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On the Slow Growth and Approximation of Entire Function Solutions of Second-Order Elliptic Partial Differential Equations on Carathéodory Domains

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Abstract

In this paper we consider the regular, real-valued solutions of the second-order elliptic partial differential equation. The characterization of generalized growth parameters for entire function solutions for slow growth in terms of approximation errors on more generalized domains, i.e., Carathéodory domains, has been obtained. Moreover, we studied some inequalities concerning the growth parameters of entire function solutions of above equation for slow growth which have not been studied so far.

Keywords: L^p-approximation error; Carathéodory domain; Fourier coefficients; generalized parameters; elliptic partial differential equation and slow growth

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1. Introduction

The linear second order elliptic partial differential equation is given in the form

$$L(v) = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + a(x,y)\frac{\partial v}{\partial x} + b(x,y)\frac{\partial v}{\partial y} + c(x,y)v = 0 \tag{1}$$

on a simply-connected domain $D \subset \mathbb{C}$ about the origin. The real analytic coefficients are entire functions on $\mathbb{C}^2$ and the coefficient $c(x,y) \leq 0$ on D. The regular, real-valued solutions of (1) in the Bergman canonical form are analytic functions of a single complex-variable.

There are so many applications of the solutions of (1) in several areas of mathematical physics; for example, its solutions arise in the Maxwell system for the modelling of electric or magnetic n-poles studied by Gilbert and Roger (1970), potential scattering by Gilbert (1969), in quasi-stationary (time independent) diffusion processes by Gilbert (1969), Gilbert and Roger (1970) and as the initial data for parabolic partial differential equations by Colton (1976).

McCoy (1979, 1980) considered certain elliptic partial differential equations and studied the growth of solutions by using function-theoretic methods found in the work of Gilbert (1969) and Gilbert and Roger (1970). These methods map the analytic function associate f onto the solution and then relate the respective domains of analyticity of associate f and regularity of the solution by approximation methods.

Using the standard procedure of Bergman (1961), Colton (1976), and Gilbert (1969), we introduce the transform of $\mathbb{C}^2$ into itself,

$$z = x + iy, z^* = x - iy, (x,y) \in \mathbb{C}^2,$$

where $z^* = \bar{z}$ if, and only if, $(x,y) \in E^2$. The continued coefficients

$$\begin{aligned} A(z,z^*) &= [\tilde{a}(z,z^*) + i\tilde{b}(z,z^*)]/4, \\ B(z,z^*) &= [\tilde{a}(z,z^*) - i\tilde{b}(z,z^*)]/4, \\ C(z,z^*) &= \tilde{c}(z,z^*)/4 \end{aligned}$$

are required as entire functions on $\mathbb{C}^2$. Then (1) takes the form

$$U_{zz^*} + A(z,z^*)U_z + B(z,z^*)U_{z^*} + C(z,z^*)U = 0, \tag{2}$$

where

$$U(z,z^*) = V((z+z^*)/2, (z-z^*)/2i), (z,z^*) \in D^2.$$

The Bergman canonical form is

$$L(V) = V_{zz^*} + D(z,z^*)V_{z^*} + F(z,z^*)V = 0, \tag{3}$$

and

$$V(z,z^*) = U(z,z^*) \exp\left\{ \int_0^{z^*} A(z,\xi)d\xi - n(z) \right\},$$

where $n(z)$ is an arbitrary entire function, and

$$\begin{aligned} -F(z,z^*) &= A_z + AB - C, \\ D(z,z^*) &= n^{'}(z) - \int_0^{z^*} A_z(z,\xi)d\xi + B(z,z^*). \end{aligned}$$

Each $V(z, z^*)$ has a locally valid integral representation shown by Bergman (1961) and Colton and Gilbert (1968), $V = b_2(f)$ and

$$V(z, z^*) = \int_L E(z, z^*, t) f(\sigma) d\mu(t),$$
$$\sigma = z(1-t^2)/2, d\mu(t) = \frac{dt}{(1-t^2)^{\frac{1}{2}}},$$

with a unique b_2-associated analytic function $f = f(z)$. The Bergman E-function is entire for $(z, z^*) \in \mathbb{C}^2$ and continuous for $t \in \mathbb{C}$ given by Gilbert (1969).

