Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Vol. 6, Issue 1 (June 2011) pp. 292 – 303
(Previously, Vol. 6, Issue 11, pp. 2033 – 2044)

Applications and Applied Mathematics:
An International Journal
(AAM)

Analytic Investigation of the KP-Joseph-Egri Equation for Traveling Wave Solutions

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Received: February 20, 2010; Accepted: April 7, 2011

Abstract

By means of the two distinct methods, the cosine-function method and the $(G'/G)-$expansion method, we successfully performed an analytic study on the KP-Joseph-Egri (KP-JE) equation. We exhibited its further closed form traveling wave solutions which reduce to solitary and periodic waves.

Keywords: Cosine-function method; $(G'/G)-$expansion method; KP-JE equation; (3 + 1)-dimensional nonlinear evolution equation; (2 + 1)-dimensional nonlinear KP-BBM equation

MSC 2010 No.: 47F05; 35Q53; 35Q55; 35G25.

1. Introduction

Phenomena in physics and other fields are often described by nonlinear evolution equations. In order to understand the physical mechanism of such phenomena in nature, exact solutions for the

nonlinear evolution equations have to be explored, as in, for example, the wave phenomena observed in fluid dynamics, plasma and elastic media and optical fibers, etc. Thus, the methods for deriving exact solutions for the governing equations have to be developed. Recently, many powerful methods have been established and improved. Among these methods, we cite the tanh method [Malfliet (1992), Malfliet and Hereman (1996)], the inverse scattering method [Gardner et al. (1967)], Hirota's direct method [Hirota (1971)], extended tanh-function method [Fan (2000)], the Jacobian elliptic function expansion method [Liu et al. (2001)], the homogeneous balance method [Wang (1995)], $(G'/G)-$expansion method [Wang (2008)], and the cosine-function method [Ali et al. (2007)] and so on.

The aim of this paper is to find the exact solutions of some important nonlinear partial differential equations by using the cosine-function method and the $(G'/G)-$expansion method.

2. Methodology

In this section, we briefly highlight the main features of the cosine-function and the $(G'/G)-$expansion method.

We consider the nonlinear partial differential equation in the form

$$F(u,u_x,u_y,u_t,u_{xx},u_{xy},...)=0, \tag{1}$$

where $u=u(x,y,t)$ is the solution of nonlinear partial differential equation (1). We use the transformations

$$u(x,y,t)=u(\xi), \qquad \xi=k(x+ly-ct). \tag{2}$$

This enables us to use the following changes:

$$\frac{\partial}{\partial t}(.)=-kc\frac{\partial}{\partial \xi}(.), \quad \frac{\partial}{\partial x}(.)=k\frac{\partial}{\partial \xi}(.), \quad \frac{\partial}{\partial y}(.)=kl\frac{\partial}{\partial \xi}(.),$$

$$\frac{\partial^2}{\partial x^2}(.)=k^2\frac{\partial^2}{\partial \xi^2}(.),... \tag{3}$$

We use (3) to change the nonlinear partial differential equation (1) to nonlinear ordinary differential equation

$$G(u(\xi),\frac{\partial u(\xi)}{\partial \xi},\frac{\partial^2 u(\xi)}{\partial \xi^2},...)=0. \tag{4}$$

2.1. The Cosine-Function Method

The solution of equation (4) can be expressed in the form:

$$u(\xi) = \lambda \cos^{\beta}(\mu\xi), \qquad |\xi| \le \frac{\pi}{2\mu}, \tag{5}$$

where λ, β and μ are unknown parameters which will be determined. Then we have:

$$\begin{aligned} u'(\xi) &= -\lambda\beta\mu \cos^{\beta-1}(\mu\xi)\sin(\mu\xi), \\ u''(\xi) &= -\lambda\mu^2\beta^2 \cos^{\beta}(\mu\xi) + \lambda\mu^2\beta(\beta-1)\cos^{\beta-2}(\mu\xi). \end{aligned} \tag{6}$$

Substituting equation (5) and equation (6) into the nonlinear ordinary differential equation (4) gives a trigonometric equation of $\cos^{\alpha}(\mu\xi)$ terms. To determine the parameters we first balance the exponents of each pair of cosine to determine α. Then we collect all terms with the same power in $\cos^{\alpha}(\mu\xi)$ and set to zero their coefficients to get a system of algebraic equations among the unknowns λ, β and μ. Now, the problem is reduced to a system of algebraic equations that can be solved to obtain the unknown parameters λ, β and μ. Hence, the solution considered in equation (5) is obtained.

