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On the Exchange Property for the Mehler-Fock Transform

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Abstract

The theory of Schwartz Distributions opened up a new area of mathematical research, which in turn has provided an impetus in the development of a number of mathematical disciplines, such as ordinary and partial differential equations, operational calculus, transformation theory and functional analysis. The integral transforms and generalized functions have also shown equivalent association of Boehmians and the integral transforms. The theory of Boehmians, which is a generalization of Schwartz distributions are discussed in this paper. Further, exchange property is defined to construct Mehler-Fock transform of tempered Boehmians. We investigate exchange property for the Mehler-Fock transform by using the theory of Mehler-Fock transform of distributions. Algebraic properties and convergence is also proved for this relation on the tempered Boehmians which is a natural extension of tempered distribution.

Keywords: Distribution spaces; tempered Boehmians; Fourier transform; Mehler-Fock transform

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1. Introduction

The concept of Boehmians is motivated by the regular operator introduced by Boehme (1973), which forms a subalgebra of the field of Mikusiński operators and thus they include only such functions whose support is bounded from the left. The theory of Boehmians (quotient of sequences), its properties and different classes of Boehmian spaces are studied by Mikusiński *et al*. (1981), Mikusiński (1983, 1995).

Tempered Boehmians is a natural extension of tempered distribution which, therefore, makes it possible to define an extension of the Fourier transform for this class of Boehmians. The Fourier transform of a tempered Boehmian is a distribution. An infinitely differentiable function $f : R^n \to C$ is called rapidly decreasing if

$$\sup_{|\alpha|\le m} \sup_{x\in R^N} (1 + x_1^2 + ... + x_N^2)^m \,|D^\alpha f(x)| < \infty, \tag{1}$$

for every nonnegative integer m, where $x = (x_1, x_2... x_N)$, $\alpha = (\alpha_1,...\alpha_N)$, α_N's are non-negative integer, $|\alpha| = \alpha_1+...+\alpha_N$, and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x^\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}...\partial x_N^{\alpha_N}}. \tag{2}$$

The space of rapidly decreasing functions is denoted by $S(R^N)$ or simply by S. If $f \in J$ and $\varphi \in S$, then the convolution

$$(f * \varphi)(x) = \int_{R^N} f(u)\varphi(x-u)du \tag{3}$$

is well defined and $f * \varphi \in J$. A sequence $\varphi_n \in S$ is called a delta sequence if it satisfies the following conditions

(i) $\int_{R^N} \varphi_n(x)dx = 1$, for all $n \in N$,

(ii) $\int_{R^N} |\varphi_n(x)|\,dx \le M$, for some constant M and for all $n \in N$,

(iii) $\lim_{n\to\infty} \int_{||x||\ge\varepsilon} \|x\|^k |\varphi_n(x)|\,dx = 0,$ for every $k \in N$ and $\varepsilon > 0$.

If $\varphi \in S$ and $\int \varphi = 1$, then the sequence of functions φ_n is a delta sequence.

A continuous function $f : R^N \to C$ is called slowly increasing if there is a polynomial p on R^N such that $|f(x)| \le p(x)$ for all $x \in R^N$. The space of slowly increasing function will be denoted by $J(R^N)$ or simply by J. Let $f_n \in J$. $\{\varphi_n\}$ is a delta sequence under usual notation. Then the space of equivalence classes of quotients of sequence will be denoted by β_J and its elements will be called tempered Boehmians. For $F = [f_n / \varphi_n] \in \beta_J$ define

$$D^\alpha F = [(f_n * D^\alpha \varphi_n)/(\varphi_n * \varphi_n)].$$

If F is a Boehmian corresponding to differentiable function, then $D^\alpha F \in \beta_J$.