The principal branch of the function element $V(z, z^*)$ continues analytically from its initial domain of definitions by contour deformation to a (larger) domain of association as given in the *Envelope Method* by Colton and Gilbert (1968) and Gilbert (1969). Using this method, Gilbert and Colton (1968, Thm. 1) showed that the (principal branch) of $V(z, z^*)$ is singular at $z = \alpha$ if and only if the b_2 associate f is singular at $z = \frac{\alpha}{2}$.

Let S denote a Carathéodory domain, that is, a bounded simply connected domain such that the boundary of S coincides with the boundary of the domain lying in the complement of the closure of S and containing the point ∞. In other words, a domain bounded by a Jordan curve is a Carath0́eodory domain. For p-fixed, let $L^p(S), 1 \leq p \leq \infty$, be the class of all functions f holomorphic on S such that

$$|||f||| = ||f||_{S,p} < \infty, 1 \leq p \leq \infty,$$
$$||f||_{S,p} = \left\{ \begin{array}{c} [A(S) \int_S |f|^p dA_z]^{\frac{1}{p}}, 1 \leq p < \infty, \\ \sup\{|f(z)| : z \in S\}, p = \infty \end{array} \right\},$$

where $A^{-1}(S)$ is the area of S and dA_z is the Lebesgue area measure on S. Then a $||.||_{S,p}$ is called the L^p-norm on $L^p(S)$. For $f \in L^p(S)$, let us define $b_n's$ called the Fourier coefficients of f as follows:

$$b_n = \int\int_S f(z)\overline{p_n(z)} dA_z, \int\int_S p_n(z)\overline{p_m(z)} dA_z = \delta_m^n,$$

where $\delta_m^n = 1$ for $n = m$ and $\delta_m^n = 0$, otherwise and $\{p_n\}_{n=0}^{\infty}$ is a sequence of polynomials, p_n being of degree n. It follows from Smirnov and Lebedev (1968, p. 272) that $f \in L^p(S)$ is entire if and only if

$$\lim_{n\to\infty} |b_n|^{\frac{1}{n}} = 0.$$

It can be seen that $f(z) = \Sigma_{n=0}^{\infty} b_n p_n(z)$ holds in the whole complex plane. For $f \in L^p(S)$, define the measure of the best polynomial approximates of f as

$$e_n^p(f) = e_n^p(f; S) = \min_{\Pi_n} |||f - h|||, n \geq 0, 1 \leq p \leq \infty,$$
$$\Pi_n = \{h : \text{polynomial of degree at most n}\}, n \geq 0.$$

Similarly, let $L^p(S^2)$ be the class of all regular solutions $V(z,z^*)$ of (3) on $S \times S^*$ such that

$$|||V||| = ||V||_{S^2,p} < \infty, 1 \leq p \leq \infty,$$

$$||V||_{S^2,p} = \left\{ \begin{array}{c} [A(S^2) \int \cdots \int_{S \times S^*} |V|^p dA_z dA_{z^*}]^{\frac{1}{p}}, 1 \leq p < \infty, \\ \sup\{|V(z,z^*)| : (z,z^*) \in S \times S^*\}, p = \infty \end{array} \right\},$$

where $A^{-1}(S^2)$ is the area of S^2 and dA_{z^*} is the Lebesgue area measure on S^*. The best polynomial approximate solutions are satisfied the following

$$E_n^p(V) = E_n^p(V; S^2) = \inf_{P_n} |||V - \Phi|||, 1 \leq p \leq \infty. \tag{4}$$

The image under the transform b_2 are the sets

$$P_n = \{\Phi : \Phi(z,z^*) = b_2[h], h \in \Pi_n\}, n \geq 0.$$

Let L^0 denote the class of functions $\xi(x)$ satisfying conditions (H,i) and (H,ii):

(H,i) $\xi(x)$ is defined on $[a,\infty)$; is positive, strictly increasing and differentiable; and tends to ∞, as $x \to \infty$.

(H,ii)

$$\lim_{x\to\infty} \frac{\xi[x(1+\varphi(x))]}{\xi(x)} = 1,$$

for every function $\varphi(x)$ such that $\varphi(x) \to 0$, as $x \to \infty$.

Let Λ denote the class of functions $\xi(x)$ satisfying conditions (H,i) and (H,iii):

(H,iii)

$$\lim_{x\to\infty} \frac{\xi(cx)}{\xi(x)} = 1,$$

for every $0 < c < \infty$.

