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Applications of Composite Convolution Operators

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Abstract

The Composite Convolution Operator is an operator which is obtained by composing Convolution operator with Composition operator. Volterra composite convolution operator is a composition of Volterra convolution operator and Composition operator. The Composite Convolution Operators and Composite Convolution Volterra operators have been defined by using the Expectation operator and Radon-Nikodym derivative. In this paper an attempt has been made to investigate applications of Composite Convolution Operators (CCO) in Integral Convolution Type Equations (ICTE). The study may explore a new technique to solve Fredholm Convolution type integral equations and Volterra Convolution type integral equations. Some methods for solving integral convolution type equations by using Composite Convolution Operators have also been studied. For integral convolution type equations, theorems on existence, uniqueness and estimates for solution have also been proved without any restriction for the parameter. In order to determine the solution by the method of successive approximations in this paper, I have made use of the concept of the Resolvent Kernel to obtain Neumann Series. The Banach Contraction Principle has also been used to obtain some results. The method of Variational Iteration has been applied to find out the approximate solution of integral equations by using Composite Convolution Operators. In this paper Numerical Methods have also been adopted for solution of these integral equations. Fourier transform has been used to solve Integral convolution type equations and Laplace transform has been applied to solve Volterra convolution type equations.

Keywords: Composite Convolution Operator; Integral Convolution Type Equations; Expectation Operator and Radon-Nikodym Derivative

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1. Introduction

Let (X, Ω, μ) be a σ-finite measure space. For each $f \in L^p(\mu)$, $1 \leq p < \infty$, there exists a unique $\phi^{-1}(\Omega)$ measurable function $E(f)$ such that

$$\int gf\, d\mu = \int gE(f)\, d\mu$$

for every $\phi^{-1}(\Omega)$ measurable function g for which left integral exists. The function $E(f)$ is called conditional expectation of f with respect to the sub- algebra $\phi^{-1}(\Omega)$. For more details about expectation operator, one can refer to Parthasarthy (1977). Given $f, g \in L^2(R)$, then the convolution of f and g, $f{*}g$ is defined by

$$f{*}g(x) = \int g(x-y)\, f(y) d\mu,$$

where g is fixed, $k(x,y) = g(x-y)$ is a convolution kernel, and the integral operator defined by

$$I_k f(x) = \int k(x-y)\, f(y) d\mu(y)$$

is known as convolution operator. Suppose $\phi : [0,1] \rightarrow [0,1]$ is a measurable transformation, then

$$I_{k\,\phi} f(x) = \int k(x-y)\, f(\phi(y)) = \int k_\phi(x-y)\, f(y) d\mu(y)$$

is known as composite convolution operator (CCO) induced by pair (k,ϕ), where

$$k_\phi(x-y) = E(f_o(y)k(x-y)\phi^{-1}(y)).$$

The adjoint of composite convolution operator $I_{k,\phi}$ is an integral operator induced by the kernel k^*_ϕ and

$$I^*_{k,\phi} f(x) = \int k^*_\phi(x-y)\, f(y) d\mu(y),$$

where $k^*_\phi(x-y) = \overline{k_\phi(y-x)}$. Also,

$$I^n_{k,\phi} f(x) = \int k^n(x-y)\, f(\phi(y)) d\mu(y) = \int k^n_\phi(x-y)\, f(y) d\mu(y),$$

where kernel k^n_ϕ is defined as $k^n_\phi(x-y) =$

$$\iiint \ldots \int k_\phi(x-z_1)k_\phi(z_1-z_2)k_\phi(z_2-z_3)\ldots k_\phi(z_{n-1}-y)d\mu(z_1)d\mu(z_2)d\mu(z_3)\ldots d\mu(z_{n-1}). \qquad (1.1)$$

The integral operators, and in particular convolution operators, have been studied extensively over the last few decades. For more details about composition operators, integral operators, convolution operators and composite integral operators, I have referred to Singh and Manhas (1993), Halmos and Sunder (1978), Stepanov [(1978) and (1980)] Gupta and Komal (2011), Gupta [(2014) and (2015)]. Whitley (1987) established the Lyubic's conjecture (1984) and generalized it to Volterra composition operators on $L^p[0,1]$.

