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A robust uniform B-spline collocation method for solving the generalized PHI-four equation

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Abstract

In this paper, we develop a numerical solution based on cubic B-spline collocation method. By applying Von-Neumann stability analysis, the proposed technique is shown to be unconditionally stable. The accuracy of the presented method is demonstrated by a test problem. The numerical results are found to be in good agreement with the exact solution.

Keywords/Phrases: PHI-four equation; cubic B-spline; Collocation method; Von-Neumann stability

MSC 2010: 49K20, 65D07, 65M70, 35B35

1. Introduction

A large variety of phenomena are governed by nonlinear partial differential equations. The analytical study on nonlinear partial differential equations was a great interest during the last century. This theory plays a major role in the field of physics including plasma physics, fluid dynamics and quantum field theory. The PHI- four equation is considered to be a special form of the famous Klein-Gordon equation that models phenomenon in particle physics where kink and anti-kink solitary wave interacts Benjamin et al. (1972).

The general PHI-four equation has the following form

$$\frac{\partial^2 u}{\partial t^2} - \mu \frac{\partial^2 u}{\partial x^2} + \delta u^q + \eta u = \psi(x,t), \quad a \le x \le b, t \ge 0, \tag{1.1}$$

subject to initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x)\,, \tag{1.2}$$

and appropriate boundary conditions as follows

$$u(a,t) = \alpha_1(t), \qquad u(b,t) = \alpha_2(t), \qquad t \ge 0 \;\; (Dirichlet\ conditions), \tag{1.3}$$

or

$$u_x(a,t) = \alpha_3(t), \qquad u_x(b,t) = \alpha_4(t), \qquad t \ge 0 \;\; (Neumann\ conditions), \tag{1.4}$$

where $u = u(x,t)$ represents the wave displacement at position x and time t with μ, δ, η, q are known constants and $\psi(x,t)$ are the source term. Changing the values of the constants in Equation 1.1 gives various types of equations. For example, setting the value of $\mu = 1$, $\eta = 0, \delta = 1$, $q = 2$ and $\psi(x,t) = -x\cos(t) + x^2\cos^2(t)$ gives a form of the well-known nonlinear Klein -Gordon equation. Another form of the Klein-Gordon equation is obtained by assigning $\mu = -\frac{5}{2}$, $\eta = 1$, $\delta = \frac{3}{2}$, $q = 3$ and $\psi(x,t) = 0$ which was discussed in Bulut (2012). If we set the value of $\mu = 1$, $\eta = -1$, $\delta = 1$, $q = 3$ and $\psi(x,t) = 0$ gives the equation under consideration which is called the PHI-four equation Wazwaz (2005).

This equation arises in many branches of mathematical physics. Special solutions known as kink and anti-kink solitons have the following form

$$u(x,t) = \pm \tanh\left(\frac{1}{\sqrt{2(1-c^2)}}\,(x - ct)\right), \tag{1.5}$$

which have been discussed and found in Triki and Wazwaz (2009). Here, in Equation (1.5), c is the wave speed and the coefficient μ satisfies the condition $\mu = 1$ for the solution in Equation (1.3) to exist. In the same manner, they have investigated two other general forms of the PHI-Four equation as

$$(u^n)_{tt} - \mu\,(u^n)_{xx} - u^m + u^n = 0, \tag{1.6}$$

$$(u^{-n})_{tt} - \mu(u^{-n})_{xx} - u^m + u^{-n} = 0, \tag{1.7}$$

where the effect of the positive and negative exponents and the coefficient μ of the second derivative u_{xx} in the obtained solution has been studied.

