Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Vol. 10, Issue 2 (December 2015), pp. 1007-1018

Applications and
Applied Mathematics:
An International Journal
(AAM)

Extension Formulas of Lauricella's Functions by Applications of Dixon's Summation Theorem

Ahmed Ali Atash
Department of Mathematics
Aden University, Aden, Yemen
ah-a-atash@hotmail.com

Received: January 20, 2015; Accepted: October 6, 2015

Abstract

The aim of this research paper is to obtain two extension formulas for the first and second kind of Lauricella's functions of three variables with the help of generalized Dixon's summation theorem, which was obtained by Lavoie et al. In addition to this, two extension formulas for the second and third kind of Appell's functions are obtained as a consequence of the above mentioned results . Furthermore, some transformation formulas involving Exton's double hypergeometric series are obtained as an applications of our main results.

Keywords: Pochhammer symbol; Gamma function; generalized hypergeometric function; Lauricella functions; Appell functions; Exton hypergeometric series; Dixon's theorem; Extension formulas

MSC 2010 No. : 33C20, 33C65, 33C70

1. Introduction

In the usual notation, let ${}_pF_q$ denote generalized hypergeometric function of one variable with *p* numerator parameters and *q* denominator parameters, defined by (see Srivastava and Manocha (1984, p.42) and Srivastava and Karlsson (1985, p.19))

$$ {}_pF_q\left[\begin{matrix} a_1,\dots,\ a_p\,; \\ b_1,\dots,\ b_q\,; \end{matrix}\ x\right]=\sum_{n=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\frac{x^n}{n!}, \tag{1.1} $$

where $(a)_n$ denotes the Pochhammer's symbol defined by

$$ (a)_n=\begin{cases} 1, & if\ \ n=0, \\ a(a+1)(a+2)\dots(a+n-1), & if\ \ n=1,2,3,\cdots. \end{cases} \tag{1.2} $$

The series (1.1) is convergent for $|x| < \infty$ if $p \le q$ and for $|x| < 1$ if $p = q + 1$, while it is divergent for all x, $x \neq 0$ if $p > q + 1$.

Furthermore, if we set

$$w = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j \ ,$$

it is known that the series (1.1), with $p = q + 1$, is
(i) absolutely convergent for $|x| < 1$ if $\mathrm{Re}(w) > 0$,
(ii) conditionally convergent for $|x| = 1$, $x \neq 1$ if $-1 < \mathrm{Re}(w) \le 0$ and
(iii) divergent for $|x| = 1$ if $\mathrm{Re}(w) \le -1$.

The Lauricella's functions $F_A^{(n)}$ and $F_B^{(n)}$ are defined and represented as follows (Srivastava and Manocha (1984, p.60))

$$F_A^{(n)}\left(a, b_1, \cdots, b_n ; c_1, \cdots, c_n ; x_1, \cdots, x_n\right)$$

$$= \sum_{m_1, \cdots, m_n = 0}^{\infty} \frac{(a)_{m_1 + \cdots + m_n} (b_1)_{m_1} \cdots (b_n)_{m_n}}{(c_1)_{m_1} \cdots (c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}, \tag{1.3}$$

$$|x_1| + \cdots + |x_n| < 1 ;$$

$$F_B^{(n)}\left(a_1, \cdots, a_n, b_1, \cdots, b_n ; c ; x_1, \cdots, x_n\right)$$

$$= \sum_{m_1, \cdots, m_n = 0}^{\infty} \frac{(a_1)_{m_1} \cdots (a_n)_{m_n} (b_1)_{m_1} \cdots (b_n)_{m_n}}{(c)_{m_1 + \cdots + m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}, \tag{1.4}$$

$$\max\{|x_1|, \cdots, |x_n|\} < 1 .$$

Clearly, we have

$$F_A^{(2)} = F_2 \text{ and } F_B^{(2)} = F_3,$$

where F_2 and F_3 are Appell double hypergeometric functions (Srivastava and Manocha (1984, p.53))

$$F_2\left(a, b_1, b_2 ; c_1, c_2 ; x, y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b_1)_m (b_2)_n}{(c_1)_m (c_2)_n} \frac{x^m}{m!} \frac{y^n}{n!},$$