If $F=[f_n/\varphi_n]\in\beta_J$ and $f_n\in S$, for all $n\in N$, then F is called a rapidly decreasing Boehmian, the space of which is denoted by β_S. If $F=[f_n/\varphi_n]\in\beta_J$ and $G=[g_n/\gamma_n]\in\beta_S$, then we define the convolution

$$F*G=[(f_n*g_n)/(\varphi_n*\gamma_n)]\in\beta_J\ .$$

In what follows, we will denote by *S′* the space of tempered distributions; that is, the space of continuous linear functional on *S*. The Mehler-Fock transform of a tempered distribution *f*, denoted by *Mf*, is the functional defined by $Mf\ (\varphi) = f\ (M\varphi)$, where $M\varphi$ is the Mehler-Fock transform of φ defined by Banerji *et al.* (2008). The Mehler-Fock transform on generalized functions is studied by Pathak (1997). The Mehler-Fock transform of Boehmian spaces are investigated by Loonker and Banerji (2008, 2009, 2009).

In Section 2 we study Mehler-Fock transform and its properties and investigate the exchange property for the Mehler-Fock transform. In Section 3, algebraic properties and convergence is proved for this relation on the tempered Boehmians.

2. The Mehler-Fock Transform and the Exchange Property

The Mehler-Fock transformation is defined as [cf. Yakubovich and Luchko (1994, p. 149)]:

$$M[f(x)]=F(r)=\int_1^\infty P_{-\frac{1}{2}+ir}(x)f(x)dx\,,\quad r>0\,, \tag{4}$$

and its inversion is given by

$$f(x)=\int_0^\infty r\tanh(\pi r)P_{-\frac{1}{2}+ir}(x)F(r)dr,\quad x>1\,. \tag{5}$$

The generalization of the Mehler-Fock transformation is given by [cf. Pathak (1997, p. 343)]

$$F(r)=\int_0^\infty f(x)P^{m,n}_{-\frac{1}{2}+ir}(\cosh x)\sinh x dx\ , \tag{6}$$

where $P^{m,n}_{-\frac{1}{2}+ir}(\cosh x)$ is the generalized Legendre function, defined for complex values of the parameters k,m and n by

$$P_k^{m,n}(z)=\frac{(z+1)^{n/2}}{\Gamma(1-m)(z-1)^{m/2}}\ {}_2F_1\left[k+\frac{n-m}{2}+1;-k+\frac{n-m}{2};1-m;\frac{1-z}{2}\right]\ , \tag{7}$$

for complex *z* not lying on the cross-cut along the real *x*-axis from 1 to $-\infty$.

The inversion formula of (6) is

$$f(x)=\int_0^\infty \chi(r)P^{m,n}_{-\frac{1}{2}+ir}(\cosh x)F(r)dr\ , \tag{8}$$

Where

$$\chi(r)=\Gamma\left(\frac{1-m+n}{2}+ir\right)\Gamma\left(\frac{1-m+n}{2}-ir\right)\Gamma\left(\frac{1-m-n}{2}+ir\right)\Gamma\left(\frac{1-m-n}{2}-ir\right)$$
$$\times\left[\Gamma(2ir)\Gamma(-2ir)\pi 2^{n-m-2}\right]^{-1}. \qquad (9)$$

When $m=n$, (6) and (8) can be written as

$$F(r)=\int_0^\infty P_{-\frac{1}{2}+ir}(\cosh\alpha)\sinh\alpha\ f(\alpha)d\alpha\,, \qquad (10)$$

and

$$f(\alpha)=\int_0^\infty r\tanh(\pi r)P_{-\frac{1}{2}+ir}(\cosh\alpha)F(r)dr\,, \qquad (11)$$

whereas for $m=n=0$, (6) and (8) reduce to (4) and (5), respectively.

The Parseval relation for the Mehler-Fock transformation is defined as [cf. Sneddon (1974, pp. 393-94)]

$$\int_0^\infty r\tanh(\pi r)F(r)G(r)dr=\int_1^\infty f(x)g(x)dx\,, \qquad (12)$$

whose convolution is

$$M[f*g]=M[f]\cdot M[g]. \qquad (13)$$

The asymptotic behavior for (7) is defined by Pathak (1997, p. 345) as

$$P^{m,n}_{-1/2+ir}(\cosh x)=\begin{cases}O(x^{-\operatorname{Re} m})\,, & x\to 0+,\\ O(e^{-(1/2)x}), & x\to\infty,\end{cases} \qquad (14)$$

and

$$P^{m,n}_{-1/2+ir}(\cosh x)=\begin{cases}O(1)\,, & r\to 0+,\\ 2^{1/2(n-m-1)}\pi^{-1/2}(\sinh x)^{-1/2}(ir)^{m-1/2}\{e^{irx}+ie^{-i(m\pi+xt)}+O(r^{-1})\}, & r\to\infty.\end{cases} \qquad (15)$$