An integral equation can be defined as equation which is a result of transformation of points in a given vector space of integrable functions by the use of certain specific integral operators to points in the same space. A computational approach for solving integral equation is an essential work in scientific research. For integral convolution type equations, theorems on existence, uniqueness and estimates for solution in $L^p[0,1]$, $1 \leq p < \infty$, were proved without any restriction for the parameter λ. In recent years, many different methods have been used to approximate the solution of integral equations and Volterra integral equations of convolution type by Srivastava and Buschman (1992), Estrada and Kanwal (2000), Kathe and Puri (2002), Mishra (2007), Mishra and Mishra (2012), Mishra, Khatri and Mishra (2013), Deepmala (2014), Deepmala and Mishra (2015) and Saeedi, Tari and Momeni Masuleh (2013). This paper is an extension of Gupta [(2014) and (2015)].

Here, I recall some basic notion in operator theory. Let H be a Hilbert space and $B(H)$ be the algebra of all bounded linear operators acting on H. Let $L^2(\mu)$ consist of all measurable functions $f : X \to R$ (or C) such that $(\int |f(x)|^2\, d\mu)^{½} d\mu < \infty$. The space $L^2(X, S, \mu)$ is a Banach space under the norm defined by

$$||f|| = (\int |f(x)|^2\, d\mu)^{½} .$$

Also, $L^2(\mu)$, the space of square-integrable functions, is a Hilbert space. The Laplace transform of a function $f(t)$ is defined as

$$L\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt .$$

The inverse of Laplace transform $L^{-1}\{F(s)\} = f(t)$.

A function $\hat{F}(w)$ is defined as the Fourier transform of $f(x)$ if

$$\hat{F}(w) = \hat{F}\ \{f(t)\} = \int f(t)e^{iwt}\, dt$$

exists, and

$$(\hat{F})^{-1}\{\hat{F}(w)\} = \frac{1}{2\pi} \int e^{iwx}\ \hat{F}\ (w)dw$$

is called the inverse Fourier transform of $\hat{F}(w)$.

In this paper, the prime focus is to solve Integral Convolution Type Equations (ICTE) using Composite Convolution Operators (CCO) by applying different techniques.

2. Methods to solve ICTE using CCO

In this section, an attempt has been made to use Composite Convolution Operators to solve the integral equations. The Banach Contraction Principle has been used to obtain the results. The Variational Iteration method has been successfully applied to find the approximate solution of integral equation using composite convolution operators.

Theorem 2.1.

Let $k_\phi \in L^2(\mu \times \mu)$. Suppose $I_{k,\phi} \in B(L^2[0,1))$. If λ is a complex number such that $|\lambda| K_\phi < 1$, then for any $g \in L^2[0,1]$ there exists a unique $f \in L^2[0,1]$ such that $(I - \lambda I_{k,\phi})f = g$, where I is the identity operator and $K_\phi = (\iint_0^1 |k_\emptyset(x-y)|^2 d\mu(x) d\mu(y))^{½}$.

Proof:

Let $A \in B(L^2[0,1])$ be defined as $A(f) = \lambda I_{k,\phi} f + g$, for any $g \in L^2[0,1]$. To show A is a contraction, we have

$$\|Af - Af_o\| = \|\lambda I_{k,\phi} f - \lambda I_{k,\phi} f_o\|,$$
$$\leq |\lambda| (\iint_0^1 |k_\emptyset(x-y)|^2 d\mu(x) d\mu(y))^{½} (\iint_0^1 |f(y) - f_0(y)|^2 d\mu(y))^{½} .$$

Then,

$$\|Af - Af_o\| \leq d \|f - f_0\|,$$

using the given condition $|\lambda| K_\phi < 1$, where $0 \leq d = |\lambda| K_\phi < 1$. Hence, A is a contraction. Then, A has a unique fixed point, say g, by Banach Contraction Principle. That is, $Af = f$ has a unique solution, and therefore,

$$(I - \lambda I_{k,\phi})f = g.$$

Theorem 2.2.

If $k_\phi \in L^2(\mu \times \mu)$ and $g \in L^2[0,1]$, then the Fredholm ICTE of the second kind

$$f(x) = g(x) + \lambda \int_0^1 k_\emptyset(x-y) f(y) d\mu(y) = g + \lambda I_{k,\phi} f . \tag{2.1}$$

has unique solution $f \in L^2[0,1]$ for sufficiently small values of scalar λ .