(1.5)

$$|x_1| + \cdots + |x_n| < 1 ;$$

$$F_3\left(a_1, a_2, b_1, b_2; c; x, y\right) = \sum_{m,n=0}^{\infty} \frac{(a_1)_m (a_2)_n (b_1)_m (b_2)_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{1.6}$$

$$\max\{|x_1|, \cdots, |x_n|\} < 1 .$$

In 1982, Exton (1982, p.137) defined the following double hypergeometric series :

$$X^{A:B;B'}_{C:D;D'}\left[\begin{matrix}(a):(b);(b');\\(c):(d);(d');\end{matrix} x, y\right] = \sum_{m,n=0}^{\infty} \frac{((a))_{2m+n} ((b))_m ((b'))_n}{((c))_{2m+n} ((d))_m ((d'))_n} \frac{x^m \ y^n}{m!\, n!}, \tag{1.7}$$

which is the generalization and unification of Horn's non-confluent double hypergeometric function H_4 (Srivastava and Manocha (1984, p.57)) and Horn's confluent double hypergeometric function H_7 (Srivastava and Manocha (1984, p.57)). For the sake of convenience the symbol $((a))_m$ denotes the product $\prod_{j=1}^{A}(a_j)_m$.

In the theory of hypergeometric and generalized hypergeometric series, Dixon's summation theorem for the sum of a ${}_3F_2(1)$ play an important role. Applications of the above mentioned theorem are now well known (see, for example Bailey (1932), Lavoie et al. (1994), Kim et al. (2010), Lee and Kim (2010), Shekhawat (2012) and Shekhawat and Thakore (2013)). The aim of this paper is to obtain two extension formulas for the Lauricella's functions $F_A^{(3)}$ and $F_B^{(3)}$ with the help of the following generalized Dixon's theorem Lavoie et al. (1994):

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix} a,b,c & ; \\ 1+a-b+i, 1+a-c+i+j; \end{matrix} 1\right] \\
&= \frac{2^{-2c+i+j}\,\Gamma(1+a-b+i)\Gamma(1+a-c+i+j)\Gamma(b-\tfrac{1}{2}|i|-\tfrac{1}{2}i)\Gamma(c-\tfrac{1}{2}(i+j+|i+j|)}{\Gamma(b)\Gamma(c)\Gamma(1+a-2c+i+j)\Gamma(1+a-b-c+i+j)} \\
&\quad \times\left\{A_{i,j}\,\frac{\Gamma(\tfrac{1}{2}a-c+\tfrac{1}{2}+[\tfrac{i+j+1}{2}])\Gamma(\tfrac{1}{2}a-b-c+1+i+[\tfrac{j+1}{2}])}{\Gamma(\tfrac{1}{2}a+\tfrac{1}{2})\Gamma(\tfrac{1}{2}a-b+1+[\tfrac{i}{2}])}\right. \\
&\quad \left. +B_{i,j}\,\frac{\Gamma(\tfrac{1}{2}a-c+1+[\tfrac{i+j}{2}])\Gamma(\tfrac{1}{2}a-b-c+\tfrac{3}{2}+i+[\tfrac{j}{2}])}{\Gamma(\tfrac{1}{2}a)\Gamma(\tfrac{1}{2}a-b+\tfrac{1}{2}+[\tfrac{i+1}{2}])}\right\},
\end{aligned} \tag{1.8}$$

$$(R(a-2b-2c) > -2-2i-j\ ;\ i = -3,-2,-1,0,1,2;\ j = 0,1,2,3)\,,$$

where $[x]$ denotes the greatest integer less than or equal to x and $|x|$ denotes the usual absolute value of x. The coefficients $A_{i,j}$ *and* $B_{i,j}$ are given respectively in (Lavoie et al., 1994) . When $i = j = 0$, (1.11) reduces immediately to the classical Dixon's summation theorem Rainville (1960, p.92); (see also Bailey (1932, p.13))

$$ {}_3F_2\left[\begin{matrix} a,b,c & ; \\ 1+a-b,1+a-c & ; \end{matrix} 1\right] = \frac{\Gamma\left(1+\frac{1}{2}a\right)\Gamma\left(1+a-b\right)\Gamma\left(1+a-c\right)\Gamma\left(1+\frac{1}{2}a-b-c\right)}{\Gamma\left(1+a\right)\Gamma(1+\frac{1}{2}a-b)\Gamma\left(1+\frac{1}{2}a-c\right)\Gamma\left(1+a-b-c\right)} \quad (1.9) $$

$((R(a-2b-2c)>-2)$.

