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On Some Optimal Multiple Root-Finding Methods and their Dynamics

Munish Kansal, V. Kanwar*, and Saurabh Bhatia

*Corresponding Author
University Institute of Engineering and Technology
Panjab University, Chandigarh-160 014, India
mkmaths@gmail.com, vmithil@yahoo.co.in, and s_bhatia@pu.ac.in

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Abstract

Finding multiple zeros of nonlinear functions pose many difficulties for many of the iterative methods. In this paper, we present an improved optimal class of higher-order methods for multiple roots having quartic convergence. The present approach of deriving an optimal class is based on weight function approach. In terms of computational cost, all the proposed methods require three functional evaluations per full iteration, so that their efficiency indices are 1.587 and, are optimal in the sense of Kung-Traub conjecture. It is found by way of illustrations that they are useful in high precision computing enviroments. Moreover, basins of attraction of some of the higher-order methods in the complex plane are also given.

Keywords: Basins of attraction; efficiency index; Kung-Traub conjecture; multiple roots; Newton's method

MSC 2010 No.: 65H05; 65B99

1. Introduction

Finding the multiple roots of nonlinear equations efficiently and accurately, is a very interesting and challenging problem in computational mathematics. We consider an equation of the form

$$f(x) = 0, \tag{1}$$

where $f : D \subset \mathbb{R} \to \mathbb{R}$ be a nonlinear continuous function on D. Analytical methods for solving such equations are almost non-existent and therefore, it is only possible to obtain approximate solutions by relying on numerical methods based on iterative procedures (Gutiérrez and Hernández, 1997; Petković et al., 2012). So, in this paper, we concern ourselves with iterative methods to find the multiple root r_m with multiplicity $m > 1$ of a nonlinear equation (1) , i.e. $f^i(r_m) = 0$, $i = 0,\ 1,\ 2,\ 3, \ldots, m-1$ and $f^m(r_m) \neq 0$. These multiple roots pose difficulties for root-finding methods as function does not change sign at even multiple roots, precluding the use of bracketing methods, limiting one to open methods.

Modified Newton's method (Rall, 1966) is an important and basic method for finding multiple roots of nonlinear equation (1), and is given by

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}.$$

It converges quadratically for multiple roots and requires the prior knowledge of multiplicity m.

If an initial guess x_n is sufficiently close to the required root r_m, then the following expressions: $x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}$, $x_{n+1} = x_n - (m-1)\frac{f'(x_n)}{f''(x_n)}$, $x_{n+1} = x_n - (m-2)\frac{f''(x_n)}{f'''(x_n)}$, $\ldots$, will have the same value. Another important modification of Newton's method for multiple roots appears in the work of (Schröder, 1870) which is given as

$$x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{f'^2(x_n) - f(x_n)f''(x_n)}.$$

This method has quadratic convergence and does not require the prior knowledge of multiplicity m. It may be obtained by applying Newton's method to the function $u(x) = \frac{f(x_n)}{f'(x_n)}$, which has a simple roots in each multiple root of $f(x)$.

As the order of an iterative method increases, so does the number of functional evaluations per step. The efficiency index (Ostrowski, 1973) gives a measure of the balance between those quantities, according to the formula $p^{\frac{1}{d}}$, where p is the order of convergence of the method and d the number of functional evaluations per step. According to the Kung-Traub conjecture (Ostrowski, 1973; King, 1973), the order of convergence of any multipoint method consuming n functional evaluations cannot exceed the bound 2^{n-1}, called the optimal order.

In the recent years, some optimal modifications of Newton's method for multiple roots have been proposed and analyzed by (Li et al., 2009; Li et al., 2010; Sharma and Sharma, 2012; Zhou et al., 2011; Kanwar et al., 2013) and the references cited therein. All these methods require one-function and two first order-derivative evaluations per iteration. (Osada, 1994) proposed a cubically convergent method for multiple roots. There are, however, not yet so many fourth or higher-order methods known in literature that can handle the case of multiple roots.

With this aim, we intend to propose two optimal schemes of fourth-order iterative methods dedicated only for multiple roots consuming three functional evaluations viz., $f(x_n),\ f'(x_n),\ f'(y_n)$

per full iteration. The present approach of deriving this optimal class of higher-order methods is based on weight function approach. All the proposed methods considered here are found to be more effective and comparable to the existing robust methods available in literature.

2. Construction of one-point methods and convergence analysis

In this section, we intend to develop one-point cubically convergent methods for multiple roots involving second-order derivative. In terms of computational cost, each method requires only three functional evaluations viz., $f(x_n),\ f'(x_n),$ and $f''(x_n)$ per full iteration.

Case I

Let us consider the following iterative schemes

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \tag{2}$$

and

$$x_{n+1} = x_n - (m-1)\frac{f'(x_n)}{f''(x_n)}, \tag{3}$$

which converge quadratically for multiple roots of nonlinear equation (1). Now, taking arithmetic mean of (2) and (3), we get

$$x_{n+1} = x_n - \frac{1}{2}\left(m\frac{f(x_n)}{f'(x_n)} + (m-1)\frac{f'(x_n)}{f''(x_n)}\right). \tag{4}$$

This method has quadratic convergence and satisfies the following error equation:

$$e_{n+1} = \frac{c_1 e_n^2}{m-1} + O(e_n^3).$$

In order to increase its order of convergence further, we insert the parameters a and b in (4) to obtain

$$x_{n+1} = x_n - \frac{1}{2}\left(m\frac{af(x_n)}{f'(x_n)} + (m-1)\frac{bf'(x_n)}{f''(x_n)}\right). \tag{5}$$

For finding the suitable values of free disposable parameters a and b in (5), we shall discuss the following Theorem (2.1).

Theorem 1: Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently smooth function defined on an open interval D, enclosing a multiple zero of $f(x)$, say $x = r_m$ with multiplicity $m > 1$. Then the family of iterative methods defined by (5) has third-order convergence when $a = 1 + m$ and $b = 1 - m$.

Proof: Let $x = r_m$ be a multiple zero of $f(x)$. Expanding $f(x_n)$, $f'(x_n)$ and $f''(x_n)$ about $x = r_m$ by the Taylor's series expansion, we have

$$f(x_n) = \frac{f^{(m)}(r_m)e_n^m}{m!}\left(1 + A_1 e_n + A_2 e_n^2 + A_3 e_n^3 + O(e_n^4)\right), \tag{6}$$

$$f'(x_n) = \frac{f^{(m)}(r_m)e_n^{(m-1)}}{(m-1)!}\left(1 + B_1 e_n + B_2 e_n^2 + B_3 e_n^3 + O(e_n^4)\right), \tag{7}$$

and

$$f''(x_n) = \frac{f^{(m)}(r_m)e_n^{(m-2)}}{(m-2)!}\left(1 + C_1 e_n + C_2 e_n^2 + C_3 e_n^3 + O(e_n^4)\right), \tag{8}$$

where
$A_i = \frac{m! f^{(m+i)}(r_m)}{(m+i)! f^{(m)}(r_m)}$, $B_i = \frac{(m-1)! f^{(m+i)}(r_m)}{(m+i-1)! f^{(m)}(r_m)}$, $C_i = \frac{(m-2)! f^{(m+i)}(r_m)}{(m+i-2)! f^{(m)}(r_m)}$, $i = 1, 2, \ldots$.
Using (6), (7) and (8) in (5) we get

$$\begin{aligned} e_{n+1} =& e_n - \frac{1}{2}\left(m\frac{af(x_n)}{f'(x_n)} + (m-1)\frac{bf'(x_n)}{f''(x_n)}\right) = \quad \frac{1}{2}(2-a-b)e_n + \frac{(a(m-1)+b(m+1))c_1 e_n^2}{2m(m-1)} + O(e_n^3) \\ &= B_1 e_n + B_2 e_n^2 + O(e_n^3). \end{aligned} \tag{9}$$