The generalized growth parameters $\rho(\alpha,\beta,f)$ and $\lambda(\alpha,\beta,f)$ of an entire function $f(z)$ were defined by Seremeta (1970) and Shah (1977) as

$$\begin{array}{c} \rho(\alpha,\beta,f) \\ \lambda(\alpha,\beta,f) \end{array} = \lim_{r\to\infty} \begin{array}{c} \sup \\ \inf \end{array} \frac{\alpha(\log M(r,f))}{\beta(\log r)},$$

where $\alpha(x) \in \Lambda$ and $\beta(x) \in L^0$. They generalized various results, cf. Juneja (1974), Reddy (1970, 1972) and Varga (1968).

Also the generalized orders of an entire function in terms of the coefficients in its Taylor series have been characterized by Shah (1977) and Nautiyal et al. (1982) by using the condition:

$$\frac{d[\beta^{-1}(\alpha(x))]}{d(\log x)} = O(1),$$

as $x \to \infty$.

It has been noticed that these results fail to exist for the functions, $\alpha(x) = \beta(x)$. To refine this class of functions, Kapoor and Nautiyal (1981) defined generalized growth parameters in a new setting as:

Let Ω be the class of functions $\xi(x)$ satisfying (H, i) and (H, iv):

(H,iv) There exists a $\delta(x) \in \Lambda$ and x_o, K_1 and K_2 such that

$$0 < K_1 \leq \frac{d(\xi(x))}{d(\delta(\log x))} \leq K_2 < \infty,$$

for all $x > x_o$.

Let $\overline{\Omega}$ be the class of functions $\xi(x)$ satisfying (H, i) and (H, v): (H,v)

$$\lim_{x\to\infty} \frac{d(\xi(x))}{d(\log x)} = K, 0 < K < \infty.$$

The generalized growth parameters of an entire function element $V(z, \overline{z})$ are defined as

$$\begin{matrix} \rho(\alpha,\alpha,V) \\ \lambda(\alpha,\alpha,V) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\log M(r,V))}{\alpha(\log r)}. \tag{5}$$

Similarly the generalized growth parameters of an associated entire function $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ are defined as

$$\begin{matrix} \rho(\alpha,\alpha,f) \\ \lambda(\alpha,\alpha,f) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\log M(r,f))}{\alpha(\log r)}, \tag{6}$$

where $\alpha(x)$ either belongs to Ω or $\overline{\Omega}$ and

$$\begin{aligned} M(r,V) &= \sup\{|V(z,\overline{z})| : (z,\overline{z}) \in S^2\}, \\ M(r,f) &= \sup\{|f(z)| : |z| = r\}, \\ \mu(r,f) &= \max_{n\geq 0}[|a_n| r^n]. \end{aligned}$$

It has been noticed that the characterization of generalized growth parameters for entire function solution of (1) for slow growth in terms of the sequence $\{E_n^p(V)\}, 1 \leq p \leq \infty$ has not been studied so far. Also, McCoy (1982) obtained the necessary and sufficient conditions for the function element $V(z, \overline{z})$ as the restriction to S of an entire function solution of (1) in terms of Bernstein limit $\lim_{n\to\infty}[E_n^p(V)]^{\frac{1}{n}} = 0, 2 \leq p \leq \infty$. In this paper we extended a result of McCoy (1982) for $1 \leq p < 2$ and obtained some inequalities on the generalized growth parameters of an entire function solution of (1) for slow growth in terms approximation errors $E_n^p(V)$ defined by (4), $1 \leq p \leq \infty$ on more generalized domains, i.e., Carathéodory domains.

Kadiri and Harfaoui (2013), Harfaoui (2010, 2011), Harfaoui and Kumar (2014), and Kumar (2011) studied the general growth of b_2-associate f as an entire function in several variables by means of the best polynomial approximation in L^p-norm on L-regular nonpluripolar compact set. Kumar (2007) obtained some inequalities concerning order and type of entire function solutions of (1) in terms of polynomial approximation errors in sup norm on polydisc. Mishra (2007), Mishra and Mishra (2012), and Mishra et al. (2014) studied some problems on trigonometric approximation of functions in Banach spaces by using different operators. Mishra and Sen (2016) investigated some results on the concept of existence and local attractively of solutions for some quadratic Volterra integral equation of fractional order. In her Ph.D. theses, Deepmala (2014) also studied some results on fixed point theorems for nonlinear contractions with applications, but our results and method are different from these authors.