We also require the following well-known identities Srivastava and Karlsson (1985, p.16-17)

$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad a \neq 0,-1,-2,\cdots, \quad (1.10) $$

$$ \Gamma\left(\tfrac{1}{2}\right)\Gamma\left(1+a\right) = 2^a\ \Gamma\left(\tfrac{1}{2}+\tfrac{1}{2}a\right)\Gamma\left(1+\tfrac{1}{2}a\right), \quad (1.11) $$

$$ (a)_{2n} = 2^{2n}\left(\tfrac{1}{2}a\right)_n\left(\tfrac{1}{2}a+\tfrac{1}{2}\right)_n \quad (1.12) $$

$$ \frac{\Gamma(a-n)}{\Gamma(a)} = \frac{(-1)^n}{(1-a)_n} \quad (1.13) $$

$$ (2n)! = 2^{2n}\left(\tfrac{1}{2}\right)_n n! \quad (1.14) $$

$$ (2n+1)! = 2^{2n}\left(\tfrac{3}{2}\right)_n n!. \quad (1.15) $$

2. Main Extension Formulas

In this section, the following extension formulas will be established :

Formula 1.

$$ F_A^{(3)}\left(a,a',b-i,b\,;d,\,c,\,c+i+j;\,x\,,y,-y\right) $$

$$ = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+2n}\ (a')_m\,(b-i)_{2n}\,x^m(-y^2/4)^n}{(d)_m(c)_{2n}\ m!n!} $$

$$ \times\frac{2^{2(2n+c-1)+i+j}\,\Gamma(1-b+i-2n)\ \Gamma(c+i+j)\ \Gamma(b-\frac{1}{2}\left|i\right|-\frac{1}{2}i)\Gamma(1-2n-c-\frac{1}{2}(i+j+\left|i+j\right|}{\Gamma(b)\Gamma(1-c-2n)\Gamma\left(2c-1+i+j+2n\right)\Gamma(c-b+i+j)} $$

$$ \times C_{i,j}\,\frac{\Gamma(n+c-\frac{1}{2}+[\frac{i+j+1}{2}])\Gamma(n-b+c+i+[\frac{j+1}{2}])}{\Gamma(\frac{1}{2})\Gamma(1-b-n+[\frac{i}{2}])} $$

$$ -\frac{ay}{2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a+1)_{m+2n}\ (a')_m\,(b-i)_{2n+1}\,x^m(-y^2/4)^n}{(d)_m\,(c)_{2n+1}\ m!n!} $$

$$\times \frac{2^{2(c+2n)+i+j}\Gamma(i-b-2n)\,\Gamma(c+i+j)\,\Gamma(b-\frac{1}{2}|i|-\frac{1}{2}i)\Gamma(-2n-c-\frac{1}{2}(i+j+|i+j|)}{\Gamma(b)\Gamma(-2n-c)\Gamma\left(2c+i+j+2n\right)\Gamma(c-b+i+j)}$$

$$\times D_{i,j}\,\frac{\Gamma(\frac{1}{2}+n+c+[\frac{i+j}{2}])\Gamma(1+n-b+c+i+[\frac{j}{2}])}{\Gamma(\frac{1}{2})\Gamma(-n-b+[\frac{i+1}{2}])}, \qquad (2.1)$$

for $i=-3,-2,-1,0,1,2$ and $j=0,1,2,3$.

The coefficients $C_{i,j}, D_{i,j}$ can be obtained from the coefficients $A_{i,j}, B_{i,j}$ by replacing a, c by $-2n, 1-c-2n$ and $-2n-1, -c-2n$ respectively .

Formula 2.

$$F_B^{(3)}\left(a, b-i, b, c, d-i-j, d; e; x, y, -y\right)$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_m\,(c)_m\,(b-i)_{2n}\,(d-i-j)_{2n}\,x^m(-y^2/4)^n}{(e)_{m+2n}\;\; m!n!}$$

$$\times \frac{2^{-2d+i+j}\Gamma(1-b+i-2n)\,\Gamma(1-d-2n+i+j)\,\Gamma(b-\frac{1}{2}|i|-\frac{1}{2}i)\Gamma(d-\frac{1}{2}(i+j+|i+j|}{\Gamma(b)\Gamma(d)\Gamma(1-2d+i+j-2n)\Gamma\left(1-b-d+i+j-2n\right)}$$