In order to acheive the third order convergence, the coefficients B_1 and B_2 must vanish. Solving $B_1 = 0$ and $B_2 = 0$, we obtain

$$a = 1 + m \quad and \quad b = 1 - m. \tag{10}$$

Therefore, inserting the values of a and b from equation (10) in formula (5), we get

$$x_{n+1} = x_n - \frac{1}{2}\left(m(m+1)\frac{f(x_n)}{f'(x_n)} - (m-1)^2\frac{f'(x_n)}{f''(x_n)}\right). \tag{11}$$

This is a cubically convergent method for multiple roots . It satisfies the following error equation

$$e_{n+1} = \frac{((1+m)^2 c_1^2 - 2m(m-1)c_2)e_n^3}{2m^2(m-1)} + O(e_n^4).$$

This completes the proof of the Theorem (2.1). □

Case II

Now, we consider a quadratically convergent scheme

$$x_{n+1} = x_n - (m-1)\frac{f'(x_n)}{f''(x_n)}, \tag{12}$$

and well-known Schröder method for multiple roots

$$x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{f'^2(x_n) - f(x_n)f''(x_n)}, \tag{13}$$

respectively. From equations (12) and (13), we get

$$x_{n+1} = x_n - \frac{1}{2}\left((m-1)\frac{f'(x_n)}{f''(x_n)} + \frac{f(x_n)f'(x_n)}{f'^2(x_n) - f(x_n)f''(x_n)}\right), \tag{14}$$

which can be viewed as an arithmetic mean of two factors namely: $(m-1)\frac{f'(x_n)}{f''(x_n)}$ and $\frac{f(x_n)f'(x_n)}{f'^2(x_n)-f(x_n)f''(x_n)}$.
It satisfies the following error equation

$$e_{n+1} = \frac{c_1 e_n^2}{m(m-1)} + O(e_n^3).$$

Now, to increase the order of (14) from two to three, we introduce two free disposable parameters k_1 and k_2 in (14) and get

$$x_{n+1} = x_n - \frac{1}{2}\left((m-1)\frac{k_1 f'(x_n)}{f''(x_n)} + \frac{k_2 f(x_n) f'(x_n)}{f'^2(x_n) - f(x_n) f''(x_n)}\right), \tag{15}$$

which satisfies the following error equation

$$e_{n+1} = \frac{2-k_1-k_2}{2} e_n + \frac{(a+b+am+bm)c_1 e_n^2}{2m(m-1)} + O(e_n^3) = B_3 e_n + B_4 e_n^4 + O(e_n^3). \tag{16}$$

Therefore, to get a cubically convergent method, B_3 and B_4 must be equal to zero. Solving $B_3 = 0$ and $B_4 = 0$, we get

$$k_1 = \frac{m-1}{m} \quad \text{and} \quad k_2 = \frac{m+1}{m}.$$

Hence, inserting the above values of k_1 and k_2 in formula in (15), we obtain

$$x_{n+1} = x_n - \frac{1}{2m}\left(\frac{(m-1)^2 f'(x_n)}{f''(x_n)} + \frac{(m+1) f(x_n) f'(x_n)}{(f'^2(x_n) - f(x_n) f''(x_n))}\right). \tag{17}$$

It satisfies the following error equation

$$e_{n+1} = \frac{((-3-2m+m^2)c_1^2 - 2m(m-1)c_2)e_n^3}{2m^2(m-1)} + O(e_n^4).$$

This is a new third order method for multiple roots which requires three functional evaluations viz., $f(x_n)$, $f'(x_n)$ and $f''(x_n)$.

3. Construction of multipoint methods and convergence analysis

In this section, we intend to develop multipoint optimal fourth-order methods from schemes (4) and (14), respectively. Each family require three functional evaluations viz., $f(x_n),\ f'(x_n),\ f'(y_n)$ per full iteration, and are optimal in the sense of Kung-Traub conjecture.

A. First family

Now, our main objective is to construct new multipoint optimal methods free from second-order derivative. For this, let $y_n = x_n - \theta\frac{f(x_n)}{f'(x_n)}$, be the Newton-like iterate with non-zero parameter 'θ' (i.e. $\theta \neq 0$). Moreover, we consider the Taylor series expansion of $f'(y_n)$ about a point $x = x_n$ as follows:

$$f'(y_n) \approx f'(x_n) + f''(x_n)(y_n - x_n),$$

which further yields

$$f''(x_n) \approx \frac{f'(x_n)\big(f'(x_n) - f'(y_n)\big)}{\theta f(x_n)}. \tag{18}$$

Using this approximate value of $f''(x_n)$ in our starting iterative scheme (4) and $\theta = \frac{2m}{m+2}$, we get a modified method free from second-order derivative as

$$x_{n+1} = x_n - \frac{1}{2}\left(\frac{m f(x_n)}{f'(x_n)} + \frac{2m(m-1) f(x_n)}{(m+2)(f'(x_n) - f'(y_n))}\right), \tag{19}$$

satisfying the error equation given by

$$e_{n+1} = \frac{1}{2}\left(1 + \frac{2(-1+m)m}{(2+m)\left(2\left(\frac{m}{2+m}\right)^m + m\left(-1+\left(\frac{m}{2+m}\right)^m\right)\right)}\right)e_n + O(e_n^2). \tag{20}$$

Further, to increase the order of convergence, we substitute two free disposable parameters a_1 and a_2 in (19)to obtain

$$x_{n+1} = x_n - \frac{1}{2}\left(\frac{ma_1 f(x_n)}{f'(x_n)} + \frac{2a_2 m(m-1)f(x_n)}{(m+2)(f'(x_n) - f'(y_n))}\right). \tag{21}$$

It satisfies the following error equation:

$$\begin{aligned} e_{n+1} =& \frac{1}{2}\left(2 - a_1 + \frac{2a_2(-1+m)m}{(2+m)\left(2\left(\frac{m}{2+m}\right)^m + m\left(-1+\left(\frac{m}{2+m}\right)^m\right)\right)}\right)e_n + O(e_n^2) \\ =& B_5 e_n + B_6 e_n^2 + O(e_n^3). \end{aligned} \tag{22}$$

Using computer algebra system $Mathematica$ 9, solving $B_5 = 0$ and $B_6 = 0$, we can see that for the choice of,

$$\begin{cases} a_1 = \dfrac{1}{2}\left(4 - 2m + m^2\left(-1 + \left(\dfrac{m}{2+m}\right)^{-m}\right)\right), \\ a_2 = -\dfrac{\left(\frac{m}{2+m}\right)^{-m}(2+m)\left(2\left(\frac{m}{2+m}\right)^m + m\left(-1+\left(\frac{m}{2+m}\right)^m\right)\right)^2}{4(-1+m)}, \end{cases} \tag{23}$$

method (21) is cubically convergent and satisfies the following error equation:

$$e_{n+1} = \frac{2\left(-1+\left(\frac{m}{2+m}\right)^m\right)c_1^2 e^3}{m^2\left(2\left(\frac{m}{2+m}\right)^m + m\left(-1+\left(\frac{m}{2+m}\right)^m\right)\right)} + O(e_n^4).$$

