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An Approximate Solution of the Mathieu Fractional Equation by Using the Generalized Differential Transform Method (Gdtm)

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Abstract

The generalized differential transform method (GDTM) is a powerful tool for solving fractional equations. In this paper we solve the Mathieu fractional equation by this method. The approximate solutions obtained are compared with the exact solution. We also show that if both differential orders decrease, we can still have an approximate solution in the different interval of p.

Keywords: Generalized differential transform method, Caputo derivative, Generalized Taylor's formula, Mathieu equation

MSC 2010: 34A08; 34B30; 35E15

1. Introduction

In the last few years fractional differential equations have been used in various sciences, especially in physics and mechanics. Odibat et al. (2008b) present the method of generalized differential transform (GDTM). This method is based on the Caputo fractional differential and the generalized Taylor's formula, Caputo (1967). The Caputo differential operator first computes the standard differential of the function and then evaluate the fractional integral to determine a proper order of the fractional derivative. Hardy (1945) had already written a formal version of the generalized Taylor's formula as

$$f(x+h)=\sum_{m=-\infty}^{\infty}\frac{h^{m+r}}{\Gamma(m+r+1)}(I^{m+r}f)(x), \tag{1}$$

where I^{m+r} is the Riemann–Liouville fractional integral of order $m+r$. Later, under certain conditions for f and $\alpha\in[0,\,1]$, Trujillo et al. (1999) introduced the following generalized Taylor's formula

$$f(x)=\sum_{j=0}^{n}\frac{c_j(x-a)^{(j+1)\alpha-1}}{\Gamma((j+1)\alpha)}+R_n(x,a); \tag{2}$$

with

$$R_n(x,a)=\frac{\widehat{D}^{(n+1)\alpha}f(\xi)}{\Gamma((n+1)\alpha+1)}(x-a)^{(n+1)\alpha}, \quad a\le\xi\le x, \tag{3}$$

$$c_j=\Gamma(\alpha)((x-a)^{1-\alpha}(\widehat{D}^{j\alpha}f))(a+), \quad j=0,1,...,n \tag{4}$$

where $\widehat{D}^{jk}$ is the Riemann–Liouville fractional derivative of order jk. This fractional derivative operator is defined for $\alpha>0,\ \alpha\in R$ and $x>a$ as follows

$$(\widehat{D}^{\alpha}f)(x)=\frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_0^x(x-\tau)^{m-\alpha-1}f(\tau)d\tau, \tag{5}$$

where $m-1<\alpha\le m$ and the sequential fractional derivative is denoted by

$$\widehat{D}^{n\alpha} = \widehat{D}^{\alpha}.\,\widehat{D}^{\alpha}.\cdots.\,\widehat{D}^{\alpha}\ \text{(n-times)}. \tag{6}$$

Odibat et al. (2007) presented a new generalized Taylor's formula as

$$f(x)=\sum_{i=0}^{n}\frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)}D^{i\alpha}f(a)+\frac{D^{(n+1)\alpha}f(\xi)}{\Gamma((n+1)\alpha+1)}(x-a)^{(n+1)\alpha}, \tag{7}$$

with $x\in(a,b]$,

where $0 < \alpha \leq 1$, $a \leq \xi \leq x$ and $D^{i\alpha}$ is the Caputo fractional derivative of order $i\alpha$. This derivative is given in Definition 2.2. For $\alpha = 1$, the generalized Taylor's formula reduces to the classical Taylor's formula. The radius of convergence, R, for the generalized Taylor's series

$$\sum_{i=0}^{\infty} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha} f(a), \tag{8}$$

depends on $f(x)$ and a, and is given by

$$R = |x-a|^{\alpha} \lim_{n\to\infty} \left| \frac{\Gamma(n\alpha+1)}{\Gamma((n+1)\alpha+1)} \cdot \frac{D^{(n+1)\alpha} f(a)}{D^{n\alpha} f(a)} \right|. \tag{9}$$