$$\times G_{i,j}\,\frac{\Gamma(\frac{1}{2}-d-n+[\frac{i+j+1}{2}])\,\Gamma(1-n-b-d+i+[\frac{j+1}{2}])}{\Gamma(\frac{1}{2})\Gamma(1-b-n+[\frac{i}{2}])}$$

$$-\frac{y}{2e}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_m\,(c)_m\,(b-i)_{2n+1}(d-i-j)_{2n+1}x^m(-y^2/4)^n}{(e+1)_{m+2n}\;\; m!n!}$$

$$\times \frac{2^{-2d+i+j}\Gamma(i-b-2n)\,\Gamma(i+j-d-2n)\,\Gamma(b-\frac{1}{2}|i|-\frac{1}{2}i)\Gamma(d-\frac{1}{2}(i+j+|i+j|)}{\Gamma(b)\Gamma(d)\Gamma\left(i+j-2d-2n\right)\Gamma(i+j-b-d-2n)}$$

$$\times H_{i,j}\,\frac{\Gamma(\frac{1}{2}-n-d+[\frac{i+j}{2}])\Gamma(1-n-b-d+i+[\frac{j}{2}])}{\Gamma(\frac{1}{2})\Gamma(-n-b+[\frac{i+1}{2}])}, \qquad (2.2)$$

for $i=-3,-2,-1,0,1,2$ and $j=0,1,2,3$.

The coefficients $G_{i,j}, H_{i,j}$ can be obtained from the coefficients $A_{i,j}, B_{i,j}$ by replacing a by $-2n$ and $-2n-1$, respectively .

Proof of Formula 1. : Denoting the left hand side of (2.1) by S, expanding $F_A^{(3)}$ in a power series and using the results Srivastava and Karlsson (1985, p.17)

$$(a)_{m+n} = (a)_m (a+m)_n \tag{2.3}$$

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A(n,m) = \sum_{m=0}^{\infty} \sum_{n=0}^{m} A(n,m-n)\,, \tag{2.4}$$

we get

$$S = \sum_{m=0}^{\infty} \frac{(a)_m\, (a')_m\, x^m}{(d)_m m!} \sum_{n=0}^{\infty}\sum_{p=0}^{n} \frac{(a+m)_n\ (b-i)_{n-p}(b)_p(-1)^p y^n}{(c)_{n-p}(c+i+j)_p\,(n-p)!\,p!}. \tag{2.5}$$

Next, using the identities Srivastava and Karlsson (1985, p.17)

$$(a)_{m-n} = \frac{(-1)^n (a)_m}{(1-a-m)_n}, 0 \le n \le m \quad \text{and} \quad (m-n)! = \frac{(-1)^n\, m!}{(-m)_n}, 0 \le n \le m\ , \tag{2.6}$$

we get

$$S = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n}\ (a')_m\,(b-i)_n x^m y^n}{(d)_m (c)_n\ m!n!}\ {}_3F_2\left[\begin{matrix} -n\ ,b,1-c-n & ; \\ 1-b-n+i,\ c+i+j & ; \end{matrix}\ 1\right]. \tag{2.7}$$

By the application of generalized Dixon’s theorem (1.8), (2.7) becomes

$$S = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n}\ (a')_m\,(b-i)_n x^m y^n}{(d)_m (c)_n\ m!n!}$$

$$\times \frac{2^{2(n+c-1)+i+j}\,\Gamma(1-b+i-n)\ \Gamma(c+i+j)\ \Gamma(b-\frac{1}{2}|i|-\frac{1}{2}i)\Gamma(1-n-c-\frac{1}{2}(i+j+|i+j|}{\Gamma(b)\Gamma(1-c-n)\Gamma\left(2c-1+i+j+n\right)\Gamma(c-b+i+j)}$$

$$\times\left\{A'_{i,j}A(b,c,n)+B'_{i,j}B(b,c,n)\right\}, \tag{2.8}$$

where

$$A(b,c,n) = \frac{\Gamma(\frac{1}{2}n+c-\frac{1}{2}+[\frac{i+j+1}{2}])\Gamma(\frac{1}{2}n-b+c+i+[\frac{j+1}{2}])}{\Gamma(\frac{1}{2}-\frac{1}{2}n)\Gamma(1-b-\frac{1}{2}n+[\frac{i}{2}])} \tag{2.9}$$

$$B(b,c,n) = \frac{\Gamma(\frac{1}{2}n+c+[\frac{i+j}{2}])\Gamma(\frac{1}{2}n-b+c+\frac{1}{2}+i+[\frac{j}{2}])}{\Gamma(-\frac{1}{2}n)\Gamma(-\frac{1}{2}n-b+\frac{1}{2}+[\frac{i+1}{2}])}. \tag{2.10}$$

The coefficients $A'_{i,j}, B'_{i,j}$ can be obtained from the coefficients $A_{i,j}, B_{i,j}$ by replacing a, c by $-n, 1-c-n$, respectively .

