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Weighted Inequalities for Riemann-Stieltjes Integrals

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Abstract

In this paper first we define a new functional which is a weighted version of the functional defined by Dragomir and Fedotov. Then, some inequalities involving this functional are obtained. Finally, we apply this result to establish new bounds for weighted Chebysev functional.

Keywords: Function of bounded variation; Ostrowski type inequalities; Riemann-Stieltjes integral

MSC 2010 No.: 26D15, 26A45, 26D10

1. Introduction

The following definitions will be frequently used to prove our results.

Definition 1.1.

Let $P : a = x_0 < x_1 < ... < x_n = b$ be any partition of $[a,b]$ and let $\Delta f(x_i) = f(x_{i+1}) - f(x_i)$, then f is said to be of bounded variation if the sum

$$\sum_{i=1}^{m} \left| \Delta f(x_i) \right|$$

is bounded for all such partitions.

Definition 1.2.

Let f be of bounded variation on $[a,b]$, and $\sum \Delta f(P)$ denotes the sum $\sum_{i=1}^{n} \left| \Delta f(x_i) \right|$ corresponding to the partition P of $[a,b]$. The number

$$V_f\ (a,b) := \sup\left\{\sum \Delta f(P)\ :\ P \in P([a,b])\right\}$$

is called the total variation of f on $[a,b]$. Here, $P([a,b])$ denotes the family of partitions of $[a,b]$.

Dragomir and Fedotov (1998) have established the following functional

$$D(f,u) = \int_a^b f(t)du(t) - \frac{u(b)-u(a)}{b-a} \int_a^b f(t)dt.$$

In the same paper, the authors proved the following inequality.

Theorem 1.

Let $f,u\ :\ [a,b] \to \mathrm{R}$ be such that u is of bounded variation on $[a,b]$ and f is Lipschitzian with the constant $L > 0$. Then we have

$$\left| D(f,u) \right| \le \frac{1}{2} L(b-a) V_u\ (a,b).$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller one.

Alomari (2012) gave the following inequality.

Theorem 2.

Let $x \in [a,b]$. Let $f,u\ :\ [a,b] \to \mathrm{R}$ be a continuous mappings on $[a,b]$. Assume that u is monotonic non-decreasing mapping on $[a,b]$ and $f\ :\ [a,b] \to \mathrm{R}$ is monotonic nondecreasing on both intervals $[a,x]$ and $[x,b]$. Then we have the inequality

$$\left|D(f,u)\right| \le 2u(b)\left[f(b)-\int_a^b f(t)dt\right].$$

Dragomir (2014) gave some new bounds for the function $D(f,u)$. One of them is following inequality.

Theorem 3.

Assume that $f,u:[a,b]\to\mathbb{R}$ are of bounded variation and such that the Riemann-Stieltjes integral $\int_a^b f(t)du(t)$ exist. Then,

$$\begin{aligned}
&\left|D(f,u)\right| \\
&\le \frac{1}{b-a}\left[\int_a^b [2t-a-b]V_f(a,t)d\left(V_u(a,t)\right)\right. \\
&\quad \left. -\int_a^b V_f(a,t)dtV_u(a,b)+2\int_a^b V_f(a,t)V_u(a,t)dt\right] \\
&\le \frac{1}{b-a}\int_a^b [2t-a-b]V_f(a,t)d\left(V_u(a,t)\right)+\frac{1}{b-a}\int_a^b \left(V_f(a,t)V_u(a,t)\right)dt \\
&\le \frac{1}{b-a}\int_a^b [2t-a-b]V_f(a,t)d\left(V_u(a,t)\right)+V_f(a,b)V_u(a,b).
\end{aligned}$$

Lemma 1.

Let $f,u:[a,b]\to\mathbb{C}$. If f is continuous on $[a,b]$ and u is of bounded variation on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b f(t)du(t)$ exist and

$$\left|\int_a^b f(t)du(t)\right| \le \int_a^b \left|f(t)\right| dV_u(a,t) \le \max_{t\in[a,b]}\left|f(t)\right| V_u(a,b).$$

A great many authors worked on inequalities for Riemann-Stieltjes integral via functions of bounded variation (or derivatives of bounded variation). For some of them, please see in Alomari (2012)-Liu (2004).