Similarly, the function $\chi(r)$, defined by (9), possesses the following asymptotic behavior [cf. Pathak 1997, p. 345)]

$$\chi(r)=\begin{cases}O(r^2), & r\to 0+\quad |\operatorname{Re} n|<1-\operatorname{Re} m,\\ \dfrac{(ir)^{1-2m}}{\pi 2^{n-m-+2}}[1+O(r^{-1})], & r\to\infty.\end{cases} \qquad (16)$$

The distributional generalized Mehler-Fock transform $f \in \mathbb{M}'^{\alpha}_{\beta}(R_+)$, where R_+ denotes the set of positive real numbers and $\alpha \geq \mathrm{Re}(m)$, $\beta \leq 1/2$, *is* defined as [cf. Pathak (1997, p. 346)]

$$F(r) := \left\langle f(x), P^{m,n}_{-\frac{1}{2}+ir}(\cosh x) \right\rangle, \quad r \geq 0, \tag{17}$$

where the space $\mathbb{M}'^{\alpha}_{\beta}(R_+)$ is the dual of the space $\mathbb{M}^{\alpha}_{\beta}(R_+)$, which is the collection of all infinitely differentiable complex valued function φ defined on open interval $(0, \infty)$ denoted by R_+ such that for every non-negative integer q,

$$\gamma_q(\varphi) = \sup_{0<x<\infty} \left| \varsigma(x) \nabla_x^q \varphi(x) \right| < \infty, \tag{18}$$

where

$$\nabla_x = \left(D_x^2 + (\coth x) D_x + \frac{m^2}{2(1-\cosh x)} + \frac{n^2}{2(1+\cosh x)} \right), \tag{19}$$

and

$$\varsigma(x) = \varsigma_{\alpha,\beta}(x) = \begin{cases} O(x^{\alpha}), & x \to 0, \\ O(x^{\beta}), & x \to \infty. \end{cases} \tag{20}$$

The topology over $\mathbb{M}'^{\alpha}_{\beta}(R_+)$ is generated by separating collection of seminorms $\{\gamma_q\}_{q=0}^{\infty}$ and is a sequentially complete locally convex topological vector space. $D(R_+)$, the space of infinitely differentiable functions of compact support with the usual topology, is a linear subspace of $\mathbb{M}'^{\alpha}_{\beta}(R_+)$.

From the properties of the hypergeometric functions, the generalized Legendre function [cf. Pathak (1997, p. 346)], satisfies the following differential equation

$$D^2 y + (\coth x) Dy + \left[\frac{m^2}{2(1-\cosh x)} + \frac{n^2}{2(1+\cosh x)} + \left(r^2 + \frac{1}{4} \right) \right] y = 0. \tag{21}$$

Therefore,

$$\nabla_x P^{m,n}_{-1/2+ir}(\cosh x) = -\left(r^2 + \frac{1}{4} \right) P^{m,n}_{-1/2+ir}(\cosh x). \tag{22}$$

Relations (14) and (15) prove the boundedness for the Legendre function [cf. Pathak (1997, pp. 346-347, Lemma 11.3.1, Equation (11.3.2))]

$$\left| \varsigma(x) \left(\frac{\partial}{\partial r} \right)^q P^{m,n}_{-1/2+ir}(\cosh x) \right| \leq C \varsigma(x) x^q \left(\frac{\pi}{2} \right)^{1/2} \Gamma\left(\frac{1}{2} - m_r \right) P^{m_r,0}_{-1/2}(\cosh x), \tag{23}$$

where C is a constant independent of x and r and, m_r is $\mathrm{Re}(m)$.