Now, separating (2.8) into even and odd powers, we get

$$S=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+2n}\,(a')_m\,(b-i)_{2n}\,x^m y^{2n}}{(d)_m (c)_{2n}\,m!(2n)!}$$

$$\times\frac{2^{2(2n+c-1)+i+j}\,\Gamma\,(1-b+i-2n)\,\Gamma(c+i+j)\,\Gamma(b-\tfrac{1}{2}|i|-\tfrac{1}{2}i)\Gamma(1-2n-c-\tfrac{1}{2}(i+j+|i+j|}{\Gamma(b)\Gamma(1-c-2n)\Gamma\left(2c-1+i+j+2n\right)\Gamma(c-b+i+j)}$$

$$\times\left\{C_{i,j}A(b,c,2n)+F_{i,j}B(b,c,2n)\right\}$$

$$+\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+2n+1}\,(a')_m\,(b-i)_{2n+1}\,x^m y^{2n+1}}{(d)_m (c)_{2n+1}\,m!(2n+1)!}\right.$$

$$\frac{2^{2(c+2n)+i+j}\,\Gamma\,(i-b-2n)\,\Gamma(c+i+j)\,\Gamma(b-\tfrac{1}{2}|i|-\tfrac{1}{2}i)\Gamma(-2n-c-\tfrac{1}{2}(i+j+|i+j|)}{\Gamma(b)\Gamma(-2n-c)\Gamma\left(2c+i+j+2n\right)\Gamma(c-b+i+j)}$$

$$\left.\times\left\{E_{i,j}\,A(b,c,2n+1)+D_{i,j}\,B(b,c,2n+1)\right\}\right). \tag{2.11}$$

Since $$\frac{1}{\Gamma(-n)}=0,\quad (n=0,1,2,\cdots),$$

where $\Gamma(\,)$ denotes the Gamma function, Srivastava and Karlsson (1985, p.16). We have seen from (2.11) that

$$B(b,c,2n)=A(b,c,2n+1)=0.$$

Therefore, (2.11) becomes

$$S=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+2n}\,(a')_m\,(b-i)_{2n}\,x^m y^{2n}}{(d)_m (c)_{2n}\,m!(2n)!}$$

$$\frac{2^{2(2n+c-1)+i+j}\,\Gamma\,(1-b+i-2n)\,\Gamma(c+i+j)\,\Gamma(b-\tfrac{1}{2}|i|-\tfrac{1}{2}i)\Gamma(1-2n-c-\tfrac{1}{2}(i+j+|i+j|}{\Gamma(b)\Gamma(1-c-2n)\Gamma\left(2c-1+i+j+2n\right)\Gamma(c-b+i+j)}$$

$$\times C_{i,j}\,\frac{\Gamma\,(n+c-\tfrac{1}{2}+[\tfrac{i+j+1}{2}])\Gamma\,(n-b+c+i+[\tfrac{j+1}{2}])}{\Gamma\,(\tfrac{1}{2}-n)\Gamma\,(1-b-n+[\tfrac{i}{2}])}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+2n+1}\,(a')_m\,(b-i)_{2n+1}\,x^m y^{2n+1}}{(d)_m (c)_{2n+1}\,m!(2n+1)!}$$

$$\frac{2^{2(c+2n)+i+j}\,\Gamma(i-b-2n)\,\Gamma(c+i+j)\,\Gamma(b-\frac{1}{2}|i|-\frac{1}{2}i)\Gamma(-2n-c-\frac{1}{2}(i+j+|i+j|)}{\Gamma(b)\Gamma(-2n-c)\Gamma\left(2c+i+j+2n\right)\Gamma(c-b+i+j)}$$

$$\times D_{i,j}\,\frac{\Gamma(c+n+\frac{1}{2}+[\frac{i+j}{2}])\Gamma(1+n-b+c+i+[\frac{j}{2}])}{\Gamma(-\frac{1}{2}-n)\,\Gamma(-n-b+[\frac{i+1}{2}])}. \qquad (2.12)$$

Finally, in (2.12) if we use the results (1.13)-(1.15), then after some simplification we arrive at the right hand side of (2.1). This completes the proof of formula 1. In exactly the same manner, formula 2 can also be proved.