Recently a large number of papers dealing with the solution of this kind of equations have been presented. Assas et al. (2013) presented an orthogonal spectral collocation scheme for solving the PHI-four equation based on Jacobi family. Ethanes et al. (2013) succeeded in finding the analytical solution of the equation using HPM (Homotopy perturbation method) in different forms of the equation. Wazwaz and Triki (2013) adapted the Anstaz method for finding the solitary wave solution of the equation at hand. Neyrami et al. (2013) applied the $tanh$ method to find the exact and explicit traveling wave solutions of the equation along with other system equations. Soliman and Abdo (2009) use the modified extended direct algebraic method to find a new exact solution of both RWL and PHI-four equations. Sassaman and Biswas (2009) solved the non-linear Klein-Gordon equation using the soliton perturbation theory to find the analytical solution of these equations. Also, Biswas and Cao (2012) presented the phase portrait using the bifurcation analysis and the Anstaz method revealing several solutions of the equation. Lastly, Najafi (2012) found the soliton solution of this equation using the HPM method for solving this type of nonlinear equations which will be solved later.

In this paper, we aim to adapt cubic B-spline collocation method for solving the PHI-four type equations with initial and boundary conditions. The paper is arranged as in Section 2; we apply the temporal discretization to solve the PHI-four type equations where we use cubic B-spline as a basis at the collocation points. In Section 3, we implement the method for the equation producing a system of linear equations to be solved. In Section 4, we propose a stability analysis using Von-Neumann stability analysis to the equation under study proving it to be unconditionally stable. Finally, Section 6 is the closing stage where we present the numerical results of our method and prove that the method is capable of providing accurate results.

2. Temporal discretization

Consider a uniform mesh Δ with the grid points λ_{ij} to discretize the region $\Omega = [a,b] \times [0,\ T]$. Each λ_{ji} is the vertex of the grid points (x_j, t_i) where $x_j = a + jh, j = 0,1,2,\dots,N$ and $t_i = ik, i = 0,1,2,\dots,M$. The quantities h and k are the mesh size in space and time directions, respectively.

Approximating the time derivative by a usual finite difference formula

$$\frac{\partial^2 u^n}{\partial t^2} = \frac{u^{n+1} - 2u^n + u^{n-1}}{k^2} + O(k^2), \tag{2.1}$$

and then substituting the above approximation into Equation (1.1) and discretize in time variable the equation becomes

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{k^2} = \mu \frac{\partial^2 u}{\partial x^2} - \delta u^q - \eta u + \psi(x,t), \tag{2.2}$$

applying the θ-weighted scheme to the space derivatives to Equation (2.2) where $(0 \le \theta \le 1)$

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{k^2} = \theta \left[\mu \frac{\partial^2 u^{n+1}}{\partial x^2} - \delta (u^{n+1})^q - \eta u^{n+1} + \psi(x,t) \right]$$

$$+(1-\theta)\left[\mu\frac{\partial^2 u^n}{\partial x^2}-\delta(u^n)^q-\eta u^n+\psi(x,t)\right], \tag{2.3}$$

where the subscription $n-1, n, n+1$ denote the adjacent time levels. Taking θ to be $\frac{1}{2}$, which corresponds to Crank -Nicholson technique, the above equation becomes

$$\frac{u^{n+1}-2u^n+u^{n-1}}{k^2}=\frac{1}{2}\left[\mu\frac{\partial^2 u^{n+1}}{\partial x^2}-\delta(u^{n+1})^q-\eta u^{n+1}+\psi(x,t)\right]$$

$$+\frac{1}{2}\left[\mu\frac{\partial^2 u^n}{\partial x^2}-\delta(u^n)^q-\eta u^n+\psi(x,t)\right]. \tag{2.4}$$

By rearranging the terms in Equation (2.4), we get the final form of the discretized equation

$$\left(1-\frac{\eta k^2}{2}\right)u^{n+1}+\frac{\mu k^2}{2}u_{xx}^{n+1}-\frac{\delta k^2}{2}(u^{n+1})^q$$

$$=\left(2-\frac{\eta k^2}{2}\right)u^n-u^{n-1}+\frac{\mu k^2}{2}u_{xx}^n-\frac{\delta k^2}{2}(u^n)^q+k^2\psi(x,t). \tag{2.5}$$

The space derivatives are approximated by cubic B-splines which are presented in the next section.