The differentiability of the Mehler-Fock transform is defined by [cf. Pathak (1997, p. 347)]

$$F'(r) := \left\langle f(x), \left(\frac{\partial}{\partial r}\right) P^{m,n}_{-\frac{1}{2}+ir}(\cosh x) \right\rangle, \tag{24}$$

where $f \in \mathbb{M}'^{\alpha}_{\beta}(R_+)$, $\alpha \geq \mathrm{Re}(m), \mathrm{Re}(m) < 1/2, \quad \beta \leq 1/2, r \geq 0$.

When q is a non-negative integer depending on f, the asymptotic behavior of the Mehler-Fock transform is

$$F(r) = \begin{cases} O(1), & r \to 0, \\ O(r^{2q}), & r \to \infty, \end{cases} \tag{25}$$

where

$$|F(r)| \leq C' \max_q \left(r^2 + \frac{1}{4} \right)^q . \tag{26}$$

For the operator $\nabla^*_x : \mathbb{M}'^{\alpha}_{\beta}(R_+) \to \mathbb{M}'^{\alpha}_{\beta}(R_+)$ under already stated symbols and for $f \in \mathbb{M}'^{\alpha}_{\beta}(R_+)$, $\varphi \in \mathbb{M}^{\alpha}_{\beta}(R_+)$, we define the operator transformation formula by

$$\left\langle (\nabla^*_x)^q f(x), \varphi(x) \right\rangle = \left\langle f(x), \nabla^q_x \varphi(x) \right\rangle, \tag{27}$$

and for f being the generalized Mehler-Fock transformation,

$$M\left[(\nabla^*_t)^q f(x) \right] = (-1)^q \left(\frac{1}{4} + r^2 \right)^q M\left[f(x) \right].$$

If $f \in \mathbb{M}'^{\alpha}_{\beta}(R_+)$ and $\varphi \in \mathbb{M}^{\alpha}_{\beta}(R_+)$ we have by transposition by Banerji *et al.* (2008)

$$\left\langle M(f), \phi \right\rangle = \left\langle f, M(\phi) \right\rangle , \tag{28}$$

where function f is absolutely integrable and φ is a testing function of rapid descent.

For a family $\{\varphi_i\}_{i \in I} = \{\varphi_i\}_I$, where I is an index set and $\varphi_i \in \mathbb{M}^{\alpha}_{\beta}(R_+) \subset S, \forall i \in I$, we define [Atanasiu and Mikusiński (2005)]:

$$\Psi(\{\varphi_i\}_I) = \left\{ x \in R^N_+ \ : M\varphi_i(x) = 0, \ \ \forall \ i \in I \right\}. \tag{29}$$

A family of pairs $\{(f_i, \varphi_i)\}_I$, where $f_i \in \mathbb{M}'^{\alpha}_{\beta}(R_+) \subset S'$ and $\varphi_i \in \mathbb{M}^{\alpha}_{\beta}(R_+) \subset S, \forall i \in I$, is said to have the exchange property if

$$f_i * \varphi_k = f_k * \varphi_i, \ \forall \ i,k \in I. \tag{30}$$

We will denote by A the collection of all families of pairs $\{(f_i, \varphi_i)\}_I$, where I is an index set, $f_i \in \mathbb{M}'^{\alpha}_{\beta}(R_+) \subset S'$ and $\varphi_i \in \mathbb{M}^{\alpha}_{\beta}(R_+) \subset S, \forall i \in I$, satisfying the exchange property such that $\Psi(\{\varphi_i\}_I) = \Phi$. If (φ_i) is a delta sequence, then $\Psi(\{\varphi_i\}_N) = \Phi$.

Definition 1.

If $\{(f_i, \varphi_i)\}_I \in A$, then the unique $F \in D'(R)$ such that $Mf_i = M\varphi_i F$ for all $i \in I$ will be denoted by $M\{(f_i, \varphi_i)\}_I$.

Let $\{(f_i,\varphi_i)\}_I, \{(g_k,\psi_k)\}_k \in A$. If $f_i * \psi_k = g_k * \varphi_i$ for all $i \in I$ and $k \in K$, then we write $\{(f_i,\varphi_i)\}_I \sim \{(g_k,\psi_k)\}_K$. This relation is clearly symmetric and reflexive. We will show that it is also transitive.