The main purpose of this paper is to obtain some weighted inequalities for Riemann-Stieltjes integral. First of all, we define a weighted version of the functional $D(f,u)$. Then , we establish some bounds for this functional according to cases of the functions f and u. Finally, some new bounds for the weighted Chebysev functional are also given.

This paper is divided into the following six sections. In Section 2, we establish some identities that will be used to prove our results. In Section 3 and Section 4, some weighted integral inequalities for the case when the function u is bounded variation and when the function f is bounded variation are given, respectively. In the next section, we give an inequality for the case when u is $(l, L)-$Lipschitzan. Finally, in Section 6, we present some applications for weighted Chebysev functional using the results given in previous sections.

2. Some Identities

Let $w : [a,b] \to \mathbb{R}$ be nonnegative and continuous on $[a,b]$. We define

$$m(a,b) = \int_a^b w(s)ds \text{ and } m_1(a,t) = \int_a^t w(s)ds, \text{ so that } m(a,t) = 0 \text{ for } t < a.$$

Now we give some representations.

Weighted version of the functional defined by Dragomir and Fedotov:

$$D(w, f, u) = m(a,b)\int_a^b f(t)du(t) - [u(b) - u(a)]\int_a^b w(t) f(t)dt.$$

Weighted Chebysev functional:

$$T(w, f, g) = \frac{1}{m(a,b)}\int_a^b w(t) f(t)g(t)dt - \left(\frac{1}{m(a,b)}\int_a^b w(t) f(t)dt\right)\left(\frac{1}{m(a,b)}\int_a^b w(t)g(t)dt\right).$$

Weighted Ostrowski transform:

$$\Theta_{f,w}(t) = m(a,b) f(t) - \int_a^b w(s) f(s)ds.$$

Weighted generalized trapezoid transform:

$$\Phi_{g,w}(t) = m(t,b)g(a) + m(a,t)g(b) - m(a,b)g(t).$$

Before we start our main results, we state and prove following lemmas:

Lemma 2.

If $f, u : [a,b] \to \mathbb{R}$ are bounded functions such that the Riemann-Stieltjes integral $\int_a^b f(t)du(t)$

and the Riemann integral $\int_a^b w(t) f(t)dt$ exist, then we have

$$D(w, f, u) = \int_a^b \Theta_{f,w}(s)du(s) = \int_a^b \left(\int_a^b Q_w(t,s)df(t) \right) du(s), \tag{1}$$

where

$$Q_w(t,s) = \begin{cases} \int_b^t w(s)ds, & a \le s \le t \le x, \\ \\ \int_a^t w(s)ds, & x < t < s \le b. \end{cases}$$

***Proof*:**

Using the integration by parts in Riemann-Stieltjes integral, we have

$$\begin{aligned}
&\int_a^b \left(\int_a^b Q_w(t,s)df(t) \right) du(s) \\
&= \int_a^b \left[\int_a^s \left(\int_a^t w(\xi)d\xi \right) df(t) + \int_s^b \left(\int_b^t w(\xi)d\xi \right) df(t) \right] du(s) \\
&= \int_a^b \left[\left(\int_a^t w(\xi)d\xi \right) f(t) \Big|_a^s - \int_a^s w(t) f(t)dt \right. \\
&\qquad \left. + \left(\int_b^t w(\xi)d\xi \right) f(t)\Big|_s^b - \int_s^b w(t) f(t)dt \right] du(s) \\
&= \int_a^b \left[\left(\int_a^b w(\xi)d\xi \right) f(s) + \int_a^b w(t) f(t)dt \right] du(s) \\
&= \int_a^b \Theta_{f,w}(s)du(s).
\end{aligned}$$

This completes the proof of the second equality. The first identity is obvious.

Corollary 1.