3. Cubic B-splines collocation method

In cubic B-spline collocation method, the approximate solution can be written as a combination of cubic B-splines basis functions for the approximation of space variables under consideration. We consider a mesh $a=x_0<x_1,\dots,x_{N-1},x_N=b$ as a uniform partition of the solution domain $a\le x\le b$ by the knots x_j with $h=x_{j+1}-x_j=\frac{b-a}{N}$, $j=0,\dots,N-1$. Our numerical treatment for solving Equation (2.1) using the collocation method with cubic B-splines is to find an approximate solution $U^N(x,t)$ to the exact $u(x,t)$ in the form

$$U^N(x,t)=\sum_{j=-1}^{n+1}c_j(t)\,B_j(x), \tag{3.1}$$

where $c_j(t)$ are unknown time-dependent quantities to be determined from the boundary conditions and collocation for the differential equation.

The cubic B-splines $B_j(x)$ at the knots is given as follows from Bhatta and Munguia (2015)

$$B_j(x)=\frac{1}{h^3}\begin{cases}(x-x_{j-2})^3, & x\in[x_{j-2},x_{j-1}),\\ (x-x_{j-2})^3-4(x-x_{j-1})^3, & x\in[x_{j-1},x_j),\\ (x_{j+2}-x)^3-4(x_{j+1}-x)^3, & x\in[x_j,x_{j+1}),\\ (x_{j+2}-x)^3, & x\in[x_{j+1},x_{j+2}),\\ 0, & otherwise,\end{cases} \tag{3.2}$$

where $\{B_{-1}, B_0, B_1, \dots, B_{N-1}, B_N, B_{N+1}\}$ form a basis of $a \le x \le b$. The concept is that each cubic B-spline covers four elements so that each element is covered by four cubic B-splines. The values of $B_j(x)$ and its derivative may be tabulated as in Table 1.

Table 1. Coefficients of cubic B-splines and its derivatives at x_j

x	x_{j-2}	x_{j-1}	x_j	x_{j+1}	x_{j+2}
$B_j(x)$	0	1	4	1	0
$B_j'(x)$	0	$3/h$	0	$-3/h$	0
$B_j''(x)$	0	$6/h^2$	$-12/h^2$	$6/h^2$	0

Using approximate function in Equation (3.1) and cubic B-splines functions from Equation (3.2), the approximate values of $U^N(x_j)$ and its two derivatives at the knots are determined in terms of the time parameters c_j as follows

$$U_j = c_{j-1} + 4c_j + c_{j+1},$$

$$U_j' = \frac{3}{h}\left(c_{j-1} + c_{j+1}\right),$$

$$U_j'' = \frac{6}{h^2}\left(c_{j-1} - 2\,c_j + c_{j+1}\right). \tag{3.3}$$

Substituting Equation (3.1) into Equation (2.5) yields the following equation

$$\left(1 - \frac{\eta k^2}{2}\right)\sum_{j=-1}^{n+1} c_j^{n+1}(t)\,B_j(x) + \frac{\mu k^2}{2}\sum_{j=-1}^{n+1} c_j^{n+1}(t)\,B_j''(x) - \frac{\delta k^2}{2}\left(\sum_{j=-1}^{n+1} c_j^{n+1}(t)\,B_j(x)\right)^q$$

$$= \left(2 - \frac{\eta k^2}{2}\right)\sum_{j=-1}^{n+1} c_j^{n}(t)\,B_j(x) - \sum_{j=-1}^{n+1} c_j^{n-1}(t)\,B_j(x) + \frac{\mu k^2}{2}\sum_{j=-1}^{n+1} c_j^{n}(t)\,B_j''(x)$$

$$-\frac{\delta k^2}{2}\left(\sum_{j=-1}^{n+1} c_j^{n}(t)\,B_j(x)\right)^q + k^2\psi(x,t). \tag{3.4}$$