Proof:

For every $f \in L^2[0,1]$, define

$$T : L^2[0,1] \to L^2[0,1] \text{ as } Tf = h \text{ for each } h \in L^2[0,1],$$

where

$$h(x) = g(x) + \lambda \int_0^1 k_\emptyset(x-y)f(y)d\mu(y).$$

To show $\psi(x) \in L^2[0,1]$, where

$$\psi(x) = \int_0^1 k_\emptyset(x-y)f(y)d\mu(y)$$

for every $f \in L^2[0,1]$. Now,

$$\left| \int_0^1 k_\emptyset(x-y)f(y)d\mu(y)\right| \le \left(\int_0^1 |k_\emptyset(x-y)|^2\, d\mu(y)\right)^{½}\left(\int_0^1 |f(y)|^2 d\mu(y)\right)^{½}.$$

(by using Cauchy-Schwartz's inequality)

Therefore,

$$\int_0^1 |\psi(x)|^2 dx \le \iint_0^1 |k_\emptyset(x-y)|^2\, d\mu(y)d\mu(x) \iint_0^1 |f(y)|^2\, d\mu(y)d\mu(x) < \infty,$$

since $k_\phi \in L^2(\mu \times \mu)$ and $f \in L^2[0,1]$. Thus, $\psi(x) \in L^2[0,1]$.

To show T is a contraction mapping, we have

$$\| T f - Tf_o \| = \| h - h_o \|,$$

And

$$\| h - h_o \| = \left\| \lambda \int_0^1 k_\emptyset\, (x-y)[f(y) - f_0(y)]d\mu(y)\right\| ,$$
$$\le |\lambda| \left(\iint_0^1 |k_\emptyset(x-y)|^2\, d\mu(y)d\mu(x)\right)^{½} \iint_0^1 |f(y) - f_0\,(y)|^2\, d\mu(y))^{½}.$$

If

$$|\lambda| < \frac{1}{\left(\iint_0^1 |k_\emptyset(x-y)|^2\, d\mu(y)d\mu(x)\right)^{½}}.$$

Then,

$$\| Tf - Tf_o \| \leq M \| f - f_0 \|,$$

where

$$0 \leq M = [|\lambda|(\iint_0^1 |k_\emptyset(x-y)|^2 \, d\mu(y) d\mu(x))^{½}] < 1.$$

This proves that T is a contraction and hence it has a unique fixed point by Banach Contraction Principle, say $f^* \in L^2[0,1]$, i.e. $Tf^* = f^*$. Thus, f^* is a unique solution of Equation (2.1).

Theorem 2.3.

Let

$$k_\phi \in L^2(\mu \times \mu) \text{ and } g \in L^2[0,1].$$

Then, the solution of integral Equation (2.1) is given by

$$f(x) = g(x) + \lambda \int_0^1 \Gamma_{\lambda,\emptyset}(x-y) f(y) d\mu(y). \tag{2.2}$$

where $\lambda \in C$ is a parameter, the resolvent kernel

$$\Gamma_{\lambda,\phi}(x-y) = \sum_{m=1}^{\infty} \lambda^{m-1} k_\emptyset^m(x-y)$$

and iterated kernel $k_\phi^m(x\text{-}y)$ is defined in Equation (1.1).

Proof:

To determine the solution by the method of successive approximation and obtain Neumann series, assume that the kernel k_ϕ is a bounded function in the square

$$[0,1] = \{ (x,y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\},$$

so that $|k_\phi(x,y)| \leq C$ for all $x,y \in [0,1]$. Also,

$$B_1 = \int_0^1 |k_\emptyset(x-y)|^2 d\mu(y).$$

Suppose zero-order approximation is given by $f_0(x) = g(x)$. When this value is substituted into the right side of Equation (2.1), we get first-order approximation

$$f_1(x) = g(x) + \lambda \int_0^1 k_\emptyset(x-y) f_0(y) d\mu(y).$$

This function, when substituted into the right side of Equation (2.1) yields second-order approximation

$$f_2(x) = g(x) + \lambda\int_0^1 k_\emptyset(x-y)f_1(y)d\mu(y).$$

Continuing the above process, the $(n+1)^{th}$ approximation can be obtained by substituting the n^{th} approximation in the right side of Equation (2.1).

The general recurrence relation has the form

$$f_{n+1}(x) = g(x) + \lambda\int_0^1 k_\emptyset(x-y)f_n(y)d\mu(y).$$

If $f_n(x)$ tends uniformly to a limit as $n \to \infty$, then a limit is required solution. For this, consider first- and second-order approximations

$$f_1(x) = g(x) + \lambda\int_0^1 k_\emptyset(x-y)f_0(y)d\mu(y),$$

and

$$f_2(x) = g(x) + \lambda\int_0^1 k_\emptyset(x-y)[g(y) + \lambda\int_0^1 k_\emptyset(y-z)g(z)d\mu(z)]\, d\mu(y)$$
$$= g(x) + \lambda\int_0^1 k_\emptyset(x-y)g(y)d\mu(y) + \lambda^2 \iint_0^1 k_\emptyset(x-y)k_\emptyset(y-z)d\mu(y)g(z)d\mu(z).$$

$$f_2(x) = g(x) + \lambda\int_0^1 k_\emptyset(x-y)g(y)d\mu(y) + \lambda^2 \iint_0^1 k_\emptyset^2(x-z)g(z)d\mu(z).$$