Let $g:[a,b]\to \mathrm{R}$ a function such that g is Riemann integrable on $[a,b]$. If we choose $u(t)=\int_a^t w(s)g(s)$ in Lemma 2, then we have

$$T(w,f,g)=\frac{1}{m^2(a,b)}\int_a^b \Theta_{f,w}(s)w(s)g(s)ds=\frac{1}{m^2(a,b)}\int_a^b\left(\int_a^b Q_w(t,s)df(t)\right)w(s)g(s)ds.$$

Lemma 3.

With the assumptions in Lemma 2, we have

$$D(w,f,u)=\int_a^b \Phi_{u,w}(t)df(t)=\int_a^b\left(\int_a^b Q_w(t,s)du(s)\right)df(t), \tag{2}$$

where the mapping $Q_w(t,s)$ is defined by as in Lemma 2.

Proof:

By the Fubini type theorem for the Riemann-Stieltjes integral, we get

$$\int_a^b\left(\int_a^b Q_w(t,s)du(s)\right)df(t)=\int_a^b\left(\int_a^b Q_w(t,s)df(t)\right)du(s).$$

This completes the proof of the first and the last terms in (2).

Integrating by parts, we obtain

$$\begin{aligned}\int_a^b Q_w(t,s)du(s)&=\int_a^t\left(\int_b^t w(\xi)d\xi\right)du(s)+\int_t^b\left(\int_a^t w(\xi)d\xi\right)du(s)\\&=\left(\int_b^t w(\xi)d\xi\right)[u(t)-u(a)]+\left(\int_a^t w(\xi)d\xi\right)[u(b)-u(a)]\\&=m(t,b)u(a)+m(a,t)u(b)-m(a,b)u(t)\\&=\Phi_{u,w}(t).\end{aligned}$$

This completes the proof.

Corollary 2.

Assume that $g : [a,b] \to \mathbb{R}$ Riemann integrable on $[a,b]$, then we have

$$\Phi_{g,w}(t) = \Phi_{\int_a^{\cdot} g,w}(t) = m(a,t)\int_a^b w(s)g(s)ds - m(a,b)\int_a^t w(s)g(s)ds.$$

Remark 1.

If we choose $w(t) = t$ in Lemma 1 and Lemma 2, then our results reduce Lemma 1 and Lemma 2 proved by Dragomir (2014), respectively.

3. Inequalities in the Case when u is of bounded variation

Now using the above identities, we state and prove the following inequalities in the case when u is of bounded variation.

Theorem 4.

Let $w : [a,b] \to \mathbb{R}$ be nonnegative and continuous on $[a,b]$. If $f, u : [a,b] \to \mathbb{R}$ are of bounded variation on $[a,b]$, then we have the inequalities

$$\begin{aligned}
&|D(w,f,u)| \\
&\leq \int_a^b [m(a,t) - m(t,b)] V_f(a,t) d(V_u(a,t)) \\
&\quad - \left(\int_a^b w(t) V_f(a,t)(f) dt\right) V_u(a,b)) + 2\int_a^b w(t) V_f(a,t) V_u(a,t) dt
\end{aligned}$$

$$\begin{aligned}
&\leq \int_a^b [m(a,t) - m(t,b)] V_f(a,t) d(V_u(a,t)) + \int_a^b w(t) V_f(a,t) V_u(a,t) dt \\
&\leq \int_a^b [m(a,t) - m(t,b)] V_f(a,t) d(V_u(a,t)) + m(a,b) V_f(a,b) V_u(a,b).
\end{aligned}$$