According to the Kung-Traub conjecture (Kung and Traub, 1974) the order of convergence of any multipoint method using n functional evaluations cannot exceed the bound 2^{n-1}, called the optimal order. For the choice of a_1 and a_2 given by the equation (23), method defined by (21) is not an optimal method because it has third-order convergence and requires three evaluations of function, viz. $f(x_n)$, $f'(x_n)$, $f'(y_n)$ per full iteration. Therefore to build an optimal fourth-order method consuming three function evaluations, we suggest the following iterative scheme by using weight function approach

$$\begin{cases} y_n = x_n - \dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{1}{2}\left[\dfrac{a_1 m f(x_n)}{f'(x_n)} + \dfrac{2a_2 m(m-1)f(x_n)}{(m+2)(f'(x_n) - f'(y_n))}\right] Q\left(\dfrac{f'(y_n)}{f'(x_n)}\right), \end{cases} \tag{24}$$

where a_1 and a_2 are defined as follows:

$$\begin{cases} a_1 = \dfrac{1}{2}\left(4 - 2m + m^2\left(-1 + \left(\dfrac{m}{2+m}\right)^{-m}\right)\right), \\ a_2 = -\dfrac{\left(\frac{m}{2+m}\right)^{-m}(2+m)\left(2\left(\frac{m}{2+m}\right)^m + m\left(-1+\left(\frac{m}{2+m}\right)^m\right)\right)^2}{4(-1+m)}, \end{cases}$$

and Q(.) $\in C^2(\mathbb{R})$ is any real-valued weight function such that the order of convergence reaches at the optimal level without consuming any more functional evaluations. Theorem (3.1) indicates that under what conditions on the weight function in (24), the order of convergence will reach the optimal level four.

Convergence Analysis

Theorem 3.1 *Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently smooth function defined on an open interval D, enclosing a multiple zero of $f(x)$, say $x = r_m$ with multiplicity $m > 1$. Then the family of iterative methods defined by* (24) *has fourth-order convergence when*

$$\begin{cases} Q(\mu) = 1, \\ Q'(\mu) = 0, \\ Q''(\mu) = \dfrac{m^4\left(\frac{m}{2+m}\right)^{-2m}\left(-1+\left(\frac{m}{2+m}\right)^m\right)}{4\left(2\left(\frac{m}{2+m}\right)^m + m\left(-1+\left(\frac{m}{2+m}\right)^m\right)\right)}, \\ |Q'''(\mu)| < \infty, \end{cases} \tag{25}$$

where $\mu = \left(\frac{m}{m+2}\right)^{m-1}$ and it satisfies the following error equation

$$e_{n+1} = \frac{(p_1 c_1^3 - p_2 c_3)e^4}{3m^9(2+m)^2\left(2p^m + m\left(-1+p^m\right)\right)^2} + O(e^5), \tag{26}$$

where

$$\begin{aligned} p =& \frac{m}{m+2}, \\ p_1 =& (2+m)^2\left(128Q'''(\mu)p^{5m} - 4m^6\left(-3+p^m\right) + 128Q'''(\mu)mp^{4m}\left(-1+p^m\right) + m^{10}\left(-1+p^m\right)^2\right. \\ &+ 32Q'''(\mu)m^2p^{3m}\left(-1+p^m\right)^2 + 8m^5p^m\left(-6+5p^m\right) + 8m^7\left(-1+p^m+p^{2m}\right) + m^9\left(2-8p^m\right. \\ &\left.\left.+6p^{2m}\right) + 2m^8\left(1-6p^m+7p^{2m}\right)\right), \\ p_2 =& 3m^8(2+m)^2\left(2p^m + m\left(-1+p^m\right)\right)^2 c_1c_2 + 3m^{10}\left(2p^m + m\left(-1+p^m\right)\right)^2. \end{aligned}$$

Proof: Let $x = r_m$ be a multiple zero of $f(x)$. Expanding $f(x_n)$ and $f'(x_n)$ about $x = r_m$ by the Taylor's series expansion, we have

$$f(x_n) = \frac{f^{(m)}(r_m)}{m!}e_n^m\left(1 + c_1e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4\right) + O(e_n^5), \tag{27}$$

and

$$f'(x_n) = \frac{f^{(m-1)}(r_m)}{(m-1)!}e_n^{m-1}\left(1 + \frac{m+1}{m}c_1e_n + \frac{m+2}{m}c_2e_n^2 + \frac{m+3}{m}c_3e_n^3 + \frac{(m+4)}{m}c_4e_n^4\right) + O(e_n^5), \tag{28}$$

respectively.

Using computer algebra system $Mathematica$ 9, we get

$$\frac{1}{2}\left\{\frac{a_1mf(x_n)}{f'(x_n)} + \frac{2a_2m(m-1)f(x_n)}{(m+2)(f'(x_n)-f'(y_n))}\right\} = \frac{1}{2}\left(a_1 - \frac{2a_2m(m-1)}{(m+2)(-m+\frac{m}{m+2}(m+2))}\right)e_n + O(e_n^2). \tag{29}$$

Furthermore, we have

$$\begin{aligned}\frac{f'(y_n)}{f'(x_n)} =& \left(\frac{m}{2+m}\right)^{m-1} - \frac{4\left(\left(\frac{m}{2+m}\right)^m c_1\right) e_n}{m^3} + \frac{4\left(\frac{m}{2+m}\right)^m \left(\left(2+m^2\right)c_1^2 - 2m^2 c_2\right) e_n^2}{m^5} \\ & - \left(8\left(\left(\frac{m}{2+m}\right)^m \left((2+m)^2\left(6+m+5m^2-m^3+m^4\right)c_1^3 - 3m^2(2+m)^2\left(4+m^2\right)c_1c_2\right.\right.\right. \\ & \left.\left.\left. +3m^4\left(6+6m+m^2\right)c_3\right)\right) e_n^3\right) / \left(3\left(m^7(2+m)^2\right)\right) + O(e_n)^4. \end{aligned} \tag{30}$$

Let $\frac{f'(y_n)}{f'(x_n)} = \mu + v$, where $\mu = \left(\frac{m}{2+m}\right)^{m-1}$. Then from (30), the remainder $v = \frac{f'(y_n)}{f'(x_n)} - \mu$ is with the same order of e_n. Thus we can consider the Taylor's expansion of the weight function $Q\left(\frac{f'(y_n)}{f'(x_n)}\right) = Q(\mu + v)$ in the neighborhood of μ and obtain

$$Q\left(\frac{f'(y_n)}{f'(x_n)}\right) = Q(\mu) + Q'(\mu)v + \frac{Q''(\mu)v^2}{2!} + \frac{Q'''(\mu)v^3}{3!} + O(e_n^4). \tag{31}$$

Using (29) and (31) in the scheme (24) , we obtain the following error equation

$$\begin{aligned} e_{n+1} =& e_n - \frac{1}{2}\left[\frac{a_1 m f(x_n)}{f'(x_n)} + \frac{2a_2 m(m-1)f(x_n)}{(m+2)(f'(x_n)-f'(y_n))}\right] Q\left(\frac{f'(y_n)}{f'(x_n)}\right) \\ =& K_1 e_n + K_2 e_n^2 + \frac{1}{2}(K_3+K_4+K_5+K_6)e_n^3 + e_n^4 + O(e_n^5), \end{aligned} \tag{32}$$

where

$$\begin{aligned} K_1 =& 1 - \frac{a_1 Q(\mu)}{2} + \frac{a_2 m(m-1)Q(\mu)}{(m+2)(2p^m + m(-1+p^m))}, \\ K_2 =& \frac{1}{2m^3}\left[4Q'(\mu)\left(\frac{m}{2+m}\right)^m\left(a_1 - \frac{2a_2(-1+m)m}{(2+m)\left(2\left(\frac{m}{2+m}\right)^m + m\left(-1+\left(\frac{m}{2+m}\right)^m\right)\right)}\right)\right. \\ & \left. - Q(\mu)m^2\left(-a_1 + \frac{2a_2(-1+m)\left(-4\left(\frac{m}{2+m}\right)^m + 2m\left(\frac{m}{2+m}\right)^m + m^2\left(-1+\left(\frac{m}{2+m}\right)^m\right)\right)}{(2+m)\left(2\left(\frac{m}{2+m}\right)^m + m\left(-1+\left(\frac{m}{2+m}\right)^m\right)\right)^2}\right)\right], \end{aligned}$$