By changing order of integration and setting

$$k_\phi^2(x\text{-}y) = \int_0^1 k_\emptyset(x-z)k_\emptyset(z-y)d\mu(z).$$

The third approximation is given by

$$\mathrm{f}_3(\mathrm{x}) = \mathrm{g}(\mathrm{x}) + \lambda\int_0^1 k_\phi(x-y)g(y)d\mu(y) + \lambda^2\int_0^1 k_\phi^2(x-y)g(y)d\mu(y) + \lambda^3\int_0^1 k_\phi^3(x-y)g(y)d\mu(y),$$

where

$$k_\phi^3(x\text{-}y) = \int_0^1 k_\emptyset(x-z)k_\emptyset^2(z-y)d\mu(z).$$

By repeating this process, the n^{th} approximate solution of Equation (2.1) is

$$f_n(x) = g(x) + \sum_{m=1}^n \lambda^m \int_0^1 k_\emptyset^m(x-y)g(y)d\mu(y).$$
$$f(x) = \lim_{n\to\infty} f_n(x) = g(x) + \sum_{m=1}^\infty \lambda^m \int_0^1 k_\emptyset^m(x-y)g(y)d\mu(y). \quad (2.3)$$

To find the condition under which the series (2.3) converges, we have

$$|\int_0^1 k_\emptyset^m(x-y)g(y)d\mu(y)|^2 \leq \int_0^1 k_\emptyset^m\,(x-y)|^2 d\mu(y) \int_0^1 |g(y)|^2\, d\mu(y).$$

Hence,

$$|\int_0^1 k_\emptyset^m(x-y)g(y)d\mu(y)|^2 \leq B_m D^2,$$

Where

$$B_m = \int_0^1 k_\emptyset^m\,(x-y)|^2 d\mu(y) \text{ and } D^2 = \int_0^1 |g(y)|^2\, d\mu(y).$$

Again,

$$|\, k_\phi^m\,(x-y)\,|^2 \leq \int_0^1 k_\emptyset^m\,(x-z)|^2 d\mu(z) \int_0^1 |k_\emptyset(z-y)|^2\, d\mu(z),$$

$$\int_0^1 k_\emptyset^m\,(x-y)|^2 d\mu(y) \leq B_{m-1}\, M^2,$$

where

$$M = \iint_0^1 |k_\emptyset(z-y)|^2 d\mu(z) d\mu(y).$$

Thus,

$$|\int_0^1 k_\emptyset^m(x-y)g(y)d\mu(y)|^2 \leq B_1\, D^2\, M^{2m-2}.$$

The infinite series (2.3) converges uniformly if $|\lambda|\, M < 1$. For a given λ, it trivially follows that Equation (2.1) has a unique solution. That is,

$$f(x) = g(x) + \lambda\int_0^1 \sum_{m=1}^{\infty} \lambda^{m-1}\, k_\emptyset^m(x-y) f(y) d\mu(y).$$

$$f(x) = g(x) + \lambda \int_0^1 \Gamma_{\lambda,\emptyset}(x-y) f(y) d\mu(y),$$

where Resolvent Kernel is given by

$$\Gamma_{\lambda,\phi}(x-y) = \sum_{m=1}^{\infty} \lambda^{m-1}\, k_\emptyset^m(x-y).$$

Theorem 2.4.

Let

$k_\phi \in L^2(\mu\times\mu)$ and $I_{k,\phi} \in B(L^2(\mu))$.

Then, for any $g \in L^2[0,1]$ and $\lambda \in C$ there exists a unique $f \in L^2[0,1]$ such that solution of integral Equation (2.1) is given by

$$f = g + \sum_{n=1}^{\infty} \lambda^n \, I_{k,\emptyset}^n \, g, \tag{2.4}$$

where $I_{k,\emptyset}^n$ is a composite convolution operator with kernel k_ϕ^n as defined in (1.1).

Proof:

Define a mapping T: $L^2[0,1] \to L^2[0,1]$ as

$$Tf \;=\; g + \lambda\, I_{k,\phi} f\,.$$

Then, the goal is to prove that T^n is a contraction for some $n \in N$.