The concept of differential transform was first introduced by Zhou (1986) who solved linear and non-linear initial value problems in electric circuit analysis. The Mathieu differential equation is:

$$D^2 y + [p - 2q\cos(2x)]y = 0, \tag{10}$$

where p and q are the characteristic number and parameter, respectively. It is a linear differential equation with variable coefficients, which commonly occurs in nonlinear vibration problems in two different ways: (I) in systems in which there is periodic forcing and (II) in stability studies of periodic motions in nonlinear autonomous systems. This equation can be written generally as

$$D^{\beta} y + [p - 2q\cos(2x)]y + cD^{\alpha} y = 0, \tag{11}$$

where D^{α} and D^{β} are Caputo derivative operators and $0 < \alpha \leq 1$, $0 < \beta \leq 2$. In the case when $\alpha = 1$, $\beta = 2$, Equation (11) represents the familiar damped Mathieu equation. We refer the interested reader to Rand (1996, 2007), Rand et al. (2010). Also the stability analysis of distributed order fractional differential equations has been recently investigated by Saberi Najafi et al. (2011). The structure of the paper is organized as follows:

Some definitions and theorems are presented in section 2. In section 3 the Mathieu equation is introduced and solved by using the GDTM. Finally in section 4 concluding remarks are made.

2. Definitions and Theorems

In this section we provide some important definitions and theorems.

Definition 2.1. The Riemann-Liouville integral operator is defined by

$$\begin{aligned} I^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau)\, d\tau, \\ I^0 f(x) &= f(x), \end{aligned} \tag{12}$$

where $\alpha \in R^{+}$, $a \geq 0$ and $x > a$.

Definition 2.2. The Caputo fractional derivative of order α, is defined by

$$D^{\alpha} f(x) = I^{m-\alpha} \frac{d^{m}}{dx^{m}} f(x) = \frac{1}{\Gamma(-\alpha+m)} \int_{a}^{x} (x-\tau)^{-\alpha+m-1} f^{(m)}(\tau) d\tau, \tag{13}$$

where $m-1 < \alpha \leq m$, $m \in Z^{+}$. For more information one may refer to Kilbas et al. (2006), Podlubny (1999), Oldham and Spanier (1974).

Definition 2.3. The differential transform of function $f(x)$ is defined by Arikoglu and Ozkol, (2006), Zhou (1986) is

$$F(k) = \frac{1}{k!} \frac{d^{k} f(x)}{dx^{k}} \bigg|_{x=x_0}, \tag{14}$$

where $f(x)$ is the original function and $F(k)$ is the transformed function.
The differential inverse transform of $F(k)$ is

$$f(x) = \sum_{k=0}^{\infty} x^{k} F(k). \tag{15}$$

If we substitute Equation (14) into Eq. (15), we get

$$f(x) = \sum_{k=0}^{\infty} \frac{x^{k}}{k!} \frac{d^{k} f(x)}{dx^{k}} \bigg|_{x=x_0}. \tag{16}$$

The concept of differential transform arises in the Taylor series expansion.

Definition 2.4. We define the generalized differential transform for the k^{th} derivative of a function $f(x)$ as follows

$$F_{\alpha}(k) = \frac{1}{\Gamma(\alpha k+1)} \left[(D^{\alpha})^{k} f(x) \right]_{x=x_0}, \tag{17}$$

where $0 < \alpha \leq 1$ and

$$(D^{\alpha})^{k} = D^{\alpha}.D^{\alpha}.\cdots.D^{\alpha} \text{ (k-times)}. \tag{18}$$

Also, the differential inverse transform of $F_{\alpha}(k)$ is defined as

$$f(x) = \sum_{k=0}^{\infty} F_{\alpha}(k)(x-x_0)^{\alpha k}. \tag{19}$$

Substituting Equation (17) into Equation (19) and using the generalized Taylor' formula, we obtain

$$f(x)=\sum_{k=0}^{\infty}F_{\alpha}(k)(x-x_0)^{\alpha k}=\sum_{k=0}^{\infty}\frac{(x-x_0)^{\alpha k}}{\Gamma(\alpha k+1)}\left(\left(D^{\alpha}\right)^{k}f\right)(x_0). \tag{20}$$

Using theorem (4) in Odibat and Shawagfeh (2007), we will obtain an approximate function $f(x)$ from the finite series as

$$f(x)=\sum_{k=0}^{n}F_{\alpha}(k)(x-x_0)^{\alpha k}. \tag{21}$$

The following theorems help us to solve fractional differential equations.

Theorem 2.1. If $f(x)=g(x)\pm h(x)$, then $F_{\alpha}(k)=G_{\alpha}(k)\pm H_{\alpha}(k)$.

Theorem 2.2. If $f(x)=a\,g(x)$, then $F_{\alpha}(k)=a\,G_{\alpha}(k)$.

Theorem 2.3. If $f(x)=g(x)h(x)$, then $F_{\alpha}(k)=\sum_{l=0}^{k}G_{\alpha}(l)H_{\alpha}(k-l)$.