Then simplifying the above equation leads to the following system of linear equations:

$$a'\,c_{j+1}^{n+1} + b'\,c_j^{n+1} + a'\,c_{j-1}^{n+1} = c'\,c_{j+1}^{n} + d'\,c_j^{n} + c'\,c_{j-1}^{n} - 2\left(c_{j-1}^{n-1} + 4c_j^{n-1} + c_{j+1}^{n-1}\right), \tag{3.5}$$

where $j = 0,1,2,\dots,N$ and

$$a' = 1 - k^2(\frac{\eta+\delta}{2} - \frac{3\mu}{h^2})$$
$$b' = 4 - k^2(2\eta + \frac{6\mu}{h^2} + \frac{\delta\,(4)^q}{2})$$
$$c' = 2 - k^2(\frac{\eta+\delta}{2} - \frac{3\mu}{h^2})$$
$$d' = 8 - k^2\left(2\eta + \frac{6\mu}{h^2} + \frac{\delta\,(4)^q}{2}\right).$$

In order to obtain a unique solution of the above system, two additional constraints are required. These constraints are obtained from the boundary conditions from Equation (1.3) or (1.4) depending on the type of conditions. Imposing the boundary conditions enables us to add the parameters c_{-1} and c_{N+1} in the system and the system of Equations (3.5) can be reduced to $(N+1)$ linear equation in $(N+1)$ unknowns. Solving the resulting system gives the unknown coefficients c_j and the approximate solution then can be calculated from Equation (3.1) as well.

4. Stability analysis

The stability of the proposed scheme is investigated by using Von -Neumann stability. The proposed scheme takes the following form

$$a'\,c_{j+1}^{n+1} + b'\,c_j^{n+1} + a'\,c_{j-1}^{n+1} = c'\,c_{j+1}^{n} + d'\,c_j^{n} + c'\,c_{j-1}^{n} - 2\left(c_{j-1}^{n-1} + 4c_j^{n-1} + c_{j+1}^{n-1}\right), \qquad (4.1)$$

setting $c_j^n = \xi^n e^{i\beta jh}, i = \sqrt{-1}$ in Equation (4.1) after simplifying it can be written as

$$\xi^2[2a'\cos(\beta h) + b' + \cos(\beta h)] - \xi[2c'\cos(\beta h) + d'] + [4\cos(\beta h) + 8] = 0. \qquad (4.2)$$

Then, by letting

$$Q = 2a'\cos(\beta h) + b' + \cos(\beta h)\,,\;\; D = 2c'\cos(\beta h) + d',\;\; Z = 4\cos(\beta h) + 8, \qquad (4.3)$$

Equation (4.2) becomes

$$Q\xi^2 - D\xi + Z = 0, \qquad (4.4)$$

by applying the Routh-Hurwitz criterion on Equation (4.4). By using the transformation $\xi = \frac{1+\nu}{1-\nu}$ and simplifying the above equation, it produces

$$(Q + D + Z)\,\nu^2 + 2(Q - Z)\nu + (Q - D + Z) = 0. \qquad (4.5)$$

The necessary and sufficient condition for $|\xi| \le 1$ is that

$$Q + D + Z \ge 0, \qquad Q - Z \ge 0, \qquad Q - D + Z \ge 0, \qquad (4.6)$$

and after substituting, we get the values as

$$Q + D + Z = (2a' + 2c' + 4)\cos(\beta h) + (b' + d' + 8),$$
$$Q - Z = (2a' - 4)\cos(\beta h) + (b' - 8),$$
$$Q - D + Z = (2a' - 2c' + 4)\cos(\beta h) + (b' - d' + 8). \qquad (4.7)$$

By letting $\varphi = \frac{1}{2}\beta h$ and the relation $\cos(2\varphi) = 2\cos^2(\varphi) - 1$ and substituting into Equation (4.7) with the values of a', b', c', and then calculating the terms, we get

$$Q + D + Z = (4a' + 4c' + 8)\cos^2(\varphi) - 2a' - 2c' + b' + d' + 4 \geq 0,$$
$$Q - Z = (4a' - 8)\cos^2(\varphi) - 2a' + b' - 4 \geq 0\,,$$
$$Q - D + Z = (4a' - 4c' + 8)\cos^2(\varphi) - 2a' + 2c' + b' - d' + 4 \geq 0\,. \tag{4.8}$$

The conclusion is that all the values are greater than 0 which is evidence that the scheme is unconditionally stable. It means that there is no restriction on the grid size but the grid size should be chosen in such a way that the accuracy of the scheme is not degraded.