The text has been divided into three parts. Section 1 consists of an introductory exposition of the paper. In Section 2, we prove four lemmas, with first lemma connecting the generalized growth parameters of an entire function f to the maximum modulus and in others we obtained some inequalities on generalized growth parameters of the entire function solution of (1) in L^p-norm, $1 \leq p \leq \infty$ on Carathéodory domains. Also, in this section we extend a theorem of McCoy (1982) for $1 \leq p < 2$. Finally, in Section 3, we prove some inequalities concerning the generalized growth parameters in terms of approximation errors $E_n^p(V)$.

Notation:

$$
\begin{aligned}
\underset{\eta}{\bigodot} &= \max\{1, \nu\}, \text{ if } \ \alpha(x) \in \Omega, \\
&= \eta + \nu, \text{ if } \ \alpha(x) \in \overline{\Omega}.
\end{aligned}
$$

We shall write $\bigodot(\nu)$ for $\bigodot_1(\nu)$.

2. Auxiliary Results

In this section we shall prove some preliminary results which will be used in the sequel.

Let S' be the component of the complement of the closure of the Carathéodory domain S that contains the point ∞. Set $S_r = \{z : |\phi(z)| = r\}, r > 1$, where the function $w = \phi(z)$ maps S' conformally on to $|w| > 1$ such that $\phi(\infty) = \infty$ and $\phi'(\infty) > 0$.

First we prove the following.

Lemma 2.1.

Let the associated entire function f having generalized growth parameters $\rho(\alpha, \alpha, f)$ and $\lambda(\alpha, \alpha, f)$. Then

$$
\begin{matrix} \rho(\alpha,\alpha,f) \\ \lambda(\alpha,\alpha,f) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\log \overline{M}(r,f))}{\alpha(\log r)} ,
$$

where

$$\overline{M}(r,f)=\max\{|f(z)|:z\in S_r\}.$$

Proof:

Let $z_o \in S, r>1$. Then from Winiarski (1970),

$$r-2|S|-z_o\leq |z|\leq r+|S|+|z_o|, z\in S_r.$$

For $\xi<1$ and $\eta>1$, using $\log Kx\simeq \log x$ as $x\rightarrow\infty, 0<K<\infty$, we get

$$\log M(\xi r,f)\leq \log \overline{M}(r,f)\leq \log M(\eta r,f).$$

Now the proof follows from (1.6). □

Lemma 2.2.

Let $V(z,z^*)\in L^p(S\times S^*), 1\leq p\leq\infty$. If the function element $V(z,\overline{z})$ has an analytic continuation as an entire function solution of (1) then there exists a sequence of mini-max polynomials $\{p_n^*\}$such that

$$\parallel f(z)-p_n^*(z)\parallel_{S,p}\leq K(\delta,p)M(\delta,f)(5/4\delta)^n, \delta>5/4, n\geq 0. \tag{7}$$

Proof:

Let the function element $V(z,\overline{z})$ have an analytic continuation as an entire function. Let p_n^* be the mini-max polynomial for $e_n(f)$ and $\Phi_n^*=b_2[p_n^*]$. The entire function $f-p_n^*$ extends on [-1,1] in a series of Chebyshev polynomials

$$T_n(z)=\frac{1}{2}\sum_{k=0}^{[n/2]}\frac{n}{n-k}\left(\begin{array}{c}n-k\\k\end{array}\right)(2z)^{n-2k}, n=0,1,2,\cdots$$

and continue analytically as

$$f(z)-p_n^*(z)=2\sum_{k=n+1}^{\infty}\alpha_k T_k(z), \alpha_k=\alpha_k(f)$$

to the ellipse

$$\overline{E}_\delta\equiv\{z\in\mathbb{C}:|z-1|+|z+1|<2\delta\}, \delta>4.$$

The Chebyshev coefficients $\alpha_k=\alpha_k(f)$ defined as contour integrals of f over the boundary $\partial\overline{E}_\delta$ are defined as

$$|\alpha_k|\leq\overline{M}(\delta,f)\delta^{-k}, k=0,1,2,\cdots,$$

where

$$\overline{M}(\delta, f) = \sup\left\{|f(z)|, z \in \overline{E}_\delta\right\}.$$

We have

$$\begin{aligned} e_n^p(f) = \|f(z) - p_n^*(z)\|_{S,p} &= \left\{\left\|2\sum_{k=n+1}^{\infty}\alpha_k T_k(z)\right\|_{S,p}, p \geq 1,\right\} \\ &\leq 2\sum_{k=n+1}^{\infty}|\alpha_k|\|T_k(z)\|_{S,p} \\ &\leq 2\sum_{k=n+1}^{\infty}|\alpha_k|\sup|T_k(z)| \\ &\leq \frac{2\overline{M}(\delta, f)}{(\delta - 1)}\sum_{k=n+1}^{\infty}(5/4\delta)^k. \end{aligned}$$