Remark 1.

Taking $x \to 0$ in (2.1) and (2.2), we deduce, respectively, the following extension formulas for Appell's functions F_2 and F_3:

Formula 3.

$$F_2\left(a, b-i, b\,;\, c,\, c+i+j\,;\, y, -y\right)$$

$$=\sum_{n=0}^{\infty}\frac{(a)_{2n}\,(b-i)_{2n}(-y^2/4)^n}{(c)_{2n}\,n!}$$

$$\times\frac{2^{2(2n+c-1)+i+j}\,\Gamma(1-b+i-2n)\,\Gamma(c+i+j)\,\Gamma(b-\frac{1}{2}|i|-\frac{1}{2}i)\Gamma(1-2n-c-\frac{1}{2}(i+j+|i+j|}{\Gamma(b)\Gamma(1-c-2n)\Gamma\left(2c-1+i+j+2n\right)\Gamma(c-b+i+j)}$$

$$\times C_{i,j}\,\frac{\Gamma(n+c-\frac{1}{2}+[\frac{i+j+1}{2}])\Gamma(n-b+c+i+[\frac{j+1}{2}])}{\Gamma(\frac{1}{2})\Gamma(1-b-n+[\frac{i}{2}])}$$

$$-\frac{ay}{2}\sum_{n=0}^{\infty}\frac{(a+1)_{2n}\,(b-i)_{2n+1}(-y^2/4)^n}{(c)_{2n+1}\,n!}$$

$$\times\frac{2^{2(c+2n)+i+j}\,\Gamma(i-b-2n)\,\Gamma(c+i+j)\,\Gamma(b-\frac{1}{2}|i|-\frac{1}{2}i)\Gamma(-2n-c-\frac{1}{2}(i+j+|i+j|)}{\Gamma(b)\Gamma(-2n-c)\Gamma\left(2c+i+j+2n\right)\Gamma(c-b+i+j)}$$

$$\times D_{i,j}\,\frac{\Gamma(\frac{1}{2}+n+c+[\frac{i+j}{2}])\Gamma(1+n-b+c+i+[\frac{j}{2}])}{\Gamma(\frac{1}{2})\,\Gamma(-n-b+[\frac{i+1}{2}])}. \qquad (2.13)$$

Formula 4.

$$F_3\left(b-i, b, d-i-j, d; e; y, -y\right)$$

$$=\sum_{n=0}^{\infty}\frac{(b-i)_{2n}\,(d-i-j)_{2n}(-y^2/4)^n}{(e)_{2n}\;\; n!}$$

$$\times\frac{2^{-2d+i+j}\,\Gamma(1-b+i-2n)\,\Gamma(1-d-2n+i+j)\,\Gamma(b-\tfrac{1}{2}|i|-\tfrac{1}{2}i)\Gamma(d-\tfrac{1}{2}(i+j+|i+j|)}{\Gamma(b)\Gamma(d)\Gamma(1-2d+i+j-2n)\Gamma\left(1-b-d+i+j-2n\right)}$$

$$\times G_{i,j}\,\frac{\Gamma(\tfrac{1}{2}-d-n+[\tfrac{i+j+1}{2}])\,\Gamma(1-n-b-d+i+[\tfrac{j+1}{2}])}{\Gamma(\tfrac{1}{2})\Gamma(1-b-n+[\tfrac{i}{2}])}$$

$$-\frac{y}{2e}\sum_{n=0}^{\infty}\frac{(b-i)_{2n+1}(d-i-j)_{2n+1}(-y^2/4)^n}{(e+1)_{2n}\;\; n!}$$

$$\times\frac{2^{-2d+i+j}\,\Gamma(i-b-2n)\,\Gamma(i+j-d-2n)\,\Gamma(b-\tfrac{1}{2}|i|-\tfrac{1}{2}i)\Gamma(d-\tfrac{1}{2}(i+j+|i+j|)}{\Gamma(b)\Gamma(d)\Gamma\left(i+j-2d-2n\right)\Gamma(i+j-b-d-2n)}$$

$$\times H_{i,j}\,\frac{\Gamma(\tfrac{1}{2}-n-d+[\tfrac{i+j}{2}])\Gamma(1-n-b-d+i+[\tfrac{j}{2}])}{\Gamma(\tfrac{1}{2})\Gamma(-n-b+[\tfrac{i+1}{2}])}\,. \tag{2.14}$$