2.2. The $(G'/G)-$Expansion Method

We initially predict the structure of the solution $u = u(\xi)$ to equation (4) in the finite series form

$$u(\xi) = \sum_{i=0}^{N} a_i \left(\frac{G'}{G}\right)^i, \quad G'' + \lambda G' + \mu G = 0, \tag{7}$$

where $G = G(\xi)$ and primes denote derivatives with respect to ξ; $a_i, \lambda,$ and μ are constants to be specified later. The positive integer N can be determined by the homogeneous balance method. Substituting (7) into equation (4) yields a system of nonlinear algebraic equations for a_i, λ, μ, k, l and c.

Finally, substitution of the system's solutions into (7) gives traveling wave solutions to equation (1).

Remark 1:

The second order LODE (7) has the following solutions:

When

$$\lambda^2 - 4\mu > 0,$$

$$\frac{G'}{G} = -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2}\left(\frac{C_1 \sinh(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi) + C_2 \cosh(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi)}{C_1 \cosh(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi) + C_2 \sinh(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi)}\right).$$

When

$$\lambda^2 - 4\mu < 0,$$

$$\frac{G'}{G} = -\frac{\lambda}{2} + \frac{\sqrt{4\mu - \lambda^2}}{2}\left(\frac{-C_1 \sin(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi) + C_2 \cos(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi)}{C_1 \cos(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi) + C_2 \sin(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi)}\right). \tag{8}$$

When

$$\lambda^2 - 4\mu = 0,$$

$$\frac{G'}{G} = -\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2\xi},$$

where C_1 and C_2 are arbitrary constants.

3. Analysis

Example 3.1.

In Yu and Ma (2010), a (2 + 1)-dimensional nonlinear KP-BBM equation

$$(u_t + u_x - \alpha(u^2)_x - \beta u_{xxt})_x + \gamma u_{yy} = 0 \tag{9}$$

was introduced with the aid of the BMM equation and the KP equation. Hereman et al. (1986) used direct search method to obtain exact solutions of the BBM equation

$$u_t + u_x - \alpha(u^2)_x - \beta u_{xxt} = 0, \tag{10}$$

and the Joseph-Egri (JE) equation

$$u_t + u_x + \alpha u u_x + u_{xtt} = 0, \tag{11}$$

and the KP equation:

$$(u_t + \alpha u u_x + u_{xxx})_x + \gamma u_{yy} = 0. \tag{12}$$

In this paper, we obtain the (2 + 1)-dimensional nonlinear KP-Joseph-Egri (JE) equation

$$(u_t + u_x + a u u_x + u_{xtt})_x + b u_{yy} = 0$$

with the aid of Joseph-Egri(JE) equation (11) and the KP equation (12).

Let us now consider the (2+1)-dimensional KP-JE equation

$$(u_t + u_x + a u u_x + u_{xtt})_x + b u_{yy} = 0. \tag{13}$$

We take the transformation

$$u(x, y, t) = u(\xi), \qquad \xi = k(x + ly - ct).$$

The substitution of the transformation into (13) yields the ODE

$$(bl^2 + 1 - c)u(\xi) + \frac{a}{2}u^2(\xi) + k^2c^2u''(\xi) = 0, \tag{14}$$

obtained by integrating the resulting equation and by considering each constant of integration to be zero.