7. Numerical experiments and discussion

To illustrate the performance of the presented method, a numerical example is given in this section. We use L_∞ norm described by the following relation for finding the maximum error of the solution

$$L_\infty = \max_{j\geq 0}\left|u_j^{exact} - U_j^{approximate}\right|.$$

We consider Equation (1.1) under the initial and boundary conditions given in Equation (1.2) with the values of the constants as $\mu = 1, \eta = -1, \delta = 1, q = 3$ and $\psi(x,t) = 0$ which gives the PHI-four equation as

$$u_{tt} - u_{xx} - u + u^3 = 0. \tag{7.1}$$

Since there is no exact solution, we will compare our method with the analytical solution which is given in Eshani et al. (2013) as follows.

$$u(x,t) = xt + \frac{xt^3}{6} - \frac{x^3t^5}{20} + \frac{xt^5}{120} - \frac{xt^7}{140} - \frac{x^3t^7}{840} - \frac{x^3t^{11}}{23760} + \\ + \frac{x^5t^{13}}{37440} - \frac{x^7t^{15}}{168000} + \frac{x^9t^{17}}{2176000}. \tag{7.2}$$

The results are computed for different time levels. Comparisons of approximate and exact solutions at different nodes x and different time levels are reported in Table 4 and L_∞ error norm is reported in Table 2 and Table 3. The graphs of exact and approximate solutions for different time levels are depicted in Figure 3 and Figure 4. A comparison between the errors is presented in Figure 1 and Figure 2. It is evident from these tables and figures that the presented method gives numerical results in good agreement with the exact solutions. To calculate the point wise rate of convergence, the algorithm has been run for various space steps on the tables with each equal interval, respectively. Also, the order of the convergence for the numerical method is illustrated in Table 5 for different time levels and different space levels, and is computed by the formula

$$\text{order of convergance} = \frac{log_{10}\left(\frac{\left|u-U_{h_i}\right|}{\left|u-U_{h_{i+1}}\right|}\right)}{\log_{10}\left(\left|\frac{h_i}{h_{i+1}}\right|\right)}.$$

Table 2. Maximum error with $0 < x < 1$ and $0 \leq t \leq 0.01$

x/t	**0.001**	**0.003**	**0.005**	**0.007**	**0.009**	**0.01**
0.0	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
0.1	1.67E-11	1.30E-10	6.79E-10	1.63E-09	3.00E-09	3.83E-09
0.2	3.33E-11	9.35E-10	2.69E-09	5.23E-09	8.55E-09	1.05E-08
0.3	5.00E-11	6.81E-10	8.07E-11	1.77E-09	4.90E-09	6.94E-09
0.4	6.67E-11	5.59E-09	1.28E-08	2.14E-08	3.14E-08	3.69E-08
0.5	8.33E-11	1.30E-08	2.37E-08	3.21E-08	3.79E-08	3.98E-08
0.6	1.00E-10	4.42E-08	9.03E-08	1.38E-07	1.87E-07	2.11E-07
0.7	1.17E-10	1.18E-07	2.31E-07	3.40E-07	4.44E-07	4.93E-07
0.8	1.33E-10	3.00E-07	6.00E-07	9.01E-07	1.20E-06	1.35E-06
0.9	1.50E-10	5.53E-07	1.10E-06	1.64E-06	2.17E-06	2.43E-06
1.0	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00

Table 3. Maximum error with $0 < x < 1$ and $0 \leq t \leq 0.1$

x/t	0.01	0.03	0.05	0.07	0.09	0.1
0.0	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
0.1	1.67E-08	2.01E-07	9.26E-07	2.05E-06	3.48E-06	4.30E-06
0.2	3.33E-08	6.92E-07	1.83E-06	3.71E-06	6.73E-06	8.71E-06
0.3	5.00E-08	5.05E-08	2.62E-06	6.81E-06	1.13E-05	1.35E-05
0.4	6.67E-08	3.48E-06	4.68E-06	5.47E-06	9.64E-06	1.39E-05
0.5	8.33E-08	7.32E-06	9.19E-07	1.55E-05	3.21E-05	3.75E-05
0.6	1.00E-07	2.99E-05	3.08E-05	6.87E-06	2.12E-05	2.87E-05
0.7	1.17E-07	8.65E-05	9.56E-05	2.41E-05	9.58E-05	1.58E-04
0.8	1.33E-07	2.46E-04	3.57E-04	3.05E-04	1.14E-04	1.56E-05
0.9	1.50E-07	5.02E-04	8.63E-04	1.04E-03	1.04E-03	9.86E-04
1.0	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00