***Proof*:**

Taking the modulus in Lemma 2 and using the Lemma 1, we have

$$|D(w,f,u)|$$

$$= \left|\int_a^b \left(\int_a^b Q_w(t,s) df(t)\right) du(s)\right|$$

$$
\begin{aligned}
&\leq \int_a^b \left| \int_a^b Q_w(t,s)\, df(t) \right| d\left(V_u\left(a,s\right)\right) \\
&= \int_a^b \left| \int_a^s \left(\int_a^t w(\xi)\, d\xi \right) df(t) + \int_s^b \left(\int_b^t w(\xi)\, d\xi \right) df(t) \right| d\left(V_u\left(a,s\right)\right) \qquad (4) \\
&\leq \int_a^b \left[\left| \int_a^s \left(\int_a^t w(\xi)\, d\xi \right) df(t) \right| + \left| \int_s^b \left(\int_b^t w(\xi)\, d\xi \right) df(t) \right| \right] d\left(V_u\left(a,s\right)\right).
\end{aligned}
$$

Since f is of bounded variation, using Lemma l again, we obtain

$$
\begin{aligned}
\left| \int_a^s \left(\int_a^t w(\xi)\, d\xi \right) df(t) \right| &\leq \int_a^s \left(\int_a^t w(\xi)\, d\xi \right) d\left(V_f\left(a,t\right)\right) \\
&= \left(\int_a^s w(\xi)\, d\xi \right) V_f\left(a,t\right) - \int_a^s w(t) V_f\left(a,t\right) dt \qquad (5) \\
&= m(a,s) V_t\left(a,s\right) - \int_a^s w(t) V_f\left(a,t\right) dt
\end{aligned}
$$

and

$$
\begin{aligned}
\left| \int_s^b \left(\int_b^t w(\xi)\, d\xi \right) df(t) \right| &\leq \int_s^b \left| \int_b^t w(\xi)\, d\xi \right| d\left(V_f\left(a,t\right)\right) \\
&= \int_s^b \left(\int_t^b w(\xi)\, d\xi \right) d\left(V_f\left(a,t\right)\right) \\
&= \int_s^b w(t) V_f\left(a,t\right) dt - m(s,b) V_f\left(a,s\right).
\end{aligned}
$$

(6)

If we substitute the inequalities (5) and (6) in (4), we establish

$$
\left| D(w,f,u) \right|
$$

$$
\leq \int_a^b \left[\left[m(a,s) - m(s,b) \right] V_f\left(a,s\right) \right.
$$

$$-\int_a^s w(t)V_f(a,t)\,dt+\int_s^b w(t)V_f(a,t)\,dt\Bigg]\,d\left(V_u(a,s)\right)$$

$$=\int_a^b\left[\left[m(a,s)-m(s,b)\right]\bigvee_a^s(f)\right]d\left(V_u(a,s)\right)$$

$$+\int_a^b\left[\int_a^b w(t)V_f(a,t)\,dt-2\int_a^s w(t)V_f(a,t)\,dt\right]d\left(V_u(a,s)\right)$$

$$=\int_a^b\left[\left[m(a,s)-m(s,b)\right]V_f(a,s)\right]d\left(V_u(a,s)\right)$$

$$+\int_a^b w(t)\bigvee_a^t(f)\,dt\,V_f(a,b)-2\int_a^b\left(\int_a^s w(t)V_f(a,t)\,dt\right)d\left(V_u(a,s)\right). \tag{7}$$

In last line of (7), we have

$$\int_a^b\left(\int_a^s w(t)V_f(a,t)\,dt\right)d\left(V_u(a,s)\right)$$

$$=\left(\int_a^s w(t)V_f(a,t)\,dt\right)V_u(a,s)\Bigg|_a^b-\int_a^b w(s)V_f(a,s)V_u(a,s)\,ds$$

$$=\left(\int_a^b w(t)V_f(a,t)\,dt\right)V_u(a,b)-\int_a^b w(s)V_f(a,s)V_u(a,s)\,ds\,. \tag{8}$$

If we put the equality (8) in (7), we obtain the first inequality in (3).

The other inequalities are obvious from the fact that

$$\int_a^b w(s)V_f(a,s)V_u(a,s)\,ds\le V_u(a,b)\int_a^b w(s)V_f(a,s)\,ds\le m(a,b)V_f(a,b)V_u(a,b).$$

Remark 2.

If we choose $w(t)=t$ in Theorem 4, then we obtain Theorem 1 in Dragomir (2014).