Theorem 2.4. If $f(x)=D^{\alpha}g(x)$, then $F_{\alpha}(k)=\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}G_{\alpha}(k+1)$.

Theorem 2.5. If $f(x)=D^{\beta}g(x)$, $m-1<\beta\leq m$ and the function $g(x)$ satisfies the conditions of theorem (2-5) in Odibat et al. (2008b), then

$$F_{\alpha}(k)=\frac{\Gamma(\alpha k+\beta+1)}{\Gamma(\alpha k+1)}G_{\alpha}\left(k+\beta/\alpha\right).$$

Theorem 2.6. If $f(x)=(x-x_0)^{n\alpha}$, then $F_{\alpha}(k)=p(k-n)$ as $p(k)=\begin{cases}1 \;;\; k=0\\ 0 \;;\; k\neq 0\end{cases}$.

The proofs may be found in Odibat et al. (2008b).

3. Discussion

In this section, we introduce the fractional Mathieu equation. Which we solve by the GDTM. We also compare different cases of approximate solutions with the exact solution. All results were obtained using MAPLE software.

3.1. Fractional Mathieu equation

The homogeneous second order form of the Mathieu equation is

$$D^2 y + [p - 2q\cos(2x)]y = 0, \tag{22}$$

where p and q are the characteristic number and parameter, respectively. If we add the term $cD^{\alpha}y$ to the Eq. (22), we obtain the following equation

$$D^2 y + [p - 2q\cos(2x)]y + cD^{\alpha}y = 0, \tag{23}$$

where $0 < \alpha \le 1$. This equation can be written generally as

$$D^{\beta} y + [p - 2q\cos(2x)]y + cD^{\alpha}y = 0, \tag{24}$$

where $0 < \alpha \le 1$, $1 < \beta \le 2$, with initial conditions

$$y(0) = 1,\ y'(0) = 0. \tag{25}$$

3.2. Particular Cases

Case I. Here we suppose $c = \alpha = 1$ and $\beta = 2$. The exact solution of Equation (24) with initial conditions (25) will be

$$y(x) = \frac{1}{2}e^{-\frac{1}{2}x} Mathieu\, S(-\frac{1}{4} + p, q, x) + e^{-\frac{1}{2}x} Mathieu\, C(-\frac{1}{4} + p, q, x), \tag{26}$$

where the functions Mathieu C(p, q, x) and Mathieu S(p, q, x) are the Mathieu equation solutions. They are even and odd functions, respectively and their alternation period order is 2π, see Frenkel and Portugal (2001) and McLachlan (1947). Using the GDTM and the above theorems, Eq. (24) is transformed as follows

$$\frac{\Gamma(k+3)}{\Gamma(k+1)} Y_1(k+2) + (p - 2qC_1)Y_1(k) + \frac{\Gamma(k+2)}{\Gamma(k+1)} Y_1(k+1) = 0, \tag{27}$$

that is

$$Y_1(k+2) = \frac{1}{\Gamma(k+3)}[Y_1(k)\,\Gamma(k+1)\,(2qC_1 - p) - \Gamma((k+2)Y_1(k+1)], \tag{28}$$

where

$$C_1 = p(k) - 2p(k-2), \tag{29}$$

is the generalized differential transform obtained from the $\cos(2x)$ expansion. We can also write the generalized differential transform of the initial conditions (25) as

$$Y_1(0) = 1,\ Y_1(1) = 0. \tag{30}$$

Considering $k = 0,1,2,...,n$ in Eq. (28) other generalized differential transform components are as follows

$$Y_1(2) = 0.5p + q\ ,\ Y_1(3) = 0.1666666667p - 0.3333333333q. \tag{31}$$

Therefore, we have the solution $y(x)$ up to $O(x^3)$ as

$$y(x) = 1 + (-0.5p + q)x^2 + (0.1666666667p - 0.3333333333q)x^3. \tag{32}$$

Figure 1 (a) shows the exact solution and (b) shows the approximate solution of Equation (24) with initial conditions (25). The interval p in (a) is [0, 3] and in (b) is [0.75, 2.70]. As we see, the obtained approximate solution compares well with the exact solution. It should be noted that including more components of the series solution results in increased error and changes the solution.