The coefficients $C_{i,j}, D_{i,j}$ can be obtained from the coefficients $A_{i,j}, B_{i,j}$ by replacing a, c by $-2n, 1-c-2n$ and $-2n-1, -c-2n$ respectively and the coefficients $G_{i,j}, H_{i,j}$ can be obtained from the coefficients $A_{i,j}, B_{i,j}$ by replacing a by $-2n$ and $-2n-1$, respectively .

Remark 2.

It is interesting to mention here that, the results (2.13) and (2.14) are a generalization of the following well-known transformations due to Bailey (1953, p.239) :

$$F_2\left[a, b, b; c, c; y, -y\right] = {}_4F_3\left[\tfrac{1}{2}a, \tfrac{1}{2}a+\tfrac{1}{2}, b, c-b; c, \tfrac{1}{2}c, \tfrac{1}{2}c+\tfrac{1}{2}; y^2\right] \tag{2.15}$$

and

$$F_3\left[b, b, d, d; e; y, -y\right] = {}_4F_3\left[\begin{matrix} b, d, \tfrac{1}{2}(b+d), \tfrac{1}{2}(b+d+1); \\ \tfrac{1}{2}e, \tfrac{1}{2}e+\tfrac{1}{2}, b+d \quad ; \end{matrix}\; y^2\right]. \tag{2.16}$$

3. Applications

Taking $i = j = 0$ in (2.1) and (2.2) and using the results (1.10)-(1.13), we get

$$F_A^{(3)}\left(a, a', b, b; d, c, c; x, y, -y\right)$$

$$=X\,{}^{1:2;1}_{0:3;1}\left[\begin{matrix} a\ : & b,c-b & ;a'; \\ -\ : & c,\ \tfrac{1}{2}c,\tfrac{1}{2}c+\tfrac{1}{2} & ;d\,; \end{matrix}\ \frac{y^2}{4},x\right] \tag{3.1}$$

$F_B^{(3)}\left(a,b,b,c,d,d\ ;e;\ x\ ,y,-y\right)$

$$=X\,{}^{0:4;2}_{1:1;0}\left[\begin{matrix} -\ : & b,d\,,\tfrac{1}{2}(b+d),\tfrac{1}{2}(b+d+1);a,c; \\ e\ : & b+d\qquad ;\ -\ ; \end{matrix}\ 4y^2,x\right], \tag{3.2}$$

respectively .

Further, taking $c=2b$ in (3.1), we get

$F_A^{(3)}\left(a,a',b,b\ ;d,\ 2b,\ 2b;\ x\ ,y,-y\right)$

$$=X\,{}^{1:1;1}_{0:2;1}\left[\begin{matrix} a\ : & b & ;a'; \\ -\ : & 2b,b+\tfrac{1}{2} & ;d\,; \end{matrix}\ \frac{y^2}{4},x\right]. \tag{3.3}$$

Taking $i=1, j=0$ in (2.1) and (2.2) and using the results (1.10)-(1.15), we get

$F_A^{(3)}\left(a,a',b-1,b\ ;d,\ c,\ c+1;\ x\ ,y,-y\right)$

$$=X\,{}^{1:2;1}_{0:3;1}\left[\begin{matrix} a\ : & b,c-b+1 & ;a'; \\ -\ : & c,\tfrac{1}{2}c+\tfrac{1}{2},\tfrac{1}{2}c+1 & ;d\,; \end{matrix}\ \frac{y^2}{4},x\right]$$

$$+\frac{a(b-1)y}{2c}X\,{}^{1:2;1}_{0:3;1}\left[\begin{matrix} a+1\ : & b,c-b+2 & ;\ a'; \\ -\ : & c+1,\tfrac{1}{2}c+1,\tfrac{1}{2}c+\tfrac{3}{2} & ;\ d\,; \end{matrix}\ \frac{y^2}{4},x\right] \tag{3.4}$$