Theorem 5.

Let $w : [a,b] \to \mathbb{R}$ be nonnegative and continuous on $[a,b]$. If $u : [a,b] \to \mathbb{R}$ is of bounded variation on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ is monotonic nondecreasing, then we have the inequality

$$
\begin{aligned}
&\left|D(w,f,u)\right| \\
&\le \int_a^b [m(a,t) - m(t,b)]\, f(t)\, d\left(V_u(a,t)\right) \\
&\quad + 2\int_a^b w(t) f(t) V_u(a,t)\,dt - \left(\int_a^b w(t) f(t)\,dt\right) V_u(a,b) \\
&\le \int_a^b [m(a,t) - m(t,b)]\, f(t)\, d\left(V_u(a,t)\right) + \left(\int_a^b w(t) f(t)\,dt\right) V_u(a,b).
\end{aligned} \tag{9}
$$

Proof:

It is well known that if the Stieltjes integrals $\int_\alpha^\beta p(t)dv(t)$ and $\int_\alpha^\beta |p(t)|dv(t)$ exist and v is monotonic non-decreasing on $[\alpha,\beta]$, then

$$
\int_\alpha^\beta p(t)dv(t) \le \int_\alpha^\beta |p(t)|dv(t). \tag{10}
$$

Using the inequality (10), we have

$$
\left|\int_a^s \left(\int_a^t w(\xi)d\xi\right) df(t)\right| \le \int_a^s \left(\int_a^t w(\xi)d\xi\right) df(t) = m(a,s) f(s) - \int_a^s w(t) f(t)dt \tag{11}
$$

and

$$\left|\int_s^b\left(\int_b^t w(\xi)d\xi\right)df(t)\right| \le \int_s^b\left|\int_b^t w(\xi)d\xi\right|df(t)$$
$$= \int_s^b\left(\int_t^b w(\xi)d\xi\right)df(t) \tag{12}$$
$$= \int_s^b w(t)f(t)dt - m(s,b)f(s).$$

If we substitute the inequalities (11) and (12) in (4), we obtain

$$\begin{aligned} &|D(w,f,u)| \\ &\le \int_a^b [m(a,s)-m(s,b)]\,f(s)d\left(V_u\,(a,s)\right) \\ &\quad + \int_a^b\left[\int_s^b w(t)f(t)dt - \int_a^s w(t)f(t)dt\right]d\left(V_u\,(a,s)\right) \end{aligned} \tag{13}$$

$$= \int_a^b [m(a,s)-m(s,b)]\,f(s)d\left(V_u\,(a,s)\right)$$
$$+ \int_a^b\left(\int_s^b w(t)f(t)dt\right)d\left(V_u\,(a,s)\right) - \int_a^b\left(\int_a^s w(t)f(t)dt\right)d\left(V_u\,(a,s)\right).$$

Using the integration by parts in Riemann-Stieltjes integral, we have

$$\int_a^b\left(\int_s^b w(t)f(t)dt\right)d\left(V_u\,(a,s)\right) = \int_a^b w(s)f(s)V_u\,(a,s)ds \tag{14}$$

and

$$\int_a^b\left(\int_a^s w(t)f(t)dt\right)d\left(V_u\,(a,s)\right) = \int_a^b w(t)f(t)dtV_u\,(a,b) - \int_a^b w(s)f(s)V_u\,(a,s)ds. \tag{15}$$

Putting the equalities (14) and (15) in (13), we complete the proof the first inequality in (9).

The second inequality is obvious.

Remark 3.

If we choose $w(t)=t$ in Theorem 5, then the first inequality in (9) reduces to the inequality (3.7) in Dragomir (2014).

4. Inequalities in the Case when f is of bounded variation

In this section, we give same inequality in the case when f is of bounded variation using the identities presented in Section 2.

Theorem 6.