Since

$$\sum_{k=n+1}^{\infty}(5/4\delta)^k < (5/4\delta)^n \quad for \quad \delta > (5/2).$$

Hence,

$$e_n^p(f) \leq \frac{2\overline{M}(\delta, f)}{(\delta - 1)}(5/4\delta)^n, n \geq 0, \delta > 5/2. \tag{8}$$

Now define $D_1 = \{z : |z| \leq \delta\}$ then we have $\overline{E}_\delta \subset D_1$. It gives

$$\overline{M}(\delta, f) \leq M(\delta, f). \tag{9}$$

In view of (2.2) and (2.3), the result is immediate.

Lemma 2.3.

Let $v(x, y) = V(z, \overline{z})$ be a real-valued solution of (1) on S and let $V(z, z^* \in L^p(S \times S^*), 1 \leq p \leq \infty$, be the analytic continuation of $V(z, \overline{z})$ to $S \times S^*$. If the function element $V(x, y)$ is the restriction to S of an entire function solution of (1) having generalized growth parameters $\rho(\alpha, \alpha, V)$ and $\lambda(\alpha, \alpha, f)$. Then $g(z) = \sum_{n=0}^{\infty}|b_n|z^n, b_n's$ are given as earlier, is an entire function. Further

$$\rho(\alpha, \alpha, V) \leq \rho(\alpha, \alpha, g), \lambda(\alpha, \alpha, V) \leq \lambda(\alpha, \alpha, g)$$

and

$$\rho(\alpha, \alpha, g) \leq \rho(\alpha, \alpha, f), \lambda(\alpha, \alpha, g) \leq \lambda(\alpha, \alpha, f)$$

also hold.

Proof:

Equation (2) implies that g is entire. In view of Smirnov and Lebedev (1968, p. 272) we have

$$\max_{z\in S_{r'}} |p_n(z)| \leq K(r')^n, n = 1, 2, \ldots,$$

where K is a constant independent of $n, r'(> 1)$ is a fixed number. Thus, applying Bernstein's inequality, e.g. Bernstein (1952, p. 21), Markushevich (1967, p. 112), for each term of the series $\sum_{n=0}^{\infty} b_n p_n(z)$, we obtain

$$|f(z)| \leq |b_o| + K \sum_{n=1}^{\infty} |b_n|(rr')^n, z \in S_r.$$

We see that $\sigma = \frac{z(1-t^2)}{2}$, it gives

$$|f(\sigma)| \leq |b_o| + K \sum_{n=1}^{\infty} |b_n|(rr'\frac{(1-t^2)}{2})^n, z \in S_r. \tag{10}$$

Now consider $V \in L^p(S^2)$, the analytic continuation to S^2 of the real-valued solution of (1) on S. Let f be the unique b_2-associate for which Bergman (1961) and Colton and Gilbert (1968)

$$V(z, z^*) = \int_L E(z, z^*, t) f(\sigma) d\mu(t). \tag{11}$$

S. Bergman (1961) has shown that on S^2, V satisfies the Goursat data

$$V(z, 0) = g(z) = \int_L f(\sigma) d\mu(t), V(0, z^*) = g(0). \tag{12}$$

Therefore, $f \in L^p(S)$, in view of Holder's inequality from (11) we get

$$|V(z, z^*|^p \leq M_{p,q} \int_L |f(\sigma)|^p d\mu(t), 2 \leq p < \infty, \tag{13}$$

where

$$M_{p,q} = \sup_{t\in L}\{\int_L |E(z, z^*, t)|^q d\mu(t) : (z, z^*) \in S \times S^*\}, \frac{1}{p} + \frac{1}{q} = 1, 2 \leq p < \infty.$$

By the application of Fubini's theorem and maximum principle, (13) gives

$$\begin{aligned}
A^{-1}(S^2)||V||^p &= \int \ldots \int_{S\times S^*} |V(z, z^*)|^p dA_z dA_{z^*} \\
&\leq \int \cdots \int_{S\times S^*} \{M_{p,q} \int_L |f(\sigma)|^p d\mu(t)\} dA_z dA_{z^*} \\
&\leq K'_{p,q} \int_S |f|^p dA_z.
\end{aligned} \tag{14}$$