Table 4. Comparison of the Approximate and exact solution at $t \leq 1.0$

x	t = 0.5		t = 0.7		t = 1.0	
	Exact	**presented**	**Exact**	**presented**	**Exact**	**presented**
0.1	0.05210	0.05188	0.07579	0.07322	0.11673	0.11274
0.2	0.10420	0.10324	0.15153	0.14597	0.23316	0.22595
0.3	0.15627	0.15352	0.22716	0.21819	0.34897	0.33964
0.4	0.20831	0.20214	0.30265	0.29048	0.46386	0.45297
0.5	0.26032	0.24894	0.37793	0.36421	0.57753	0.56406
0.6	0.31228	0.29500	0.45295	0.44063	0.68965	0.67045
0.7	0.36419	0.34382	0.52767	0.51910	0.79993	0.76996
0.8	0.41603	0.40026	0.60203	0.59591	0.90806	0.86128
0.9	0.46779	0.46225	0.67598	0.66705	1.01374	0.94335
1.0	0.51947	0.51632	0.74948	0.73247	1.11665	1.01426

Table 5. The order of convergence, $k = 0.01$

h_i	$\lvert u_j - U_j \rvert_{L_\infty}$	Order(t=0.08)	$\lvert u_j - U_j \rvert_{L_\infty}$	Order(t=0.1)
0.01	0.00304401		0.0077414	
0.02	0.00669762	1.137674552	0.0158813	1.036662335
0.03	0.01150527	1.334400618	0.0246507	1.08433985
0.04	0.01783823	1.524387417	0.0340141	1.119178568
0.05	0.02574425	1.644087369	0.0436986	1.122780344
0.06	0.03484264	1.659871346	0.0533605	1.09562413
0.07	0.04443221	1.577176487	0.0627918	1.05580443
0.08	0.05388875	1.445031546	0.0719981	1.024589251
0.09	0.06304699	1.332608206	0.0810903	1.009686001