$$\begin{aligned} K_3 =& \frac{4Q'(\mu)\left(\frac{m}{2+m}\right)^m\left(-a_1 + \frac{2a_2(-1+m)\left(-4\left(\frac{m}{2+m}\right)^m + 2m\left(\frac{m}{2+m}\right)^m + m^2\left(-1+\left(\frac{m}{2+m}\right)^m\right)\right)}{(2+m)\left(2\left(\frac{m}{2+m}\right)^m + m\left(-1+\left(\frac{m}{2+m}\right)^m\right)\right)^2}\right)c_1^2}{m^4}, \\ K_4 =& -\frac{4\left(\frac{m}{2+m}\right)^m\left(a_1 - \frac{2a_2(-1+m)m}{(2+m)\left(2\left(\frac{m}{2+m}\right)^m + m\left(-1+\left(\frac{m}{2+m}\right)^m\right)\right)}\right)\left(\left(2Q''(\mu)\left(\frac{m}{2+m}\right)^m + Q'(\mu)m\left(2+m^2\right)\right)c_1^2 - 2Q'(\mu)m^3c_2\right)}{m^6}, \\ K_5 =& -Q(\mu)\left(\frac{2a_2(-1+m)m\left(1+m-\frac{\left(\frac{m}{2+m}\right)^m\left(-4+2m+3m^2+m^3\right)}{m^2}\right)c_1^2}{(2+m)\left(m-\left(\frac{m}{2+m}\right)^m(2+m)\right)^2} - \frac{2a_2(-1+m)mc_2}{(2+m)\left(2\left(\frac{m}{2+m}\right)^m + m\left(-1+\left(\frac{m}{2+m}\right)^m\right)\right)}\right) \\ & + Q(\mu)\frac{a_1((1+m)c_1^2 - 2mc_2)}{m^2}, \end{aligned}$$

and

$$\begin{aligned} K_6 =& -\frac{Q(\mu)}{(2+m)\left(m-\left(\frac{m}{2+m}\right)^m(2+m)\right)^2\left(2\left(\frac{m}{2+m}\right)^m + m\left(-1+\left(\frac{m}{2+m}\right)^m\right)\right)}\left(2a_2m(m-1)\left(\left(1+m-\frac{\left(\frac{m}{2+m}\right)^m\left(-4+2m+3m^2+m^3\right)}{m^2}\right)^2c_1^2\right.\right. \\ & \left.\left. - \left(m-\left(\frac{m}{2+m}\right)^m(2+m)\right)\left(2c_2 + mc_2 + \frac{\left(\frac{m}{2+m}\right)^m\left(4(-2+m)c_1^2 - m^2\left(-8+4m+4m^2+m^3\right)c_2\right)}{m^4}\right)\right)\right). \end{aligned}$$

For obtaining an optimal general class of fourth-order iterative methods, the coefficients of e_n, e_n^2, and e_n^3 in the error equation (32) must be zero simultaneously. After simplifying the equation

(32), we have the following equations involving of $Q(\mu)$, $Q'(\mu)$, and $Q''(\mu)$.
Solving $K_1 = 0, K_2 = 0, K_3 = 0, K_4 = 0,\ K_5 = 0$ and $K_6 = 0$, we get

$$\begin{cases} Q(\mu) = 1, \\ Q'(\mu) = 0, \\ Q''(\mu) = \dfrac{m^4 \left(\frac{m}{2+m}\right)^{-2m} \left(-1 + \left(\frac{m}{2+m}\right)^m\right)}{4\left(2\left(\frac{m}{2+m}\right)^m + m\left(-1+\left(\frac{m}{2+m}\right)^m\right)\right)}, \end{cases} \tag{33}$$

where $\mu = \left(\frac{m}{m+2}\right)^{m-1}$.
Using the above conditions, the scheme (24) satisfy the error equation (26). This reveals that the general two-step class of higher-order methods (24) reaches the optimal order of convergence four by using only three functional evaluations per full iteration. This completes the proof. □

Finally, under the conditions of Theorem (3.1), we get

$$\begin{aligned} x_{n+1} =& x_n - \frac{m}{4} f(x_n) \left(1 + \frac{1}{6} Q'''(\mu) \left(\frac{f'(y_n)}{f'(x_n)} - p^{-1+m}\right)^3 + \frac{m^4 p^{-2m} \left(-\frac{f'(y_n)}{f'(x_n)} + p^{-1+m}\right)^2 (-1+p^m)}{8\left(2p^m + m\left(-1+p^m\right)\right)}\right) \\ & \times \left(\frac{4 - 2m + m^2\left(-1+p^{-m}\right)}{f'(x_n)} - \frac{p^{-m}\left(2p^m + m\left(-1+p^m\right)\right)^2}{f'(x_n) - f'(y_n)}\right), \end{aligned} \tag{34}$$

where $|Q'''(\mu)| < \infty$ and $p = \frac{m}{m+2}$.

This is a new optimal family of fourth-order methods for multiple roots.

B. Second family

Similarly, in order to develop multipoint methods from formula (14), which requires the computation of second-order derivative, we shall make use of the following approximation

$$f''(x_n) \approx \frac{f'(x_n)\big(f'(x_n) - f'(y_n)\big)}{\theta f(x_n)}. \tag{35}$$

Using this approximate value of $f''(x_n)$ in formula (14) and $\theta = \frac{2m}{m+2}$, we get a modified method free from second-order derivative as

$$x_{n+1} = x_n - \frac{1}{2}\left[\frac{2m f(x_n)\left(\frac{(-1+m)}{f'(x_n)-f'(y_n)} + \frac{(2+m)}{f'(x_n)(-2+m)+f'(y_n)(2+m)}\right)}{m+2}\right]. \tag{36}$$

It satisfies the following error equation

$$e_{n+1} = \left(1 - \frac{m\left(\frac{(-1+m)}{m-\left(\frac{m}{2+m}\right)^m (2+m)} + \frac{(2+m)}{(-2+m)m + \left(\frac{m}{2+m}\right)^m (2+m)^2}\right)}{2+m}\right) e_n + O(e_n^2).$$

On inserting parameters α_1 and α_2 in (36), we get

$$x_{n+1}=x_n-\frac{1}{2}\left[\frac{2mf(x_n)\left(\frac{\alpha_1(-1+m)}{f'(x_n)-f'(y_n)}+\frac{\alpha_2(2+m)}{f'(x_n)(-2+m)+f'(y_n)(2+m)}\right)}{m+2}\right]. \tag{37}$$

It satisfies the following error equation

$$e_{n+1}=\left(1-\frac{m\left(\frac{(-1+m)\alpha_1}{m-\left(\frac{m}{2+m}\right)^m(2+m)}+\frac{(2+m)\alpha_2}{(-2+m)m+\left(\frac{m}{2+m}\right)^m(2+m)^2}\right)}{2+m}\right)e_n+O(e_n^2). \tag{38}$$