Let

$$\{(f_i,\varphi_i)\}_I, \{(g_k,\psi_k)\}_K, \{(h_l,\gamma_l)\}_L \in A.$$

If

$$\{(f_i,\varphi_i)\}_I \sim \{(g_k,\psi_k)\}_K \text{ and } \{(g_k * \psi_k)\}_K \sim \{(h_l * \gamma_l)\}_L,$$

then

$$f_i * \psi_k = g_k * \varphi_i, \qquad g_k * \gamma_l = h_l * \psi_k, \tag{31}$$

for all $i \in I$, $k \in K$, $l \in L$. Therefore,

$$f_i * \psi_k * \gamma_l = g_k * \varphi_i * \gamma_l, \quad g_k * \gamma_l * \varphi_i = h_l * \psi_k * \varphi_i, \tag{32}$$

for all $i \in I$, $k \in K$, $l \in L$. Since $*$ is commutative, we have

$$f_i * \gamma_l * \psi_k = h_l * \varphi_i * \psi_k. \tag{33}$$

Now fix $i \in I$ and $l \in L$. Since $\Psi(\{\psi_k\}_K) = \Phi$ and (32) holds for every $k \in K$, we conclude that $f_i * \gamma_l = h_l * \varphi_i$ for all $i \in I$ and $l \in L$, which means that $\{(f_i,\varphi_i)\}_I \sim \{(h_l,\gamma_l)\}_L$.

Theorem 1.

If a family of pair $\{(f_i,\varphi_i)\}_I$ has the exchange property and $\Omega = \Psi\{(\varphi_i)_I\}^c$ (the complement of $\Psi\{(\varphi_i)_I\}$ in R_+), then there exists a unique $F \in D'(\Omega)$ such that

$$M[f_i] = FM[\varphi_i], \forall \ i \in I. \tag{34}$$

Proof:

For every $x \in \Omega$ there exists $i \in I$ and $\varepsilon > 0$ such that $|M\varphi_i(x)| > \varepsilon$ in an open neighborhood of x. Then we can define $F = Mf_i / M\varphi_i$ in that neighborhood. Let for some $\varepsilon > 0$, we have $|M\varphi_i(x)| > \varepsilon$ for all $x \in U$ and $|M\varphi_k(x)| > \varepsilon$ for all $x \in V$, where U and V are open sets. Since $f_i * \varphi_k = f_k * \varphi_i$, we have

$$Mf_i M\varphi_k = Mf_k M\varphi_i \quad \text{and} \quad \frac{Mf_i}{M\varphi_i} = \frac{Mf_k}{M\varphi_k}, \tag{35}$$

on $U \cap V$. This shows that F is a unique function.

Theorem 2.

There exists $\{(f_i, \varphi_i)\}_I \in A$, for every $F \in D'(R)$ and such that $F = M(\{(f_i, \varphi_i)\}_I)$.

Proof:

Since $D(R)$ denotes the space of smooth function with compact support, there exists a total sequence $\{\varphi_i\}_N$ such that $M\varphi_i \in D(R)$ for all $I \in N$. Then for every $I \in N$, there is $f_i \in \mathbf{M}'^{\alpha}_{\beta}(R_+) \subset S'$ such that $Mf_i = M\varphi_i F$. Clearly $\{(f_i, \varphi_i)\}_N \in A$ and $F = M(\{(f_i, \varphi_i)\}_N)$. This completes the proof of the theorem.

Definition 2. [Atanasiu and Mikusiński (2005)]

Let $\{U_i\}_I$ be an open covering of R_+ and let $\{\varphi_i\}_I$ be such that $|M\varphi_i(x)| > 0$ for $x \in U_i$. A family $\{\varphi_i\}_I$ such that $\gamma(\{\varphi_i\}_I) = \Phi$ will be called total.

Lemma 1. [Atanasiu and Mikusiński (2005)]

If $\{\varphi_i\}_I$ and $\{\psi_k\}_K$ are total, then $\{\varphi_i * \psi_k\}_{I\times K}$ is total.