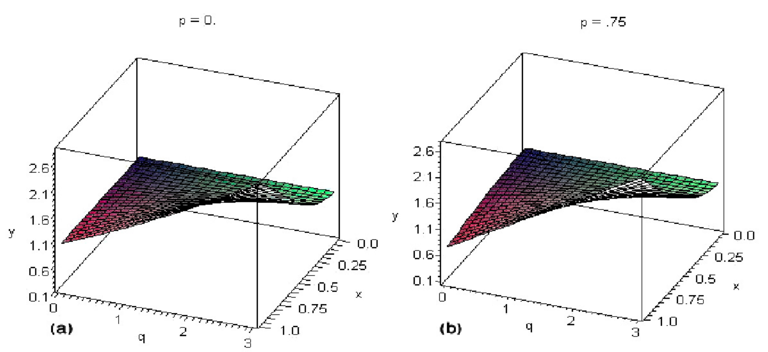

Figure 1. The surface shows the solution $y(x)$ for Equation (24) and (25) in x-q-$y(x)$ space.
(a) Exact solution and $p \in [0,3]$,
(b) Approximate solution when $\alpha = 1, \beta = 2$ and $p \in [0.75, 2.70]$.

Note that in the following we let $c = 1$.

Case II. Now, we suppose $\alpha = 0.5, \beta = 2$. Then the generalized differential transform of Equation (24) is

$$Y_{0.5}(k+4) = \frac{1}{\Gamma(0.5k+3)}[Y_{0.5}(k)\,\Gamma(0.5k+1)\,(2q\,C_{0.5} - p) - \Gamma(0.5k+1.5)Y_{0.5}(k+1)], \tag{33}$$

where

$$C_{0.5} = p(k) - 2p(k-4), \tag{34}$$

is the generalized differential transform obtained from the $\cos(2x)$ expansion. We can also write the generalized differential transform of initial conditions (25) as

$$Y_{0.5}(0) = 1\,,\ Y_{0.5}(1) = 0\,,\ Y_{0.5}(2) = 0\,,\ Y_{0.5}(3) = 0. \tag{35}$$

Therefore, we have the solution $y(x)$ up to $O(x^{3.5})$ as

$$y(x) = 1 + (-0.5p + q)x^2 + (0.08597174604p - 0.1719434921q)x^{3.5}. \tag{36}$$

Figure 2 (b) shows the approximate solution of Equations (24) and (25). On the other hand, we obtain the approximate solution of the fractional Mathieu equation by setting $\alpha = 0.5$ in the interval [1.70, 2.20].

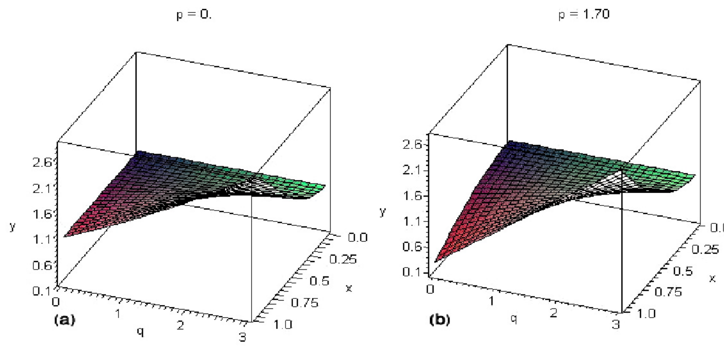

Figure 2. The surface shows the solution $y(x)$ for Equations (24) and (25) in x-q-$y(x)$ space.
(a) Exact solution and $p \in [0,3]$,
(b) Approximate solution when $\alpha = 0.5, \beta = 2$ and $p \in [1.70, 2.20]$.

Case III. In another case, we suppose $\alpha = 0.25, \beta = 2$. The generalized differential transform of Equation (24) is

$$\begin{aligned} Y_{0.25}(k+8) = \frac{1}{\Gamma(0.25k+3)}[&Y_{0.25}(k)\,\Gamma(0.25k+1)\,(2q\,C_{0.25} - p) \\ &-\Gamma(0.25(k+1.25)Y_{0.25}(k+1)], \end{aligned} \tag{37}$$

where

$$C_{0.25} = p(k) - 2p(k-8). \tag{38}$$

Also, we can write the generalized differential transform of initial conditions (25) as

$$Y_{0.25}(0) = 1, Y_{0.25}(1) = Y_{0.25}(2) = \ldots = Y_{0.25}(7) = 0. \tag{39}$$

Therefore, we have the solution $y(x)$ up to $O(x^{3.75})$ as

$$y(x) = 1 + (-0.5p + q)x^2 + (0.06029106159p - 0.1205821232q)x^{3.75}. \tag{40}$$

Figure 3 (b) shows the approximate solution of Equations (24) and (25) in the interval [1.88, 2], which is more limited than the last two forms. In this case, also the approximate solution compares well with the exact solution.