$F_B^{(3)}\left(a,b-1,b,c,d-1,d\ ;e;\ x\ ,y,-y\right)$

$$=X\,{}^{0:4;2}_{1:1;0}\left[\begin{matrix} -\ : & b,d\,,\tfrac{1}{2}(b+d),\tfrac{1}{2}(b+d-1);a,c; \\ e\ : & b+d-1\qquad ;\ -\ ; \end{matrix}\ 4y^2,x\right]$$

$$+\frac{(1-b-d)y}{e}X\,{}^{0:4;2}_{1:1;0}\left[\begin{matrix} -\ : & b,d\,,\tfrac{1}{2}(b+d),\tfrac{1}{2}(b+d+1);a,c; \\ e+1: & b+d-1\qquad ;\ -\ ; \end{matrix}\ 4y^2,x\right], \tag{3.5}$$

respectively.

Further, taking $c=2b-1$ in (3.4), we get

$F_A^{(3)}\left(a,a',b-1,b\ ;d,\ 2b-1,\ 2b;\ x\ ,y,-y\right)$

$$=X\,{}^{1:1;1}_{0:2;1}\left[\begin{matrix} a\ : & b & ;a'; \\ -\ : & 2b-1,\ b+\tfrac{1}{2} & ;d\,; \end{matrix}\ \frac{y^2}{4},x\right]$$

$$+\frac{a(b-1)y}{2(2b-1)} X^{1:1;1}_{0:2;1}\left[\begin{matrix} a+1 & : & b & ; a'; & \frac{y^2}{4}, x \\ - & : & 2b, b+\frac{1}{2} & ; d; & \end{matrix}\right]. \qquad (3.6)$$

Similarly, other results can also be obtained.

4. Conclusion

We conclude our present investigation by remarking that the main results established in this paper can be applied to obtain a large number of transformation formulas for the Lauricella's functions $F_A^{(3)}$ and $F_B^{(3)}$ of three variables and Appell's functions F_2 and F_3 in terms of Exton's double hypergeometric series and generalized hypergeometric function. Further, in the extension formulas (1) and (3), if we take $c=2b$, then we can obtain a new extension formula for Lauricella's function $F_A^{(3)}(a, a', b-i, b; d, 2b, 2b+i+j; x, y, -y)$ and Appell's function $F_2(a, b-i, b; 2b, 2b+i+j; y, -y)$. Also, many special cases of these extension formulas can be obtained in terms of Exton's double hypergeometric series and generalized hypergeometric function.

Acknowledgments

The author is highly grateful to the anonymous referees and the Editor-in-Chief Professor Aliakbar Montazer Haghighi for useful comments and suggestions towards the improvement of this paper.

REFERENCES

Bailey, W. N. (1932). *Generalized Hypergeometric Series*, Cambridge University Press, Cambridge.

Bailey, W. N. (1953). On the sum of terminating ${}_3F_2(1)$. *Quart. J. Math. Oxford,* (2) 4: 237-240.

Exton, H. (1982). Reducible double hypergeometric functions and associated integrals. *An Fac. Ci. Univ. Porto*, 63(1-4) : 137-143.

Kim, Y.S., Rathie, A.k. and Choi, J. (2010). Summation formulas derived from the Srivastava's triple hypergeometric series H_C. *Commun. Korean Math. Soc.*, 25(2): 185-191.

Lavoie, J. L., Grondin, F., Rathie, A.K. and Arora, K. (1994). Generalizations of Dixon's theorem on the sum of a ${}_3F_2$. Math. Comp. 62(205) : 267-276.

Lee, S. W. and Kim, Y. S. (2010). An extension of the triple hypergeometric series by Exton. *Honam Mathematical* J., 32(1) : 61-71.

Rainville, E.D. (1960). *Special Functions*, The Macmillan Company, New York.

Shekhawat, N. (2012). On two summations due to Ramanujan and their generalization. *Advances in Computational Mathematics and its Applications,* 2(1) : 237-238 Copyright ©World Science Publisher, United States www.worldciencepuplisher.org.

Shekhawat, N. and Thakore, V. (2013). On one of summations due to Ramanujan and its Generalization. *Advances in Computational Mathematics and its Applications,* 2(4) : 339-34. Copyright © World Science Publisher United States www.

worldciencepuplisher.org .
Srivastava, H.M. and Karlsson, P.W. (1985). *Multiple Gaussian Hypergeometric Series*, Halsted Press, New York .
Srivastava, H.M. and Manocha, H.L. (1984). *A treatise on Generating Functions*, Hasted Press, New York .