Theorem 3.

Let

$$\{(f_i, \varphi_i)\}_I,\ \{(g_k, \psi_k)\}_K \in A.$$

Then,

$$\{(f_i, \varphi_i)\}_I \sim \{(g_k, \psi_k)\}_K \text{ if and only if } M(\{(f_i, \varphi_i)\}_I) = M(\{(g_k, \psi_k)\}_K).$$

Proof:

Here,

$$F = M(\{(f_i,\varphi_i)_I\}) \text{ and } G = M(\{(g_k,\psi_k)\}_K).$$

If

$$\{(f_i,\varphi_i)\}_I \sim \{g_k,\psi_k)\}_K,$$

then

$$\begin{aligned} F\, M\varphi_i M\psi_k &= Mf_i M\psi_k \\ &= Mg_k M\varphi_i \end{aligned}$$

$$FM\varphi_i M\psi_k = GM\psi_k M\varphi_i, \quad \forall i \in I, k \in K. \tag{37}$$

Hence, $F = G$, by Lemma 1. Now assume $F = G$. Then,

$$Mf_i M\psi_k = F\, M\varphi_i M\psi_k = G\, M\psi_k M\varphi_i = Mg_k M\varphi_i, \quad \forall i \in I, k \in K. \tag{38}$$

Hence,

$$\{(f_i,\varphi_i)\}_I \sim \{(g_k,\psi_k)\}_K.$$

This completes the proof of the theorem.

Theorem 4.

There exists a delta sequence (φ_n) such that for every $T \in \beta_J$, $T = [\{(f_n,\varphi_n)\}_N]$ for some $f_n \in J$.

Proof:

Let (ψ_n) be a delta sequence such that $M\psi_n \in D(R)$. Then for any $T \in \beta_J$, we have $TM\psi_n \in \mathbb{M}'^{\alpha}_{\beta}(R_+) \subset S'$, since $MT \in D'(R)$. Consequently, $MTM\psi_n = Mg_n$ for some $g_n \in \mathbb{M}'^{\alpha}_{\beta}(R_+) \subset S'$. It is easy to check that $T = [\{(g_n * \psi_n, \psi_n * \psi_n)\}_N]$. Since $f_n = g_n * \psi_n \in J$ and $(\varphi_n) = (\psi_n * \psi_n)$ is a delta sequence, where (φ_n) does not depend on T, hence the theorem is proved.

3. Algebraic Properties and Convergence

β_J becomes a vector space with the addition operation, defined by

$$[\{(f_i,\varphi_i)\}_I] + [\{(g_k,\psi_k)\}_K] = [\{(f_i * \psi_k + g_k * \varphi_i, \quad \varphi_i * \psi_k)\}_{I\times K}]. \tag{39}$$

Moreover, multiplication by a scalar and the operation $*$ are defined by

$$\lambda[\{(f_i,\varphi_i)\}_I]=[\{(\lambda f_i,\varphi_i)\}_I], \quad \lambda\in C. \tag{40}$$

If

$$[\{(f_i,\varphi_i)\}_I], \quad [\{(g_k,\psi_k)\}_K]\in\beta_J \text{ and } g_k\in \mathbb{M}^{\alpha}_{\beta}(R_+) \text{ for all } k\in K,$$

then for the operation $*$ we can define

$$[\{(f_i,\varphi_i)\}_I]*[\{(g_k,\psi_k)\}_k]=[\{(f_i*g_k,\varphi_i*\psi_k)\}_{I\times K}]. \tag{41}$$

Definition 3. [Atanasiu and Mikusiński (2005)]

Let T_0, T_1, T_2, ... $\in \beta_J$. Then the sequence (T_n) is said to converge to T_0, which is written as $T_n\to T_0$ if there exists a total family $\{\varphi_i\}_I$ such that

(a) there exists tempered distribution $f_{i,n}$, where $i\in I$ and $n\in N$ such that

$$T_n=[\{f_{i,n},\varphi_i\}_I] \text{ for all } n=0,1,2,...,$$

(b) $f_{i,n}\to f_{i,0}$ in $\mathbb{M}'^{\alpha}_{\beta}(R_+)$ as $n\to\infty$ for every $i\in I$.