Using (10) in (14) we get for $2 \leq p < \infty$,

$$A^{-1}(S^2)||V||^p \leq K'_{p,q} \int_S |\{|b_0| + K \sum_{n=1}^{\infty} |b_n| \frac{(rr'(1-t^2))}{2}\}|^p dA_z, (z, z^*) \in (S_r \times S_r). \tag{15}$$

For the case $1 \leq p < 2$, from (11) we get

$$|V(z, z^*)| \leq [\sup_{t \in L} \int_L |E(z, z^*, t)|^{p'} d\mu(t)]^{\frac{1}{p'}} [\int_L |f|^p d\mu(t)]^{\frac{1}{p}}, \frac{1}{p} + \frac{1}{p'} = 1, 1 \leq p < 2.$$

Now we have

$$\begin{aligned} A^{-1}(S^2)||V(z, z^*)|| &= \int \ldots \int_{S \times S^*} [\sup_{t \in L} \int_L |E(z, z^*, t)|^{p'} d\mu(t)]^{\frac{1}{p'}} [\int_L |f|^p d\mu(t)]^{\frac{1}{p}} dA_z dA_{z^*} \\ &\leq \int \cdots \int_{S \times S^*} M(p', S, S^*)(\int_L |f(\sigma)|^p d\mu(t))^{\frac{1}{p}} dA_z dA_{z^*} \\ &\leq M'(p', S, S^*)(\int_L |f(\sigma)|^p d\mu(t))^{\frac{1}{p}} dA_z \\ &\leq M''(p', S, S^*) \int_S |\{|b_0| + K \sum_{n=1}^{\infty} |b_n| \frac{(rr'(1-t^2))}{2}\}|^p dA_z. \end{aligned} \tag{16}$$

For $p = \infty$, the reasoning is similar. Using (15) and (16) we obtain

$$\overline{M}(\frac{r(1-t^2)}{2}, V) \leq b_o + CM(\frac{rr'(1-t^2)}{2}, g), r > 1.$$

Now using (5) and (6) we get

$$\rho(\alpha, \alpha, V) \leq \rho(\alpha, \alpha, g), \lambda(\alpha, \alpha, V) \leq \lambda(\alpha, \alpha, g).$$

Now we have

$$b_n = \int \int_S f(z)\overline{p_n(z)} dxdy = \int \int_S (f(z) - p_n^*(z))\overline{p_n(z)} dxdy.$$

Since p_n is orthogonal to any polynomial of degree less than n, using Schwarz inequality, we get

$$|b_n| \leq ||f(z) - p_n^*(z)||_{S,p}.$$

Using (7) in above, we get

$$|b_n| \leq K(\delta, p)M(\delta, f)(\frac{5}{4\delta})^n, \delta > \frac{5}{4}, n \geq 0,$$

it gives

$$\mu\left(\frac{4\delta}{5}; g\right) \leq K(\delta, p)M(\delta, f),$$

for all sufficiently large values of δ. Thus using Theorem 3 of Kapoor and Nautiyal (1968), Lemma 2.1 and the fact that either $\alpha \in \Omega$ or $\overline{\Omega}$, we get

$$\rho(\alpha, \alpha, g) \leq \rho(\alpha, \alpha, f), \lambda(\alpha, \alpha, g) \leq \lambda(\alpha, \alpha, f).$$

McCoy (1982) proved the following theorem

Theorem A.

Let $v(x, y) = V(z, \overline{z})$ be a real-valued solution of $L[v] = 0$ on S and let $V(z, z^*) \in L^P(S \times S^*), 2 \leq p \leq \infty (p - fixed)$, be the analytic continuation of $V(z, \overline{z})$ to $S \times S^*$. The function element $v(x, y)$ is the restriction to S of an entire function solution of (1) if, and only if, the Bernstein limit

$$\lim_{n \to \infty} [E_n^p(V)]^{\frac{1}{n}} = 0 \tag{17}$$

is valid.

First we shall extend the above theorem for $1 \leq p < 2$. We are giving here only necessary steps which will fulfill our requirement.