From error equation (38), we can see that for the following choices of α_1 and α_2

$$\alpha_1=-\frac{\left(\frac{m}{2+m}\right)^{-m}(2+m)\left(2\left(\frac{m}{2+m}\right)^m+m\left(-1+\left(\frac{m}{2+m}\right)^m\right)\right)^2\left(-8\left(\frac{m}{2+m}\right)^m+m^3\left(1+\left(\frac{m}{2+m}\right)^m\right)+m^2\left(-2+4\left(\frac{m}{2+m}\right)^m\right)\right)}{8(-1+m)m^3},$$

$$\alpha_2=-\frac{\left(\frac{m}{2+m}\right)^{-m}\left(-4\left(\frac{m}{2+m}\right)^m+2m\left(\frac{m}{2+m}\right)^m+m^2\left(-1+\left(\frac{m}{2+m}\right)^m\right)\right)\left(4\left(\frac{m}{2+m}\right)^m+m^2\left(1+\left(\frac{m}{2+m}\right)^m\right)+m\left(-2+4\left(\frac{m}{2+m}\right)^m\right)\right)^2}{8m^3},$$

the method (37) attains the third order convergence and satisfies the following error equation:

$$e_{n+1}=\frac{2\left(16\left(\frac{m}{2+m}\right)^{2m}-4m^2\left(\frac{m}{2+m}\right)^{2m}+2m^3\left(-1+\left(\frac{m}{2+m}\right)^m\right)^2+m^4\left(-1+\left(\frac{m}{2+m}\right)^{2m}\right)\right)c_1^2e_n^3}{m^4\left(2\left(\frac{m}{2+m}\right)^m+m\left(-1+\left(\frac{m}{2+m}\right)^m\right)\right)\left(4\left(\frac{m}{2+m}\right)^m+m^2\left(1+\left(\frac{m}{2+m}\right)^m\right)+m\left(-2+4\left(\frac{m}{2+m}\right)^m\right)\right)}+O(e_n^4).$$

It can be easily seen that the method (37) under the above mentioned choices of α_1 and α_2 is not optimal in the sense of Kung-Traub conjecture. For this purpose, let us consider the following class of methods by the use of weight function:

$$\begin{cases} y_n=x_n-\dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)},\\ x_{n+1}=x_n-\dfrac{1}{2}\left[\dfrac{2mf(x_n)\left(\frac{\alpha_1(-1+m)}{f'(x_n)-f'(y_n)}+\frac{\alpha_2(2+m)}{f'(x_n)(-2+m)+f'(y_n)(2+m)}\right)}{m+2}\right]Q\left(\dfrac{f'(y_n)}{f'(x_n)}\right), \end{cases} \tag{39}$$

where, α_1 and α_2 are defined as above and Q(.) $\in C^2(\mathbb{R})$ is any real-valued weight function such that the order of convergence reaches at the optimal level without consuming any more functional evaluations. Theorem (3.2) indicates that under what conditions on the weight function in (39), the order of convergence will reach the optimal level four.

Theorem 2: Let $f: D\subseteq\mathbb{R}\to\mathbb{R}$ be a sufficiently smooth function defined on an open interval D, enclosing a multiple zero of $f(x)$, say $x=r_m$ with multiplicity $m>1$. Then the family of iterative methods defined by (39) has fourth-order convergence when

$$\begin{cases} Q(\mu)=1,\\ Q'(\mu)=0,\\ Q''(\mu)=\dfrac{p^{-2m}\left(16m^2p^{2m}-4m^4p^{2m}+2m^5\left(-1+p^m\right)^2+m^6\left(-1+p^{2m}\right)\right)}{4\left(2p^m+m\left(-1+p^m\right)\right)\left(4p^m+m^2\left(1+p^m\right)+m\left(-2+4p^m\right)\right)},\\ |Q'''(\mu)|<\infty, \end{cases} \tag{40}$$

where $\mu = \left(\frac{m}{m+2}\right)^{m-1}$ and $p = \frac{m}{m+2}$. It satisfies the following error equation

$$e_{n+1} = \frac{((2+m)^2\beta_1 c_1^3 - \beta_2)e_n^4}{3m^9(2+m)^2\left(2p^m + m\left(-1+p^m\right)\right)^2\left(4p^m + m^2\left(1+p^m\right) + m\left(-2+4p^m\right)\right)^2} + O(e_n^5), \tag{41}$$

where

$$\begin{aligned}
\beta_1 &= 2048Q'''(\mu)p^{7m} + 2048Q'''(\mu)mp^{6m}\left(-2+3p^m\right) + m^{14}\left(-1+p^{2m}\right)^2 \\
&\quad + 512Q'''(\mu)m^2p^{5m}\left(6-16p^m+15p^{2m}\right) + 64m^7p^m\left(-6+15p^m-23p^{2m}+14p^{3m}\right) \\
&\quad + 2m^{13}\left(-1+4p^m-6p^{2m}-4p^{3m}+7p^{4m}\right) + 8m^{11}\left(-1-2p^m+16p^{2m}\right. \\
&\quad \left.-42p^{3m}+37p^{4m}\right) + 2m^{12}\left(-1+8p^m-2p^{2m}-40p^{3m}+43p^{4m}\right) \\
&\quad + 16m^9\left(-5+5p^m+7p^{2m}-47p^{3m}+56p^{4m}\right) + 16m^8\left(3+10p^m-6p^{2m}-54p^{3m}+71p^{4m}\right) \\
&\quad + 4m^{10}\left(13-32p^m+84p^{2m}-176p^{3m}+155p^{4m}\right) \\
&\quad + 128m^4p^{3m}\left(-12+Q'''(\mu)+18p^m+4Q'''(\mu)p^{2m}-16Q'''(\mu)p^{3m}+15Q'''(\mu)p^{4m}\right) \\
&\quad + 32m^6p^{2m}\left(12-4p^m-26p^{2m}+Q'''(\mu)p^m\left(-1+p^{2m}\right)^2\right) \\
&\quad + 1024m^3p^{4m}\left(3+Q'''(\mu)\left(-1+3p^m-6p^{2m}+5p^{3m}\right)\right) + 128m^5p^{3m}\left(6-10p^mQ'''(\mu)\right. \\
&\quad \left.+\left(-1+2p^m-2p^{2m}-2p^{3m}+3p^{4m}\right)\right), \\
\beta_2 &= 3m^8(2+m)^2\left(8p^{2m}+4mp^m\left(-2+3p^m\right)+m^3\left(-1+p^{2m}\right)+m^2\left(2-4p^m+6p^{2m}\right)\right)^2c_1c_2 \\
&\quad + 3m^{10}\left(8p^{2m}+4mp^m\left(-2+3p^m\right)+m^3\left(-1+p^{2m}\right)+m^2\left(2-4p^m+6p^{2m}\right)\right)^2c_3.
\end{aligned}$$

Proof: The proof is similar to the Theorem (3.1). Hence ommited here. □

Therefore, using the conditions of Theorem (3.2) in (39), we get a new class of fourth-order methods given by

$$x_{n+1} = x_n - \frac{f(x_n)p^{-m}}{8m^2}\left(A_1+A_2\right)\left(1+\frac{1}{6}Q'''(\mu)\left(\frac{f'(y_n)}{f'(x_n)}-p^{-1+m}\right)^3+A_3\right), \tag{42}$$

$$\begin{cases}
A_1 = -\dfrac{\left(-4p^m+2mp^m+m^2\left(-1+p^m\right)\right)\left(4p^m+m^2\left(1+p^m\right)+m\left(-2+4p^m\right)\right)^2}{f'(x_n)(-2+m)+f'(y_n)(2+m)}, \\
A_2 = -\dfrac{\left(2p^m+m\left(-1+p^m\right)\right)^2\left(-8p^m+m^3\left(1+p^m\right)+m^2\left(-2+4p^m\right)\right)}{f'(x_n)-f'(y_n)}, \\
A_3 = \dfrac{p^{-2m}\left(f'(y_n)m-f'(x_n)(2+m)p^m\right)^2\left(16p^{2m}-4m^2p^{2m}+2m^3\left(-1+p^m\right)^2+m^4\left(-1+p^{2m}\right)\right)}{8f'(x_n)^2\left(2p^m+m\left(-1+p^m\right)\right)\left(4p^m+m^2\left(1+p^m\right)+m\left(-2+4p^m\right)\right)},
\end{cases}$$

where $\mu = \left(\frac{m}{m+2}\right)^{m-1}$ and $p = \frac{m}{m+2}$.

