases but the amplitude and timing of the peak decreases with increase in f. It is observed that the parameters c the loss rate of population, τ the maturation time and f the intraspecific coefficient are the key parameters which we need to control to keep the mature healthy population away from extinction and the infected individuals of the latter species from extinction respectively.

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