To verify the sufficiency of the Bernstein limit for $1 \leq p < 2$, let us assume that (17) is valid. We observe that if $1 \leq p < 2$, let $p^{'}$ such that $\frac{1}{p} + \frac{1}{p^{'}} = 1$, then $p^{'} \geq 2$. By the Hölder inequality we have

$$|V(z, z^*)| \leq \frac{1}{A^{-1}(S^2)} \int \cdots \int_{S \times S^*} |V(z, z^*)| dA_z dAz^*$$

or

$$\begin{aligned} A^{-1}(S^2)||V|| &\leq ||V(z, z^*)||_{L^p(S \times S^*)} ||dA_z dA_{z^*}||_{L^{p'}(S \times S^*)} \\ &\leq ||A^{-1}(S)A^{-1}(S^*)||_{L^{p'}(S \times S^*)} ||V(z, 0)||_{L^p(S \times S^*)} \\ &= ||A^{-1}(S)A^{-1}(S^*)||_{L^{p'}(S \times S^*)} ||g||_{L^p(S)}. \end{aligned}$$

But $||A^{-1}(S)A^{-1}(S^*)||_{L^{p'}(S \times S^*)} \leq C(p^{'}, S, S^*)$. Therefore, we get

$$||V|| \leq C^{'}(p^{'}, S, S^*)||g||_{L^p(S)}, 1 \leq p < 2, \frac{1}{p} + \frac{1}{p^{'}} = 1.$$

Now consider the case $2 \leq p \leq \infty$ McCoy (1982) with above, we get

$$||V|| \geq C(p, p^{'}, S, S^*)|||g|||, 1 \leq p \leq \infty.$$

Now sufficiency part follows on the lines of proof of McCoy (1982, p.521).

In order to prove the necessary part for $1 \leq p \leq 2$, consider

$$V(z,z^*)=\int_L E(z,z^*,t)f(\sigma)d\mu(t).$$

It gives

$$|V(z,z^*)|\leq [\sup_{t\in L}\int_L |E(z,z^*,t)|^{p'}d\mu(t)]^{\frac{1}{p'}}[\int_L |f|^p d\mu(t)]^{\frac{1}{p}}, \frac{1}{p}+\frac{1}{p'}=1, 1\leq p<2.$$

Now we have

$$\begin{aligned} A^{-1}(S^2)||V(z,z^*)|| &= \int\cdots\int_{S\times S^*}[\sup_{t\in L}\int_L |E(z,z^*,t)|^{p'}d\mu(t)]^{\frac{1}{p'}}[\int_L |f|^p d\mu(t)]^{\frac{1}{p}}dA_z dA_{z^*}\\ &\leq \int\cdots\int_{S\times S^*} M(p',S,S^*)(\int_L |f(\sigma)|^p d\mu(t))^{\frac{1}{p}}dA_z dA_{z^*}\\ &\leq M'(p',S,S^*)(\int_L |f(\sigma)|^p d\mu(t))^{\frac{1}{p}}dA_z\\ &\leq M''(p',S,S^*)||f||_{S,p}, \end{aligned} \tag{18}$$

where the constant $M''(p',S,S^*)$ depends on the domain $S\times S^*$ and p'. For $p=\infty$, the reasoning is similar. From (18) we have

$$\lim_{n\to\infty}[e_n^p(f)]^{\frac{1}{n}}=0, 1\leq p\leq\infty,$$

which satisfy the Bernstein limit and necessity.

Lemma 2.4.

Let $V(z,z^*)\in L^p(S\times S^*), 1\leq p\leq\infty$. If the function element $V(z,\overline{z})$ has an analytic continuation as an entire function solution of (1) having generalized growth parameters $\rho(\alpha,\alpha,V)$ and $\lambda(\alpha,\alpha,V)$.Then $\tilde{g}(z,z^*)=\sum_{n=0}^{\infty}E_n^p(V)z^n z^{*n}, (z,z^*)\in S\times S^*$ is an entire function. Further

$$\rho(\alpha,\alpha,\tilde{g})\leq\rho(\alpha,\alpha,f), \lambda(\alpha,\alpha,\tilde{g})\leq\lambda(\alpha,\alpha,f).$$

Proof:

In view of Lemma 2.2 and (18) we have

$$E_n^p(V)\leq e_n^p(f)\leq K(\delta,p)M(\delta,f)(5/4\delta)^n, \delta>5/2, n\geq 0. \tag{19}$$