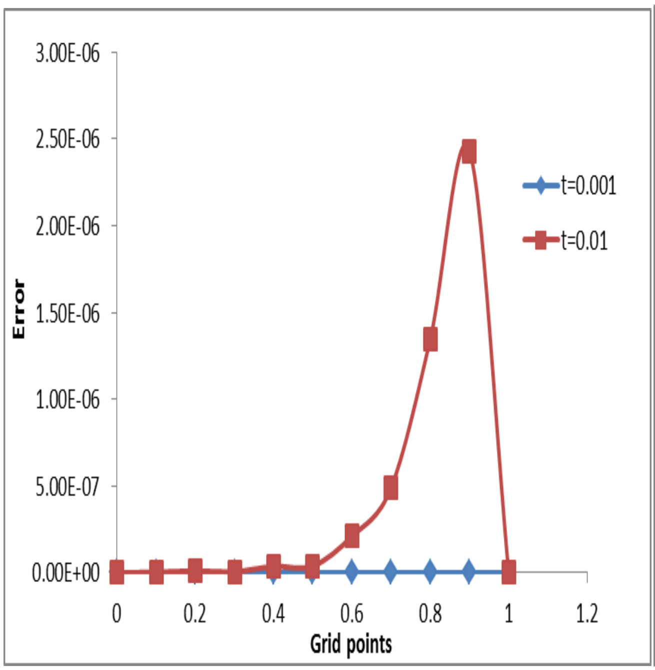


Figure 1. Maximum error comparison at $t = 0.01$ *and* $t = 0.001$

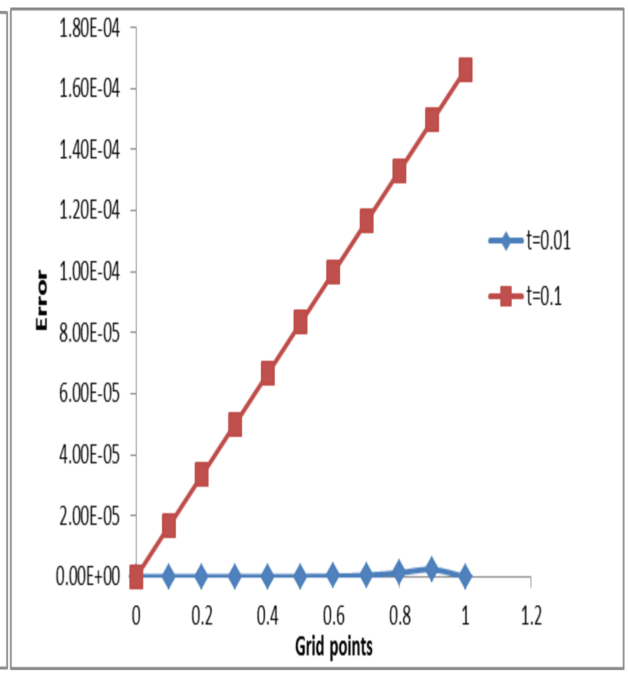
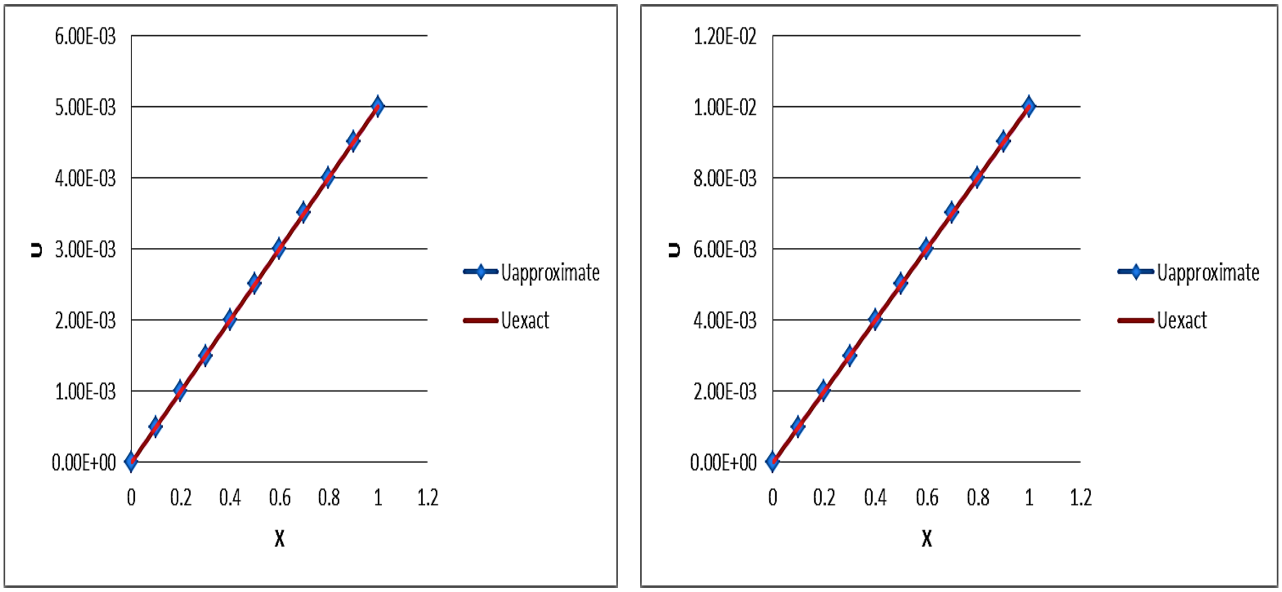
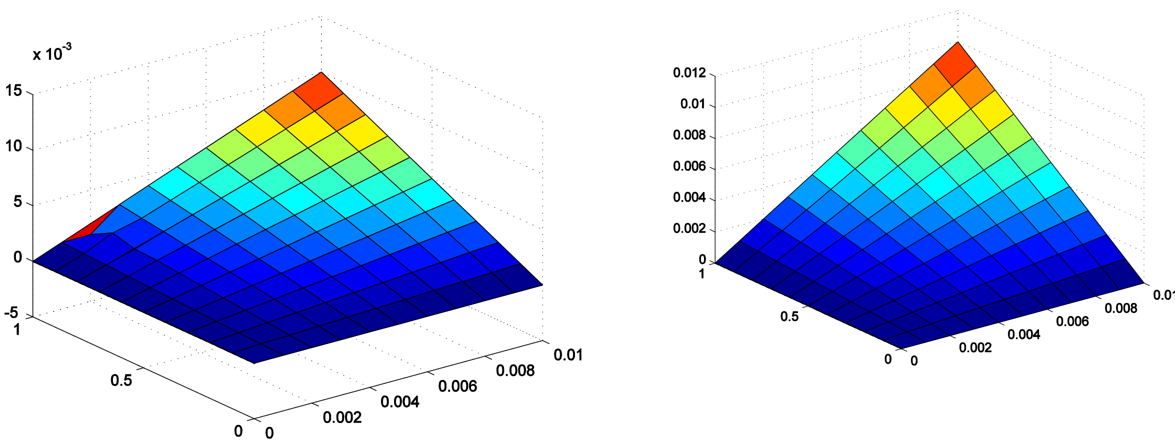