Let $w:[a,b]\to \mathbb{R}$ be nonnegative and continuous on $[a,b]$ and $f:[a,b]\to \mathbb{R}$ be a function of bounded variation on $[a,b]$. If $u:[a,b]\to \mathbb{R}$ is continuous such that there exist constant $\alpha,\beta>0$ and $L_a, L_b>0$ with

$$|u(t)-u(a)|\le L_a(t-a)^{\alpha} \tag{16}$$

and

$$|u(b)-u(t)|\le L_b(b-t)^{\beta}, \tag{17}$$

for all $t\in[a,b]$, then we have

$$\begin{aligned}|D(w,f,u)|&\\ &\le L_a\left[\int_a^b w(t)(t-a)^{\alpha}V_f(a,t)\,dt-\alpha\int_a^b m(t,b)(t-a)^{\alpha-1}V_f(a,t)\,dt\right]\\ &\quad+L_b\left[\beta\int_a^b m(a,t)(b-t)^{\beta-1}V_f(a,t)\,dt-\int_a^b w(t)(b-t)^{\beta}V_f(a,t)\,dt\right].\end{aligned} \tag{18}$$

Proof:

Taking the modulus in Lemma 3 and using Lemma 1, we have

$$\begin{aligned}|D(w,f,u)|&\\ &\le\int_a^b\left|\int_a^b Q_w(t,s)\,du(s)\right|d\left(V_f(a,t)\right)\\ &\le\int_a^b\left[\left|\int_a^t\left(\int_b^t w(\xi)d\xi\right)du(s)\right|+\left|\int_t^b\left(\int_a^t w(\xi)d\xi\right)du(s)\right|\right]d\left(V_f(a,t)\right)\end{aligned}$$

$$\leq \int_a^b \left[m(t,b)\left|u(t)-u(a)\right| + m(a,t)\left|u(b)-u(t)\right| \right] d\left(V_f\left(a,t\right)\right). \tag{19}$$

Using properties (16) and (17) in (19), we obtain

$$\begin{aligned}
&\left|D(w,f,u)\right| \\
&\quad\leq \int_a^b \left[L_a m(t,b)\left(t-a\right)^{\alpha} + L_b m(a,t)\left(b-t\right)^{\beta} \right] d\left(V_f\left(a,t\right)\right) \\
&\quad= L_a \int_a^b m(t,b)\left(t-a\right)^{\alpha} d\left(V_f\left(a,t\right)\right) + L_b \int_a^b m(a,t)\left(b-t\right)^{\beta} d\left(V_f\left(a,t\right)\right).
\end{aligned} \tag{20}$$

Integrating by parts, we have

$$\begin{aligned}
&\int_a^b m(t,b)\left(t-a\right)^{\alpha} d\left(V_f\left(a,t\right)\right) \\
&\quad= m(t,b)\left(t-a\right)^{\alpha}\left(V_f\left(a,t\right)\right)\Big|_a^b \\
&\qquad -\int_a^b \left[-w(t)\left(t-a\right)^{\alpha} + \alpha m(t,b)\left(t-a\right)^{\alpha-1} V_f\left(a,t\right) \right] dt \\
&\quad= \int_a^b w(t)(t-a)^{\alpha} V_f\left(a,t\right) dt - \alpha \int_a^b m(t,b)(t-a)^{\alpha-1} V_f\left(a,t\right) dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b m(a,t)\left(b-t\right)^{\beta} d\left(V_f\left(a,t\right)\right) \\
&\quad= m(a,t)\left(b-t\right)^{\beta}\left(V_f\left(a,t\right)\right)\Big|_a^b \\
&\qquad -\int_a^b \left[w(t)\left(b-t\right)^{\beta} + \beta m(a,t)\left(b-t\right)^{\beta-1} V_f\left(a,t\right) \right] dt \\
&\quad= \beta \int_a^b m(a,t)\left(b-t\right)^{\beta-1} V_f\left(a,t\right) dt - \int_a^b w(t)\left(b-t\right)^{\beta} V_f\left(a,t\right) dt
\end{aligned}$$

These equalities complete the proof.

Remark 4.