4. Some special cases

C. Special cases of formula (34)

We can deduce many optimal fourth-order methods from (34) for multiple roots of a nonlinear equation. For simplicity, we discuss some interesting cases as follows: (i)

(1) Let us consider the following weight function

$$Q(x) = Ax^2 + Bx + C.$$

Then $Q'(x) = 2Ax + B, \ \ Q''(x) = 2A$.
According to the theorem (3.1), we should solve the following equations:

$$\begin{cases} A\mu^2 + B\mu + C = 1, \\ 2A\mu + B = 0, \\ 2A = \dfrac{m^4\left(\frac{m}{2+m}\right)^{-2m}\left(-1+\left(\frac{m}{2+m}\right)^m\right)}{4\left(2\left(\frac{m}{2+m}\right)^m + m\left(-1+\left(\frac{m}{2+m}\right)^m\right)\right)}. \end{cases} \tag{43}$$

The solutions to the above equations are

$$A = \frac{m^4\left(\frac{m}{2+m}\right)^{-2m}\left(-1+\left(\frac{m}{2+m}\right)^m\right)}{8\left(2\left(\frac{m}{2+m}\right)^m + m\left(-1+\left(\frac{m}{2+m}\right)^m\right)\right)},$$
$$B = -\frac{m^4\left(\frac{m}{2+m}\right)^{-1-m}\left(-1+\left(\frac{m}{2+m}\right)^m\right)}{4\left(2\left(\frac{m}{2+m}\right)^m + m\left(-1+\left(\frac{m}{2+m}\right)^m\right)\right)},$$
$$C = 1 + \frac{m^2(2+m)^2\left(-1+\left(\frac{m}{2+m}\right)^m\right)}{8\left(2\left(\frac{m}{2+m}\right)^m + m\left(-1+\left(\frac{m}{2+m}\right)^m\right)\right)},$$

and thus we obtain the following iterative scheme of order four

$$\begin{aligned} x_{n+1} = x_n - \frac{m}{32}f(x_n)&\left(8 + \frac{m^2(2+m)^2\left(-1+p^m\right)}{2p^m + m\left(-1+p^m\right)} + \frac{f'(y_n)^2 m^4 p^{-2m}\left(-1+p^m\right)}{f'(x_n)^2\left(2p^m + m\left(-1+p^m\right)\right)}\right. \\ &\left.-\frac{2f'(y_n)m^3(2+m)p^{-m}\left(-1+p^m\right)}{f'(x_n)\left(2p^m + m\left(-1+p^m\right)\right)}\right)\left(\frac{4-2m+m^2\left(-1+p^{-m}\right)}{f'(x_n)} - \frac{p^{-m}\left(2p^m + m\left(-1+p^m\right)\right)^2}{f'(x_n) - f'(y_n)}\right), \end{aligned} \tag{44}$$

where $p = \frac{m}{m+2}$.
This is a new fourth-order optimal method for multiple roots.

(2) Taking $Q'''(\mu) = 0$ in (34), we get

$$\begin{aligned} x_{n+1} =& x_n - \frac{m}{4}f(x_n)\left(1 + \frac{m^4 p^{-2m}\left(-\frac{f'(y_n)}{f'(x_n)} + p^{-1+m}\right)^2\left(-1+p^m\right)}{8\left(2p^m + m\left(-1+p^m\right)\right)}\right) \\ &\times\left(\frac{4-2m+m^2\left(-1+p^{-m}\right)}{f'(x_n)} - \frac{p^{-m}\left(2p^m + m\left(-1+p^m\right)\right)^2}{f'(x_n) - f'(y_n)}\right). \end{aligned} \tag{45}$$

This is again a new fourth-order optimal method for multiple roots.

(3) Taking $Q'''(\mu) = \frac{1}{2}$ in (34), we get

$$x_{n+1} = x_n - \frac{m}{4} f(x_n) \left(1 + \frac{1}{12} \left(\frac{f'(y_n)}{f'(x_n)} - p^{-1+m} \right)^3 + \frac{m^4 p^{-2m} \left(-\frac{f'(y_n)}{f'(x_n)} + p^{-1+m} \right)^2 (-1 + p^m)}{8 \left(2p^m + m\left(-1 + p^m\right)\right)} \right) \times \left(\frac{4 - 2m + m^2 \left(-1 + p^{-m}\right)}{f'(x_n)} - \frac{p^{-m} \left(2p^m + m\left(-1 + p^m\right)\right)^2}{f'(x_n) - f'(y_n)} \right). \tag{46}$$

This is a new fourth-order optimal method for multiple roots.

(4) Taking $Q'''(\mu) = -\frac{1}{2}$ in (34), we get

$$x_{n+1} = x_n - \frac{m}{4} f(x_n) \left(1 - \frac{1}{12} \left(\frac{f'(y_n)}{f'(x_n)} - p^{-1+m} \right)^3 + \frac{m^4 p^{-2m} \left(-\frac{f'(y_n)}{f'(x_n)} + p^{-1+m} \right)^2 (-1 + p^m)}{8 \left(2p^m + m\left(-1 + p^m\right)\right)} \right) \times \left(\frac{4 - 2m + m^2 \left(-1 + p^{-m}\right)}{f'(x_n)} - \frac{p^{-m} \left(2p^m + m\left(-1 + p^m\right)\right)^2}{f'(x_n) - f'(y_n)} \right). \tag{47}$$

This is a new fourth-order optimal method for multiple roots.

Some particular cases of formula (42)

For different specific values of $Q'''(\mu)$ various optimal multipoint methods can be derived from formula (42) as follows: (i)

(1) Taking $Q'''(\mu) = 0$ in (42) , we get

$$x_{n+1} = x_n - \frac{f(x_n) p^{-m}}{8m^2} \left(A_1 + A_2\right) \left(1 + A_3\right), \tag{48}$$

where $A_1,\ A_2,\ A_3$ are defined by (42).

This is a new fourth-order optimal method for multiple roots.

(2) Taking $Q'''(\mu) = \frac{1}{2}$ in (42), we get

$$x_{n+1} = x_n - \frac{f(x_n) p^{-m}}{8m^2} \left(A_1 + A_2\right) \left(1 + \frac{1}{12} \left(\frac{f'(y_n)}{f'(x_n)} - p^{-1+m} \right)^3 + A_3 \right), \tag{49}$$

where $A_1,\ A_2,\ A_3$ are defined by (42).

This is again a new fourth-order optimal method for multiple roots. Therefore, by choosing different values of $Q'''(\mu)$, we can derive several new fourth-order optimal methods for multiple roots.