Now,

$$\begin{aligned} T^n f &= g + \lambda\, I_{k,\phi} f + \lambda^2\, I_{k,\phi}^2\, g + \ldots + \lambda^n\, I_{k,\phi}^n\, g, \\ &= \lambda^n\, I_{k,\phi}^n\, g + \sum_{m=1}^{n-1} \lambda^m\, I_{k,\emptyset}^m\, g. \end{aligned}$$

Then, for $n \geq 2$, we have

$$\|\, T^n f_1 - T^n f_2 \,\|^{2} \leq \frac{|\lambda|^{2n}}{(n-1)!}\; \|\, k_\phi^n \,\|^2 \, \|\, f_1 - f_2 \,\|^2\,.$$

Thus, there exists $m \in N$ such that T^m is a contraction. Therefore, $Tf = f$ has a unique solution by using Banach contraction theorem. Hence,

$$f = \lim_{n\to\infty} T^n f.$$

3. Numerical methods to solve ICTE

Method –I: Laplace Transform.

In this section Laplace transform has been used to solve Volterra convolution type equation VCTE. To find the solution of VCTE, first obtain the Laplace Transform of the problem and then find the inversion of Laplace transform.

Suppose the VCTE of first kind is given by

$$f(x) = \int_0^x k_\phi(x-y)g(y)d\mu(y), \tag{3.1}$$

where $k_\phi(x- y)$ is a convolution kernel and it depends only on difference $(x$-$y)$. The solution of Equation (3.1) can be obtained by using Laplace transform.

That is,

$$F(s) = G(s)K(s),$$

$$G(s) = \frac{F(s)}{K(s)},$$

and the solution can be obtained by inversion. This method is also applicable to the VCTE of second kind,

$$g(x) = f(x) + \int_0^x k_\phi(x-y)g(y)d\mu(y), \tag{3.2}$$

$$G(p) = F(p)+ K(p)\, G(p).$$

That is,

$$G(p) = \frac{F(p)}{1-K(p)}.$$

Method –II: Fourier Transform:

It is easy to see that Fourier transform has also been used to solve ICTE. To find the solution of ICTE, first obtain the Fourier Transform of the Equation (2.1).

$$f(x) = g(x) + \lambda\int_0^x k_\phi(x-y)g(y)d\mu(y).$$

Applying Fourier transform on both sides, we obtain

$$\hat{f} = \lambda\, (2\pi)^{1/2}\, \hat{I}_{k,\emptyset}\, \hat{f} + \hat{g},$$

where $\hat{I}_{k,\emptyset}$ is a Fourier transform of CCO $I_{k,\phi}$. Then, the solution of above equation is given by taking the inverse Fourier transform, that is,

$$f(x) = Ғ\left(\frac{\hat{g}}{1-\lambda(2\pi)^{½}\hat{I}_{k,\emptyset}}\right),$$

$$f(x) = \int e^{iw}\,\frac{\hat{g}(w)}{1-\lambda(2\pi)^{½}\hat{I}_{k,\emptyset}(w)}\,dw.$$

4. Conclusion

Composite Convolution Operators are defined by using expectation operator and Radon-Nikodym Derivative. In this paper, Fubini theorem is used frequently. The definition of Composite Convolution Integral Operator is motivated by Whitley, wherein he composed Composition Operator with Volterra integral operator to study Lyubic's conjecture. The Composite Convolution Operators have immense applications in dynamical systems as well as in the theory of integral and partial differential equations.

The study may be useful for investigating new technique to solve integral equations and Volterra integral equations. In this paper major results/findings have been successfully achieved in deriving applications of Composite Convolution Operators (CCO) in integral convolution type equations (ICTE). The Methods for solving ICTE using CCO were studied and obtained. The results on existence, uniqueness and estimates of integral convolution type equations have been obtained and solution of ICTE has also been proved without any restriction for the parameter. The method of successive approximations has also been used to determine the solution of integral convolution type equations (ICTE). The Variational Iteration method has been applied to find out the approximate solution of integral equations using Composite Convolution Operators. In this paper Numerical Methods are also adopted for solution of these integral equations. Fourier transform has been used to solve integral convolution type equations and Laplace transform has been applied to solve Volterra convolution type equations.

One can study other Numerical methods like Inversion method, Wavelet method, Matrix method to solve Integral Convolution Type Equations (ICTE). We can extend application of Integral Convolution Type Equations (ICTE) to partial differential equations. The study may open new horizons to solve integral equations and Volterra integral equations.

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