If we choose $w(t)=t$ in Theorem 6, then we obtain Theorem 4 in (Dragomir 2014).

Corollary 3.

Let f and w be as in Theorem 6. If u is of $r-H-$Hölder type, i.e.,

$$|u(t)-u(s)| \leq H|t-s|^r \text{ for any } t,s \in [a,b],$$

where $H>0$ and $r \in (0,1)$ are given, then

$$\begin{aligned}&|D(w,f,u)| \\ &\leq H\left[\int_a^b w(t)\left[(t-a)^r-(b-t)^r\right]V_f(a,t)\,dt\right. \\ &\qquad\left.+r\int_a^b\left[m(a,t)(b-t)^{r-1}-m(t,b)(t-a)^{r-1}\right]V_f(a,t)\,dt\right]. \qquad (21)\end{aligned}$$

Corollary 4. If u is Lipschitzian with the constant $L>0$, then we have

$$\begin{aligned}&|D(w,f,u)| \\ &\leq 2L\left[\int_a^b w(t)\left(t-\frac{a+b}{2}\right)V_f(a,t)\,dt+\int_a^b\left[\frac{m(a,t)-m(t,b)}{2}\right]V_f(a,t)\,dt\right].\end{aligned}$$

5. Inequalities for (l,L)-Lipschitzan Fuctions

The following lemma was given by Dragomir (2014).

Lemma 4.

Let $u:[a,b]\to \mathbb{R}$ and $l,L\in\mathbb{R}$ with $L>l$. The following statements are equivalent:

(i) The function $u-\frac{l+L}{2}.e$, where $e(t)=t$, $t\in[a,b]$ is $\frac{1}{2}(L-l)$-Lipschitzan;

(ii) We have the inequalities

$$l \leq \frac{u(t)-u(s)}{t-s} \leq L \text{ for each } t,s\in[a,b],\ t\neq s;$$

(iii) We have the inequalities

$$l(t-s) \leq u(t)-u(s) \leq L(t-s) \text{ for each } t,s\in[a,b],\ t>s.$$

Definition 3.

The function $u : [a,b] \to \mathbb{R}$ which satisfies one of the equivalent conditions (i) - (iii) from Lemma l3 is said to be (l,L)-Lipschitzan on $[a,b]$. If $L>0$ and $l=-L$, then $(-L,L)$-Lipschitzan means L-Lipschitzan in the classical sense.

Theorem 7.

Let $w : [a,b] \to \mathbb{R}$ be nonnegative and continuous on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$. If $u : [a,b] \to \mathbb{R}$ is an (l,L)-Lipschitzan function, then we have the inequality

$$\left| D(w,f,u) - \frac{l+L}{2} \int_a^b \int_a^b w(s)\left[f(t) - f(s) \right] ds dt \right|$$

$$\leq (L-l) \left[\int_a^b w(t) \left(t - \frac{a+b}{2} \right) V_f\left(a,t\right) dt + \int_a^b \left[\frac{m(a,t) - m(t,b)}{2} \right] V_f\left(a,t\right) dt \right].$$

Proof:

From Lemma 2, we have

$$D\left(w, f, u - \frac{l+l}{2}.e \right)$$

$$= \int_a^b \left[\left(\int_a^b w(s) ds \right) f(t) + \int_a^b w(s) f(s) ds \right] d\left[u(t) - \frac{l+l}{2} t \right]$$

$$= \int_a^b \left[\left(\int_a^b w(s) ds \right) f(t) + \int_a^b w(s) f(s) ds \right] du(t)$$

$$- \frac{l+l}{2} \int_a^b \left[\left(\int_a^b w(s) ds \right) f(t) + \int_a^b w(s) f(s) ds \right] dt$$

$$= D(w,f,u) - \frac{l+L}{2} \int_a^b \int_a^b w(s)\left[f(t) - f(s) \right] ds dt.$$

Applying Corollary 4 for the function $u - \frac{l+l}{2}.e$, which is $\frac{1}{2}(L-l)$-Lipschitzian, we have

$$\left|D\left(w, f, u-\frac{l+l}{2}.e\right)\right|$$

$$\leq (L-l)\left[\int_a^b w(t)\left(t-\frac{a+b}{2}\right)V_f\left(a,t\right)dt+\int_a^b\left[\frac{m(a,t)-m(t,b)}{2}\right]V_f\left(a,t\right)dt\right],$$

which completes the proof.