5. Numerical results

In this section, we shall check the effectiveness of newly proposed multi-point methods. We employ the present family of methods namely, method (45), (48) denoted by (MM_4^1) and (MM_4^2) respectively, to solve the following nonlinear equations. We compare them with the Rall's method

(Rall, 1966), method of (Zhou et al., 2011) namely method (11) (ZM_4), (Li et al., 2010) methods namely, method (69) and method(75) denoted by (LM_4^1) and (LM_4^2), (Sharma and Sharma, 2010) method denoted by (SM_4) respectively. For better comparisons of our proposed methods, we have given two comparison tables in each example: one is corresponding to the absolute error value of given nonlinear functions (with the same total number of functional evaluations =12) and other is with respect to number of iterations taken by each method to obtain the root correct up to 35 significant digits. All computations have been performed using the programming package $Mathematica$ 9 with multiple precision arithmetic. We use $\epsilon = 10^{-34}$ as a tolerance error. The following stopping criteria are used for computer programs: (i) $|x_{n+1} - x_n| < \epsilon$, (ii) $|f(x_{n+1})| < \epsilon$.

Example 1: Consider the following 5×5 matrix

$$B = \begin{bmatrix} 29 & 14 & 2 & 6 & -9 \\ -47 & -22 & -1 & -11 & 13 \\ 19 & 10 & 5 & 4 & -8 \\ -19 & -10 & -3 & -2 & 8 \\ 7 & 4 & 3 & 1 & -3 \end{bmatrix}.$$

The corresponding characteristic polynomial of this matrix is as follows:

$$f_1(x) = (x-2)^4(x+1).$$

It's characteristic equation has one multiple root at $x = 2.0000000000000000000000000000000$ of multiplicity four.

$f(x)$	x_0	RM_2	ZM_4	LM_4^1	LM_4^2	SM_4	MM_4^1	MM_4^2
Comparison of different iterative methods with the same total number of functional evaluations (TNFE=12)								
$f_1(x)$	1.0	7.4−244	1.2e−61	3.7e−616	1.6e−616	2.8e−615	1.29e−620	1.50e−619
	1.5	1.4e−336	5.6e−1017	4.0e−1019	1.2e−1019	7.1e−1018	2.87e−1025	8.30e−1024
	2.5	5.5e−360	1.1e−1150	6.1e−1153	1.6e−1153	1.3e−1151	4.97e−1159	1.33e−1157
	2.9	3.5e−302	7.3e−931	4.7e−933	1.3e−933	9.2e−932	7.17e−939	1.65e−937
Comparison of different iterative methods with respect to number of iteration								
$f_1(x)$	1.0	7	4	4	4	4	4	4
	1.5	6	4	4	4	4	4	4
	2.5	6	4	4	4	4	4	4
	2.9	6	4	4	4	4	4	4

Example 2: Consider the following 6×6 matrix

$$A = \begin{bmatrix} 5 & 8 & 0 & 2 & 6 & -6 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 6 & 18 & -1 & 1 & 13 & -9 \\ 3 & 6 & 0 & 4 & 6 & -6 \\ 4 & 14 & -2 & 0 & 11 & -6 \\ 6 & 18 & -2 & 1 & 13 & -8 \end{bmatrix}.$$

The corresponding characteristic polynomial of this matrix is as follows:

$$f_2(x) = (x-1)^3(x-2)(x-3)(x-4).$$

It's characteristic equation has one multiple root at $x = 1.00000000000000000000000000000000$ of multiplicity three.

$f(x)$	x_0	RM_2	ZM_4	LM_4^1	LM_4^2	SM_4	MM_4^1	MM_4^2
Comparison of different iterative methods with the same total number of functional evaluations (TNFE=12)								
$f_2(x)$	0.4	1.5e−110	4.1e−358	2.6e−365	3.7e−367	2.0e−361	6.82e−378	3.39e−372
	0.6	2.8e−136	1.0e−451	2.2e−459	2.6e−461	2.8e−455	1.61e−473	4.04e−467
	1.3	6.1e−121	8.9e−310	2.8e−313	4.7e−314	1.7e−311	5.60e−322	4.68e−318
	1.4	2.4e−88	2.5e−144	2.6e−146	8.6e−147	3.0e−145	3.61e−150	2.04e−148
Comparison of different iterative methods with respect to number of iteration								
$f_2(x)$	0.4	7	5	5	5	5	5	5
	0.6	7	4	4	4	4	4	4
	1.3	7	5	5	5	5	5	5
	1.4	8	5	5	5	5	5	5

Example 3: $f_3(x) = (5\tan^{-1} x - 4x)^8$.

This equation has finite number of roots with multiplicity eight but our desired root is $r_m = 0.94913461128828951372581521479848875$.

$f(x)$	x_0	RM_2	ZM_4	LM_4^1	LM_4^2	SM_4	MM_4^1	MM_4^2
Comparison of different iterative methods with the same total number of functional evaluations (TNFE=12)								
$f_3(x)$	0.7	2.6e−238	1.6e−248	1.8e−248	1.7e−248	1.6e−248	1.16e−248	1.22e−248
	1.0	3.6e−685	2.2e−2297	5.5e−2300	5.5e−2300	5.4e−2298	3.94e−2313	2.72e−2312
	1.2	1.4−379	2.0e−1136	1.1e−1138	1.1e−1138	6.0e−1137	8.09e−1150	4.04e−1149
Comparison of different iterative methods with respect to number of iteration								
$f_3(x)$	0.7	8	6	6	6	6	5	5
	1.0	6	4	4	4	4	4	4
	1.2	7	4	4	4	4	4	4

Example 4: $f_4(x) = \left((x-1)^3 - 1\right)^{50}$.
This equation has finite number of roots with multiplicity fifty but our desired root is $r_m = 2.00000000000000000000000000000000$.

Example 5: $f_5(x) = (x^2 - e^x - 3x + 2)^3$.
This equation has finite number of roots with multiplicity three but our desired root is $r_m = 0.25753028543986076045536730493724178$.

$f(x)$	x_0	RM_2	ZM_4	LM_4^1	LM_4^2	SM_4	MM_4^1	MM_4^2
Comparison of different iterative methods with the same total number of functional evaluations (TNFE=12)								
$f_4(x)$	1.7	$1.72e-2282$	4.7e−2269	2.6e−2269	2.6e−2269	4.6e−2269	16.90e−2276	7.49e−2276
	2.5	$5.76e-2024$	2.4e−4261	2.6e−4262	2.4e−4262	2.2e−4261	1.76e−4287	2.41e−4287
Comparison of different iterative methods with respect to number of iteration								
$f_4(x)$	1.7	8	5	5	5	5	5	5
	2.5	8	5	5	5	5	5	5

$f(x)$	x_0	RM_2	ZM_4	LM_4^1	LM_4^2	SM_4	MM_4^1	MM_4^2
Comparison of different iterative methods with the same total number of functional evaluations (TNFE=12)								
$f_5(x)$	−0.5	2.06e-179	3.6e−662	2.1e−651	5.1e−649	6.8e−657	1.60e−635	4.26e−642
	1.0	$8.60e-149$	6.0e−726	1.1e−725	1.2e−725	8.0e−726	3.58e−725	2.14e−725
Comparison of different iterative methods with respect to number of iteration								
$f_5(x)$	−0.5	7	4	4	4	4	4	4
	1.0	64	4	4	4	5	4	4

6. Attractor basins in the complex plane

We here investigate the comparison of the attained multiple root finders in the complex plane using basins of attraction. It is known that the corresponding fractal of an iterative root-finding method is a boundary set in the complex plane, which is characterized by the iterative method applied to a fixed polynomial $p(z) \in \mathbb{C}$, see e.g. (Scott et al., 2011; Neta et al., 2012). The aim herein is to use basin of attraction as another way for comparing the iteration algorithms.

From the dynamical point of view, we consider a rectangle $D = [-3, 3] \times [-3, 3] \in \mathbb{C}$ and we assign a color to each point $z_0 \in D$ according to the multiple root at which the corresponding iterative method starting from z_0 converges, and we mark the point as black if the method does not converge. In this section, we consider the stopping criterion for convergence to be less than 10^{-4} wherein the maximum number of full cycles for each method is considered to be 100. In this way, we distinguish the attraction basins by their colors for different methods.