Theorem 5.

The Mehler-Fock transform is an isomorphism from β_J to $D'(R)$.

Proof:

Since $T_n\to T_0$ in β_J if and only if $T_n-T_0\to 0$, it suffices to prove the continuity at 0. Let $T_n\to 0$ is in β_J. Then there exists tempered distribution $f_{i,n}$ where $i\in I$ and $n\in N$ such that $T_n=[\{f_{i,n},\varphi_i\}_I]$ for all $n=1,2,...$ and $f_{i,n}\to 0$ in $\mathbb{M}'^{\alpha}_{\beta}(R_+)\subset S'$ as $n\to\infty$ for every $i\in I$. If $\psi\in D(R)$, then there are $i_1,...,i_k$ such that

$$\text{supp}\psi\subset\bigcup_{m=1}^{k}\text{supp}M\varphi_{i_m}.$$

Then,

$$\lim_{n\to\infty}MT_n\psi=\lim_{n\to\infty}\sum_{m=1}^{k}(MT_nM\varphi_{i_m})\frac{\overline{M\varphi_{i_m}}\psi}{\sum_{m=1}^{k}|M\varphi_{i_m}|^2}$$

$$=\sum_{m=1}^{k}\left(\lim_{n\to\infty}Mf_{i_m,n}\right)\frac{\overline{M\varphi_{i_m}}\psi}{\sum_{m=1}^{k}|M\varphi_{i_m}|^2}=0, \tag{42}$$

because $\lim_{n\to\infty} Mf_{i,n} = 0$ for $\forall\ i \in I$, due to the continuity of the Mehler-Fock transform in $\mathbb{M}'^{\alpha}_{\beta}(R_+)$. This proves the continuity of $M:\ \beta_J \to D'(R)$, because $\lim_{n\to\infty} MT_n\psi = 0$ in $\mathbb{M}'^{\alpha}_{\beta}(R_+)$ for every $\psi \in D(R)$, implies $\lim_{n\to\infty} MT_n = 0$ in $D'(R)$.

Now, assume $\lim_{n\to\infty} MT_n = 0$ in $D'(R)$. By Theorem 4, there exists a delta sequence (φ_i), $i \in N$ such that for every $n \in N$, we have $T_n = [\{(f_{i,n}, \varphi_i)\}_N]$ for some $f_{i,n} \in J$. Let $(\psi_k), k \in N$ be a delta sequence such that $M\psi_k \in D(R)$ for every $k \in N$. Then,

$$\lim_{n\to\infty} MT_n\, M\varphi_i = M\psi_k = 0 \text{ in } \mathbb{M}'^{\alpha}_{\beta}(R_+) \text{ for every } i, k \in N.$$

Since

$$MT_n\ M\varphi_i = f_{i,n} \quad \forall\, i, k \in N,$$
$$\lim_{n\to\infty} Mf_{i,n} M\psi_k = 0 \text{ in } \mathbb{M}'^{\alpha}_{\beta}(R_+) \subset S',$$

which implies

$$\lim_{n\to\infty} f_{i,n} * \psi_k = 0, \text{ in } \mathbb{M}'^{\alpha}_{\beta}(R_+) \subset S'.$$

But,

$$T_n = [\{(f_{i,n}, \varphi_i)\}_I] = [\{(f_{i,n} * \psi_k, \varphi_k * \psi_k)\}_{I\times K}], \tag{43}$$

for all n = 0,1,2,... . Thus, we have $T_n \to 0$ in β_J. This proves the theorem.

3. Conclusions

The present paper focused on the exchange property for the Mehler-Fock transform via tempered Boehmians which is the natural extension of tempered distributions. Algebraic properties and convergence proved for this relation are useful in this area for development of the convolution properties and other operations of Mehler-Fock transform of distributions and Boehmians [Pathak *et al*. (2016)]. The formula and the property established in this paper may also be suitable for an ultraBoehmians. The aforesaid analysis can be used to develop the Calderon’s formula for Mehler-Fock transform [Pathak *et al*. (2016)].

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