Using the Cosine-Function Method

Substituting equation (6) into (14) gives:

$$(bl^2 + 1 - c)\lambda \cos^{\beta}(\mu\xi) + \frac{a\lambda^2}{2}\cos^{2\beta}(\mu\xi)$$
$$-k^2c^2\lambda\mu^2\beta^2\cos^{\beta}(\mu\xi) + k^2c^2\lambda\mu^2\beta(\beta - 1)\cos^{\beta-2}(\mu\xi) = 0.$$

By equating the exponents and the coefficients of each pair of the cosine function we obtain the following system of algebraic equations:

$$\begin{aligned}
&\frac{a\lambda^2}{2} + k^2c^2\lambda\mu^2\beta(\beta-1) = 0,\\
&(bl^2+1-c)\lambda - k^2c^2\lambda\mu^2\beta^2 = 0,\\
&2\beta = \beta - 2.
\end{aligned} \tag{15}$$

Solving the system (15), we have

$$\lambda = -\frac{3}{a}(bl^2+1-c), \qquad \mu = \pm\frac{\sqrt{(bl^2+1-c)}}{2kc}, \qquad \beta = -2. \tag{16}$$

So we obtain the exact soliton solutions of the (2+1)-dimensional KP-JE equation in the form

$$u_1(x,y,t) = -\frac{3}{a}(bl^2+1-c)\sec^2[\frac{\sqrt{(bl^2+1-c)}}{2kc}(k(x+ly-ct)+\xi_0)],$$

for

$$(bl^2+1-c) > 0.$$

$$u_2(x,y,t) = -\frac{3}{a}(bl^2+1-c)\sec h^2[\frac{\sqrt{(c-bl^2-1)}}{2kc}(k(x+ly-ct)+\xi_0)],$$

for

$$(bl^2+1-c) < 0.$$

Using the $(G'/G)-$Expansion Method

Now, we assume that the solution of equation (14) can be expressed as in (7). By the homogeneous balance principle, we determine that $N=2$. Hence, we look for solutions to equation (14) in the form

$$u(\xi) = a_0 + a_1(\frac{G'}{G}) + a_2(\frac{G'}{G})^2, \quad G'' + \lambda G' + \mu G = 0. \tag{17}$$

Substituting equation (17) into equation (14), the left-hand side of equation (14) is converted into a polynomial of $(\frac{G'}{G})^j$ $(j=0,1,2,...)$, then setting each coefficient to zero, we get a set of under-determined algebraic equations for a_i $(i=0,1,2), k, l, c, \lambda$ and μ:

$$
\begin{aligned}
&(bl^2+1-c)a_0+\frac{1}{2}aa_0^2+k^2c^2(a_1\lambda\mu+2a_2\mu^2)=0,\\
&(bl^2+1-c)a_1+k^2c^2(2a_1\mu+a_1\lambda^2+6a_2\lambda\mu)+aa_0a_1=0,\\
&(bl^2+1-c)a_2+k^2c^2(4a_2\lambda^2+3a_1\lambda+8a_2\mu)+\frac{a}{2}(2a_0a_2+a_1^2)=0,\\
&k^2c^2(2a_1+10a_2\lambda)+aa_1a_2=0,\\
&\frac{1}{2}aa_2^2+6k^2c^2a_2=0.
\end{aligned}
\tag{18}
$$

Solving these under-determined algebraic equations, we get the following result:

$$
\begin{aligned}
&a_1=\frac{4ck}{a}\sqrt{3(bl^2+1-c)-3aa_0},\quad a_2=-\frac{12k^2c^2}{a},\quad \mu=-\frac{2bl^2+2-2c+aa_0}{12k^2c^2},\\
&\lambda=-\frac{1}{3ck}\sqrt{3(bl^2+1-c)-3aa_0},
\end{aligned}
\tag{19}
$$

where a_0, k, l and c are arbitrary constants.

Substituting equation (19) into equation (17), we have

$$
u(\xi)=a_0+\frac{4ck}{a}\sqrt{3(bl^2+1-c)-3aa_0}\,(\frac{G'}{G})-\frac{12k^2c^2}{a}(\frac{G'}{G})^2. \tag{20}
$$