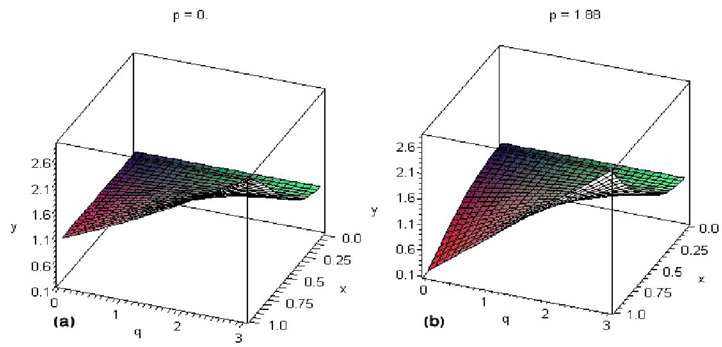

Figure 3. The surface shows the solution y(x) for Equations (24) and (25) in x-q-y(x) space.
(a) Exact solution and $p \in [0,3]$,
(b) Approximate solution when $\alpha = 0.25, \beta = 2$ and $p \in [1.88, 2]$.

Case IV. Finally, in the last case, we suppose $\alpha = 0.5, \beta = 1.5$. The generalized differential transform of Equation (24) is

$$Y_{0.5}(k+3) = \frac{1}{\Gamma(0.5k+2.5)}[Y_{0.5}(k)\,\Gamma(0.5k+1)\,(2q\,C_{0.5} - p) - \Gamma(0.5k+1.5)Y_{0.5}(k+1)], \tag{41}$$

and

$$C_{0.5} = p(k) - 2p(k-4), \tag{42}$$

with the following transformed initial conditions

$$Y_{0.5}(0) = 1,\ Y_{0.5}(1) = Y_{0.5}(2) = 0. \tag{43}$$

Therefore, we have the solution $y(x)$ up to $O(x^4)$ as

$$\begin{aligned} y(x) = {} & 1 + (-0.7522527782p + 1.504505556q)x^{1.5} + (0.3009011113p \\ & - 0.6018022226q)x^{2.5} + (-0.2215567314p(-0.7522527782p \\ & + 1.504505556q))x^{3} + (-0.08597174604p + 0.1719434921q)x^{3.5} \\ & + (-0.1384729571p(0.3009011113p - 0.6018022226q) \\ & + 0.05538918284p(-0.7522527782p + 1.504505556q))x^{4}. \end{aligned} \tag{44}$$

The above case is a test of the two fractional order equations in Odibat et al. (2008a), that had one of order zero and the other was of fractional order. In our work we changed both orders of the equation. We decreased the order of the equation; then we placed the interval p at a greater distance than the previous two distances that is, [1.55, 2.80].

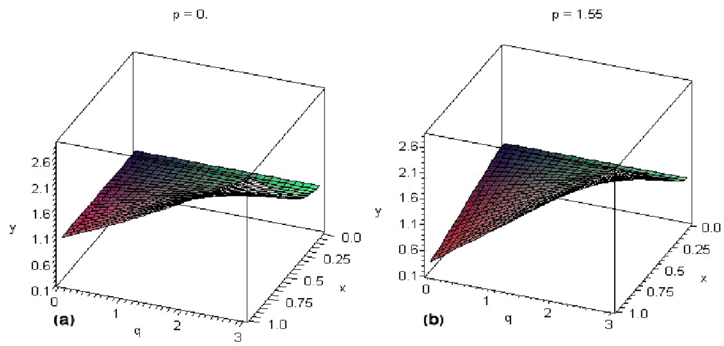

Figure 4. The surface shows the solution y(x) for Equations (24) and (25) in x-q-y(x) space.
(a) Exact solution and $p \in [0, 3]$,
(b) Approximate solution when $\alpha = 0.5, \beta = 1.5$ and $p \in [1.55, 2.80]$.

4. Conclusion

We have solved the fractional Mathieu equation by the GDTM in four different cases of differential orders. In the first three cases, decreasing α and fixing β was executed. Also by limiting the interval p, the approximate solution was obtained. In the last case, when we decreased both orders of the equation and extended the interval p, we determined the approximate solution of the equation. Finally we mention that by the GDTM not only were we able to obtain the approximate solution with integer orders, but in the different cases of fractional orders we also obtain the approximate solution close to the exact solution.

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