We have compared our methods (11), (45) (MM_4^1), (48) (MM_4^2) with (LM_4^1), (SM_4^2), (ZM_4^3), for some complex polynomials having multiple zeros with known multiplicity.

For the first test, we have taken the cubic polynomial:

Test Problem 1. $p(z) = (z^2 - 1)^3$.

Its roots are: $1.0, -1.0$ with multiplicity three. Based on Fig. 1 and Fig.2 , we can see that the following methods performed better: (45), (11) , (48), SM_4^1, ZM_4^1 while the method namely, LM_1^4 did not perform well.

The second test problem is a non polynomial function as follows:

Test Problem 2. $p(z) = (z^3 + 1/z)^8$.

Its roots are: $-0.707107+0.707107I, -0.707107-0.707107I, 0.707107+0.707107I, 0.707107-0.707107I$ with multiplicity eight. The results are shown in Fig. 3 and Fig.4. The following methods performed well:, (11), LM_4^1 while the methods namely, (48),(45), SM_4^1, and ZM_4^1 are little sensitive to the initial guess.

Test Problem 3. $p(z) = (z^3 + 2z - I)^2$.

Its roots are: $0.-1.61803I,\ 0.+1.I,\ 0.+0.618034I$ with multiplicity 2. The results are presented in Fig. 5 and Fig. 6. The methods (11) , SM_4^1 performed better as compared to the other methods namely, (45), LM_4^1 and ZM_4^1, (48) .

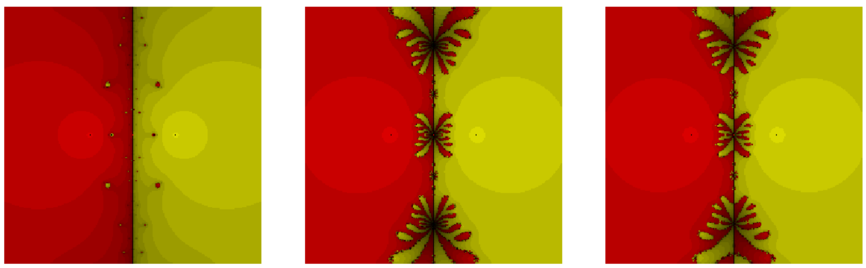

Fig. 1: The chaotic behaviour of the methods (11)$(left)$, (45)$(center)$, (48)$(right)$ for test problem 1.

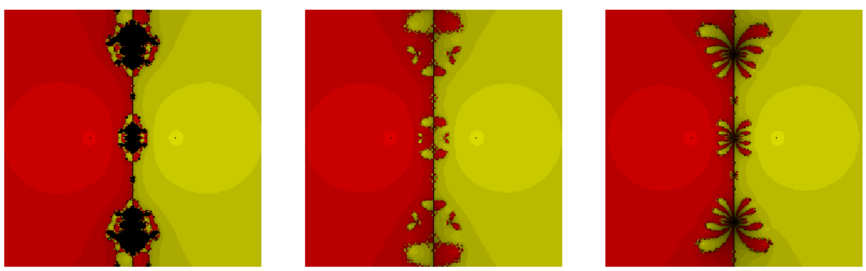

Fig. 2: The chaotic behaviour of the methods (LM_4^1), (SM_4^2), (ZM_4^3), respectively for test problem 1.

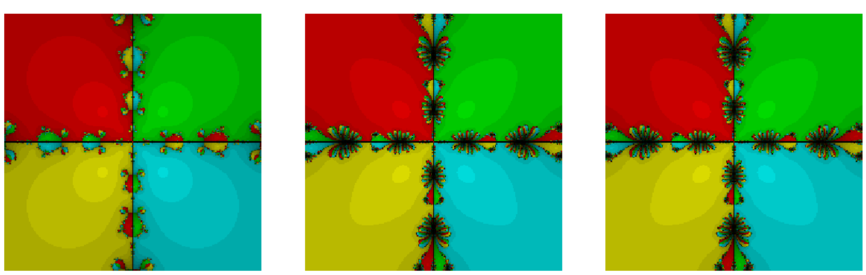

Fig. 3: The chaotic behaviour of the methods (11)$(left)$, (45)$(center)$, (48)$(right)$, for test problem 2.

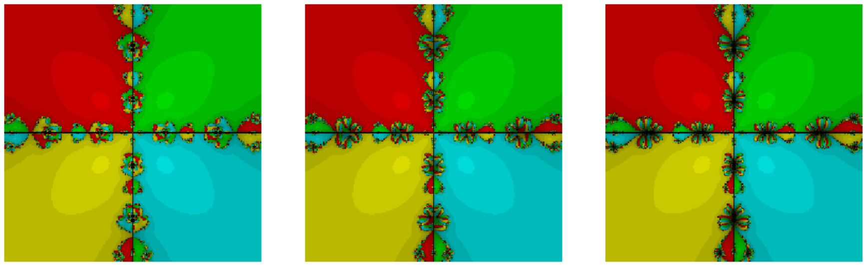

Fig. 4: The chaotic behaviour of the methods (LM_4^1), (SM_4^2), (ZM_4^3), respectively for test problem 2.

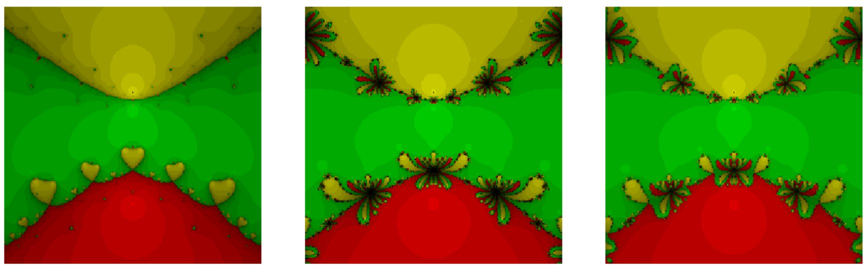

Fig. 5: The chaotic behaviour of the methods (11)$(left)$, (45)$(center)$, (48)$(right)$, for test problem 3.

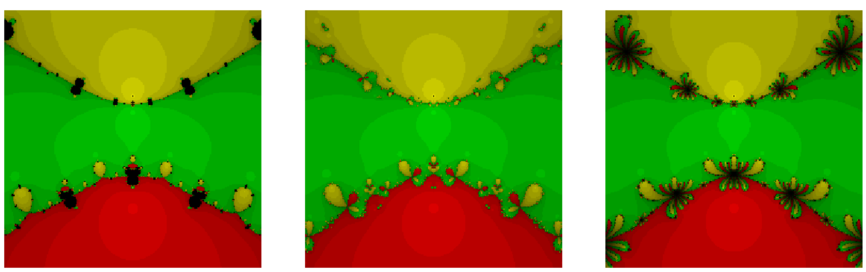

Fig. 6: The chaotic behaviour of the methods (LM_4^1), (SM_4^2), (ZM_4^3), respectively for test problem 3.

7. Conclusions

Using quadratically convergent schemes, we present two one-point iterative methods of order three for finding multiple zeros of a nonlinear equation. Based on one-point iterative schemes, we developed two optimal families of multipoint methods having quartic convergence. Each family requires three functional evaluations viz., $f(x_n),\ f'(x_n),\ f'(y_n)$ per full iteration, and are optimal in the sense of Kung-Traub conjecture. Some numerical experiments have been carried out to confirm the theoretical order of convergence of multipoint methods. Furthermore, we have also discussed the complex basins of attractions of the proposed fourth-order methods.

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