Substituting the general solutions of LODE (7) into equation (20); we have two types of travelling wave solutions of the (2 + 1)-dimensional KP-JE equation in the following:

$$
u_3(x,y,t)=\frac{(bl^2+1-c)}{a}[1-3c(\frac{C_1\sinh(\frac{\sqrt{bl^2+1-c}}{2ck}(\xi+\xi_0))+C_2\cosh(\frac{\sqrt{bl^2+1-c}}{2ck}(\xi+\xi_0))}{C_1\cosh(\frac{\sqrt{bl^2+1-c}}{2ck}(\xi+\xi_0))+C_2\sinh(\frac{\sqrt{bl^2+1-c}}{2ck}(\xi+\xi_0))})^2]
$$

for

$$\lambda^2 - 4\mu = \frac{bl^2 + 1 - c}{c^2 k^2} > 0.$$

$$u_4(x, y, t) = \frac{(bl^2 + 1 - c)}{a}[1 + 3c(\frac{-C_1 \sin(\frac{\sqrt{c - bl^2 - 1}}{2ck}(\xi + \xi_0)) + C_2 \cos(\frac{\sqrt{c - bl^2 - 1}}{2ck}(\xi + \xi_0))}{C_1 \cos(\frac{\sqrt{c - bl^2 - 1}}{2ck}(\xi + \xi_0)) + C_2 \sin(\frac{\sqrt{c - bl^2 - 1}}{2ck}(\xi + \xi_0))})^2]$$

for

$$\lambda^2 - 4\mu = \frac{bl^2 + 1 - c}{c^2 k^2} < 0.$$

If

$$C_1 \neq 0, C_2 = 0,$$

then

$$u_5(x, y, t) = \frac{(bl^2 + 1 - c)}{a}[1 - 3c \tanh^2(\frac{\sqrt{bl^2 + 1 - c}}{2ck}(k(x + ly - ct) + \xi_0))],$$

for

$$(bl^2 + 1 - c) > 0.$$

$$u_6(x, y, t) = \frac{(bl^2 + 1 - c)}{a}[1 - 3c \tan^2(\frac{\sqrt{c - bl^2 - 1}}{2ck}(k(x + ly - ct) + \xi_0))],$$

for

$$(bl^2 + 1 - c) < 0.$$

Example 2.

In [Geng (2003)], a (3 + 1)-dimensional nonlinear evolution equation

$$3w_{xz} - (2w_t + w_{xxx} - 2ww_x)_y + 2(w_x \partial_x^{-1} w_y)_x = 0, \tag{21}$$

was derived with its algebraic-geometrical solutions exactly given in terms of the Riemann theta functions. Further, Geng and Ma (2007) derived an N-soliton solution of the equation and its

Wronskian form by using the Hirota method and Wronskian technique. Let us construct traveling wave solutions of this equation with the aid of the cosine function method by using the transformation

$$w(t,x,y,z)=w(\xi), \qquad \xi=x+ay+bt+cz.$$

Then, equation (21) becomes

$$(3c-2ab)w''-aw''''+4a(w')^2+4aww''=0. \tag{22}$$

Integrating (22) twice with respect to ξ, and choosing the constants of integration as zero, we arrive at

$$(3c-2ab)w-aw''+2aw^2=0. \tag{23}$$

Using Cosine-Function Method

Substituting equation (6) into (23) gives:

$$\begin{aligned}&(3c-2ab)\lambda\cos^{\beta}(\mu\xi)+a\lambda\mu^2\beta^2\cos^{\beta}(\mu\xi)-a\lambda\mu^2\beta(\beta-1)\cos^{\beta-2}(\mu\xi)\\&+2a\lambda^2\cos^{2\beta}(\mu\xi)=0.\end{aligned} \tag{24}$$

By comparing the exponents and the coefficients of each pair of the cosine function we obtain the following system of algebraic equations:

$$\begin{aligned}&(3c-2ab)\lambda+a\lambda\mu^2\beta^2=0,\\&-a\lambda\mu^2\beta(\beta-1)+2a\lambda^2=0,\\&2\beta=\beta-2.\end{aligned} \tag{25}$$

Solving the system (25), we have

$$\lambda=\frac{6ab-9c}{4a}, \qquad \mu=\pm\sqrt{\frac{2ab-3c}{4a}}, \qquad \beta=-2. \tag{26}$$