Figure 2. Maximum error comparison at $t = 0.01$ *and* $t = 0.1$



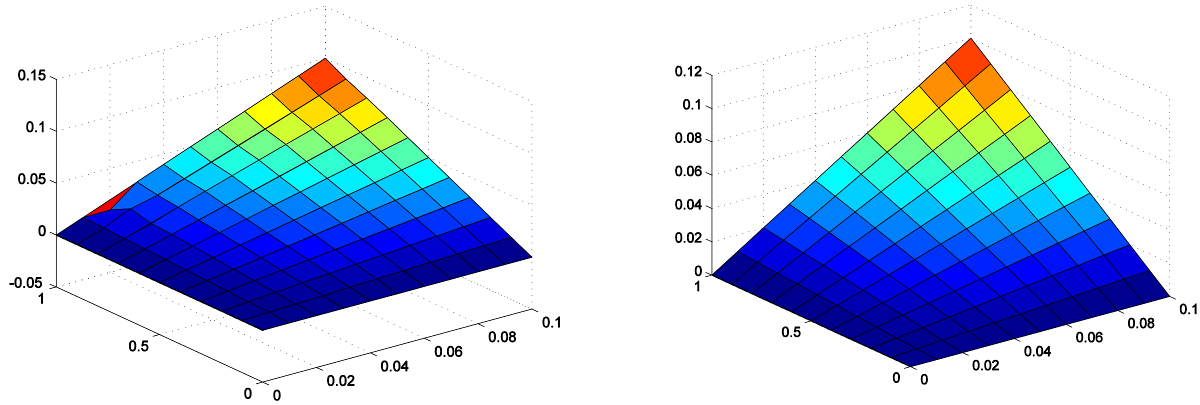
(a) (b)

Figure 3. Approximate and Exact solution at (a) $t = 0.005$, (b) $t = 0.01$



(a) (b)

Figure 4. (a) Approximate solution (b) Exact solution at $0 \le t \le 0.01$



(a) (b)

Figure 5. (a) Approximate solution (b) Exact solution at $0 \leq t \leq 0.1$.

8. Conclusion

In this paper, we developed a robust collocation method based on cubic B-spline as a basis for the calculation of an approximate solution for the famous PHI-four equation. The numerical solution is obtained using a three -time level implicit scheme based on a cubic B-spline for space derivatives and Crank-Nicholson finite difference discretization for time derivatives. During the computations, we found that the scheme is unconditionally stable using Von-Neumann stability analysis. To examine the accuracy and efficiency of the proposed algorithm we gave an example. These computational results show that our proposed algorithm is effective and accurate.

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