Remark 5.

If we choose $w(t)=t$ in Theorem 7, then we obtain Theorem 5 in Dragomir (2014).

6. Bounds For Weighted Chebysev Functional

In this section, we apply the our results for the weighted Chebysev functional. From Section 2, we know that

$$T(w, f, g)=\frac{1}{m^2(a,b)}D(w, f, u) \tag{22}$$

by choosing the $u(t)=\int_a^t w(s)g(s)$ in Lemma 2.

Moreover, u is of bounded variation on any subinterval $[a,s]$, $s\in[a,b]$, and g is continuous on $[a,b]$, then we have

$$V_u\left(a,s\right)=\int_a^s w(t)|g(t)|dt,\ s\in[a,b]. \tag{23}$$

Proposition 1.

If f is of bounded variation on $[a,b]$, then we have the inequality

$$|T(w, f, g)|$$

$$\leq\frac{1}{m^2(a,b)}\left[\int_a^b[m(a,t)-m(t,b)]w(t)|g(t)|V_f\left(a,t\right)dt\right.$$
$$\left.-\left(\int_a^b w(t)V_f\left(a,t\right)dt\right)\left(\int_a^b w(t)|g(t)|dt\right)\right.$$

$$+2\int_a^b w(t)V_f\left(a,t\right)\left(\int_a^t w(s)\left|g(s)\right|ds\right)dt\Bigg]$$

$$\leq \frac{1}{m^2(a,b)}\Bigg[\int_a^b \left[m(a,t)-m(t,b)\right]w(t)\left|g(t)\right|V_f\left(a,t\right)dt$$

$$+\int_a^b w(t)V_f\left(a,t\right)\left(\int_a^t w(s)\left|g(s)\right|ds\right)dt\Bigg]$$

$$\leq \frac{1}{m^2(a,b)}\Bigg[\int_a^b \left[m(a,t)-m(t,b)\right]w(t)\left|g(t)\right|V_f\left(a,t\right)dt$$

$$+m(a,b)\left(\int_a^b w(t)\left|g(t)\right|dt\right)V_f\left(a,b\right)\Bigg].$$

Proof:

If choose $u(t)=\int_a^t w(s)g(s)$ in Theorem 4 and use the identity (21) and (22), we can prove the required result easily.

Proposition 2.

If f is monotonic non-decreasing on $[a,b]$, then we have the inequality

$$\left|T(w,f,g)\right|$$

$$\leq \frac{1}{m^2(a,b)}\Bigg[\int_a^b \left[m(a,t)-m(t,b)\right]f(t)w(t)\left|g(t)\right|dt$$

$$+2\int_a^b w(t)f(t)\left(\int_a^t w(s)\left|g(s)\right|ds\right)dt-\left(\int_a^b w(t)f(t)dt\right)\left(\int_a^b w(t)\left|g(t)\right|dt\right)\Bigg]$$

$$\leq \frac{1}{m^2(a,b)}\Bigg[\int_a^b \left[m(a,t)-m(t,b)\right]f(t)w(t)\left|g(t)\right|dt$$

$$+\left(\int_a^b w(t)f(t)dt\right)\left(\int_a^b w(t)\left|g(t)\right|dt\right)\Bigg].$$

Proof:

The proof is obvious from Theorem 5.

7. Conclusions

Some explicit error bounds are known for Chebysev functional. In this paper, by using the ideas of Dragomir (2014), we establish some weighted versions of integral inequalities obtained in Dragomir (2014), The methods used in this paper might find some potential applications in the generalizations of some other integral inequalities. To do so, one should define some new functional as we defined in Section 2.

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