So we obtain the exact soliton solutions of the (3 + 1)-dimensional nonlinear evolution equation (21) in the form

$$w_1(t,x,y,z)=\frac{6ab-9c}{4a}\sec^2[\sqrt{\frac{2ab-3c}{4a}}(x+ay+bt+cz+\xi_0)]$$

for

$$\frac{2ab-3c}{4a}>0.$$

$$w_2(t,x,y,z)=\frac{6ab-9c}{4a}\sec h^2[\sqrt{\frac{3c-2ab}{4a}}(x+ay+bt+cz+\xi_0)]$$

for

$$\frac{2ab-3c}{4a}<0.$$

Using the $(G'/G)-$Expansion Method

Now, we assume that the solution of equation (23) can be expressed as the ansatz (7). By the homogeneous balance principle, we determine that $N=2$. Hence, we look for solutions to equation (23) in the form

$$u(\xi)=a_0+a_1(\frac{G'}{G})+a_2(\frac{G'}{G})^2,\quad G''+\lambda G'+\mu G=0. \tag{27}$$

Substituting equation (27) into equation (23), the left-hand side of equation (23) is converted into a polynomial of $(\frac{G'}{G})^j\ (j=0,1,2,...)$, then setting each coefficient to zero, we get a set of under-determined algebraic equations for $a_i\ (i=0,1,2), a, b, c, \lambda$ and μ:

$$\begin{aligned}
&(3c-2ab)a_0-a(a_1\lambda\mu+2a_2\mu^2)+2aa_0^2=0,\\
&(3c-2ab)a_1+4aa_0a_1-a(2a_1\mu+a_1\lambda^2+6a_2\lambda\mu)=0,\\
&(3c-2ab)a_2+2a(2a_0a_2+a_1^2)-a(4a_2\lambda^2+3a_1\lambda+8a_2\lambda)=0,\\
&4aa_1a_2-a(2a_1+10a_2\lambda)=0,\\
&-6aa_2+2aa_2^2=0.
\end{aligned} \tag{28}$$

Solving these under-determined algebraic equations, we get the following result:

$$a_0=\frac{-6ab+9c+aa_1^2}{12a},\quad a_2=3,\quad \mu=\frac{27c-18ab+aa_1^2}{36a},\ \lambda=\frac{a_1}{3}, \tag{29}$$

where a_1, a, b and c are arbitrary constants.

Substituting equation (29) into equation (27), we have

$$u(\xi)=\frac{-6ab+9c+aa_1^2}{12a}+a_1(\frac{G'}{G})+3(\frac{G'}{G})^2. \tag{30}$$

Substituting the general solutions of LODE (7) into equation (30); we have two types of travelling wave solutions of the (3 + 1)-dimensional nonlinear evolution equation (21) in the following:

$$w_3(t,x,y,z)=\frac{(2ab-3c)}{4a}[-1+3(\frac{C_1\sinh(\sqrt{\frac{2ab-3c}{4a}}(\xi+\xi_0))+C_2\cosh(\sqrt{\frac{2ab-3c}{4a}}(\xi+\xi_0))}{C_1\cosh(\sqrt{\frac{2ab-3c}{4a}}(\xi+\xi_0))+C_2\sinh(\sqrt{\frac{2ab-3c}{4a}}(\xi+\xi_0))})^2]$$

for

$$\lambda^2-4\mu=\frac{2ab-3c}{a}>0.$$

$$w_4(t,x,y,z)=\frac{(2ab-3c)}{4a}[-1-3(\frac{-C_1\sin(\sqrt{\frac{3c-2ab}{4a}}(\xi+\xi_0))+C_2\cosh(\sqrt{\frac{3c-2ab}{4a}}(\xi+\xi_0))}{C_1\cos(\sqrt{\frac{3c-2ab}{4a}}(\xi+\xi_0))+C_2\sin(\sqrt{\frac{3c-2ab}{4a}}(\xi+\xi_0))})^2]$$

for

$$\lambda^2-4\mu=\frac{2ab-3c}{a}<0.$$

If

$$C_1\neq 0, C_2=0,$$

then

$$w_5(t,x,y,z)=\frac{2ab-3c}{4a}[-1+3\tanh^2(\sqrt{\frac{2ab-3c}{4a}}(x+ay+bt+cz+\xi_0)],$$

for

$$\frac{2ab-3c}{a}>0.$$

$$w_6(t,x,y,z) = \frac{2ab-3c}{4a}[-1+3\tan^2(\sqrt{\frac{3c-2ab}{4a}}(x+ay+bt+cz+\xi_0)],$$

for

$$\frac{2ab-3c}{a} < 0.$$

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