If V is the restriction to S of an entire function, then $\lim_{n\to\infty}(E_n^p(V))^{\frac{1}{n}}=0$ for $\delta>5/2$ and $\delta\to\infty$. So $\tilde{g}(z,z^*)$ is entire. Further (19) gives

$$\tilde{g}(z,z^*)\leq\sum_{n=0}^{\infty}2K_{p,q}\gamma(\delta,p)M(\delta,f)(5/4\delta)^n z^n z^{*n}$$

or

$$\begin{aligned} M((\frac{4\delta}{5})^2,\tilde{g}) &\leq P(\delta)+2K_{p,q}\gamma(\delta,p)M(4\delta+1,f)\sum_{n=0}^{\infty}[\frac{4\delta}{4\delta+1}]^n\\ &= P(\delta)+2K_{p,q}\gamma(\delta,p)(4\delta+1)M(4\delta+1,f) \end{aligned}$$

or

$$\rho(\alpha,\alpha,\tilde{g}) \leq \rho(\alpha,\alpha,f), \lambda(\alpha,\alpha,\tilde{g}) \leq \lambda(\alpha,\alpha,f).$$

3. Main Results

In this section we shall prove our main results.

Theorem 3.1.

Let $V(z,z^*) \in L^p(S\times S^*), 1\leq p\leq \infty$. If the function element $V(z,\overline{z})$ has an analytic continuation as an entire function solution of (1) and $f\in L^p(S)$ be the restriction to S of an entire function, then the generalized growth parameters are given by the following relations:

(i) $\rho(\alpha,\alpha,f) \geq \rho(\alpha,\alpha,V) = \bigodot(L)$,

(ii) $\lambda(\alpha,\alpha,f) \geq \max_{\{n_k\}}[\bigodot_X\{l'\}]$,

(iii) If we take $\alpha(x)=\alpha(a)$ on $(-\infty,a)$, then

$$\lambda(\alpha,\alpha,f) \geq \lambda(\alpha,\alpha,V) \geq \bigodot(l^*),$$

and

$$\lambda(\alpha,\alpha,f) \geq \lambda(\alpha,\alpha,V) \geq \max_{\{n_k\}}[\bigodot_X\{{l'}^*\}],$$

where

$$\begin{aligned}
L &= \lim_{n\to\infty}\sup\frac{\alpha(n)}{\alpha\{\frac{1}{n}\log[E_n^p(V)]^{-1}\}},\\
X &\equiv X(\{n_k\}) = \lim_{k\to\infty}\inf\frac{\alpha(n_{k-1})}{\alpha(n_k)},\\
l' &\equiv l'(\{n_k\}) = \lim_{k\to\infty}\inf\frac{\alpha(n_{k-1})}{\alpha\{\frac{1}{n_k}\log[E_{n_k}^p(V)]^{-1}\}},\\
l^* &= \lim_{n\to\infty}\inf\frac{\alpha(n)}{\alpha\{\log(\frac{E_{n-1}^p(V)}{E_n^p(V)})\}},\\
{l'}^* &\equiv {l'}^*(\{n_k\}) = \lim_{k\to\infty}\inf\frac{\alpha(n_{k-1})}{\alpha\{\frac{1}{(n_k-n_{k-1})}\log(\frac{E_{n_{k-1}}^p(V)}{E_{n_k}^p(V)})\}}.
\end{aligned}$$

The maximum in (ii) is taken over all increasing sequences $\{n_k\}$ of positive integers. Further if $\{n_m\}$ is the sequence of principal indices of the entire function $\tilde{g}(z)=\sum_{n=0}^{\infty}E_n^p(V)z^n z^{*n}$ and $\alpha(n_k)\sim\alpha(n_{k+1})$ as $k\to\infty$, then (ii) also hold for $\alpha(x)\in\overline{\Omega}$.

Proof:

The proof follows easily from Kapoor and Nautiyal (1981, Thms. 4-6, Lemma 1) and Lemma 2.4.

4. Conclusion

When one works with real or complex numbers, there is a natural notion of the magnitude of a number x, i.e. $|x|$. The situation becomes more complicated when one deals with objects with more degree of freedom. The L^{∞}-norm is concerned with the height of a function, while the L^{p}-norms are instead concerned with a combination of height and width of a function. In this paper our study is concerned with the L^{p}-norm. Regular real-valued solutions of equation (1) have several applications in mathematical physics. These solutions arise in the Maxwell system for the modelling of electric or magnetic n-poles, in potential scattering and in quasi-stationary (time independent) diffusion processes. It has been noticed that the characterizations of generalized growth parameters for entire function solutions of (1) for slow growth in terms of L^{p}-approximation errors on Carathéodory domains have not been studied so far. Thus in this paper we have tried to fill this gap.

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