Some Results on f-Simultaneous Chebyshev Approximation

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Abstract

Let $X$ be a Hausdorff topological vector space and $f$ be a real valued continuous function on $X$. In this paper we introduce and study the concept of $f$–simultaneous approximation of a nonempty subset $K$ of $X$ as a generalization to the problem of simultaneous approximation. Further we present some results regarding $f$–simultaneous approximation in the quotient space.

Keywords: Hausdorff topological vector space; $f$-best simultaneous approximation; $f$-simultaneous Chebyshev; simultaneous approximation; quotient space

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1. Introduction

Let $K$ be a subset of a Hausdorff topological vector space $X$ and $f$ be a real valued continuous function on $X$. For $x \in X$, set $F_K(x) = \inf_{k \in K} f(x - k)$. A point $k_0 \in K$ is called $f$–best approximation to $x$ in $K$ if $F_K(x) = f(x - k_0)$. The set $P_K^f(x) = \{k_0 \in K : F_K(x) = f(x - k_0)\}$ denotes the set of all $f$–best approximations to $x$ in $K$. Note that this set may be empty. The set $K$ is said to be $f$–proximinal ($f$–Chebyshev) if for each $x \in X$, $P_K^f(x)$ is non-empty (singleton). The notion of $f$–best approximation in a vector space $X$ was given by Breckner and Brosowski and in a Hausdorff topological vector space $X$ by Narang. For a Hausdorff locally convex topological vector space and a continuous sublinear functional $f$ on $X$, 339
Breckner, Brosowski, and Govindarajulu proved certain results on best approximation relative to the functional \( f \). By using the existence of elements of \( f \)-best approximation some results on fixed point were proved by Pai and Veermani.

As a generalization to the problem of simultaneous approximation (see Saidi and Singer), we introduce the concept of best \( f \)-simultaneous approximation as follows:

**Definition 1.**

Let \( f \) be a real valued continuous function on a Hausdorff topological vector space \( X \). A subset \( A \) of \( X \) is called \( f \)-bounded if there exists \( M > 0 \) such that \( |f(x)| \leq M \) every \( x \in A \).

Note that \( f \)-bounded sets need not be bounded in the classical sense, for example if \( f(x) = e^{-x} \), the set \([0, \infty)\) is an \( f \)-bounded subset of real numbers.

**Definition 2.**

Let \( X \) be a Hausdorff topological real vector space, \( f \) be a real valued continuous function on \( X \), and \( K \) be a non-empty subset of \( X \). A point \( k_0 \in K \) is called \( f \)-best simultaneous approximation in \( K \) if there exists an \( f \)-bounded subset \( A \) of \( X \) such that

\[
F_K(A) = \inf_{k \in K} \sup_{a \in A} |f(a - k)| = \sup_{a \in A} |f(a - k_0)|.
\]

The set of all \( f \)-best simultaneous approximations to an \( f \)-bounded subset \( A \) of \( X \) in \( K \) is denoted by

\[
P^f_K(A) = \left\{ k \in K : F_K(A) = \sup_{a \in A} |f(a - k)| \right\}.
\]

The set \( K \) is called \( f \)-simultaneously proximinal (\( f \)-simultaneously Chebyshev) if for each \( f \)-bounded set \( A \) in \( X \), \( P^f_K(A) \neq \emptyset \) (singleton).

We note that if \( f(x) = \|x\| \) (\( f(x) = \|x\| + \epsilon \)), then the concept of \( f \)-best approximation is precisely best approximation, i.e. best \( \epsilon \)-approximation (see Khalil, Rezapour, Singer and others).

A set \( K \) is said to be inf-compact at a point \( x \in X \), (see Pai and Veermani), if each minimizing sequence in \( K \) (i.e. \( f(x - k_n) \to F_K(x) \)) has a convergent subsequence in \( K \). The set \( K \) is called inf-compact if it is inf-compact at each \( x \in X \). A subset \( K \) of \( X \) is called \( f \)-compact, (see Moghaddam), if for every sequence \( \{k_n\} \) in \( K \), there exist a subsequence \( \{k_{n_i}\} \) of \( \{k_n\} \) and \( k_0 \in K \) such that \( f(k_{n_i} - k_0) \to 0 \). It is easy to see that if \( K \) is \( f \)-compact or inf-compact, then \( K \) is \( f \)-simultaneously proximinal.

In this paper we introduce and study the concept of \( f \)-simultaneous approximation of a subspace \( K \) of a Hausdorff topological real vector space \( X \), and existence and uniqueness. Certain results regarding \( f \)-simultaneous approximation in quotient spaces is obtained by generalizing some of the results in Moghaddam.

Throughout this paper \( X \) is a Hausdorff topological real vector space and \( f \) is a real valued continuous function on \( X \).
2. \( f \)-Simultaneous Approximation

In this section we give some characterization of \( f \)-proximinal sets in \( X \). We begin with the following definitions:

**Definition 3.**

A function \( f : X \to \mathbb{R} \) is called

1. absolutely subadditive if \( |f(x + y)| \leq |f(x)| + |f(y)| \) for all \( x, y \in X \).
2. absolutely homogeneous if \( f(\alpha x) = |\alpha| f(x) \), for all \( x \in X \) and all \( \alpha \in \mathbb{R} \).

**Definition 4.**

A subset \( K \) of \( X \) is called \( f \)-closed if for all sequences \( \{k_m\} \) of \( K \) and for all \( x \in X \) such that \( f(x - k_m) \to 0 \), we have \( x \in K \).

**Theorem 1.**

Let \( K \) be a subset of \( X \). Then,

1. \( F_{K+y}(A + y) = F_K(A) \), for all \( f \)-bounded sets \( A \subset X \), \( y \in X \).
2. \( P_{K+y}^f(A + y) = P_K^f(A) + y \), for all \( f \)-bounded sets \( A \subset X \), \( y \in X \).
3. \( K \) is \( f \)-simultaneously proximinal (\( f \)-simultaneously Chebyshev) if and only if \( K + y \) is \( f \)-simultaneously proximinal (\( f \)-simultaneously Chebyshev) for every \( y \in X \).

Moreover if \( f \) is an absolutely homogeneous function, then

4. \( F_{\lambda K}(\lambda A) = |\lambda| F_K(A) \), for all \( f \)-bounded sets \( A \subset X \) and \( \lambda \in \mathbb{R} \).
5. \( P_{\lambda K}^f(\lambda A) = \lambda P_K^f(A) \), for all \( f \)-bounded sets \( A \subset X \) and \( \lambda \in \mathbb{R} \).
6. \( K \) is \( f \)-simultaneously proximinal (\( f \)-simultaneously Chebyshev) if and only if \( \lambda K \) is \( f \)-simultaneously proximinal (\( f \)-simultaneously Chebyshev), \( \lambda \in \mathbb{R} \).

**Proof:**

1. Let \( A \subset X \), \( f \)-bounded set. Then
   \[ F_{K+y}(A + y) = \inf_{w \in K} \sup_{a \in A} |f((a + y) - (w + y))| = F_K(A). \]

2. The equation
   \[ \sup_{a \in A} |f(a - k_0)| = \inf_{k \in K} \sup_{a \in A} |f((a + y) - (k + y))| = \inf_{k \in K} \sup_{a \in A} |f(a - k)|, \]
   implies that \( k_0 + y \in P_{K+y}^f(A + y) \) if and only if \( k_0 \in P_K^f(A) \). Thus
   \[ P_{K+y}^f(A + y) = P_K^f(A) + y. \]

3. This follows immediately from part two.
4. Let \( A \subset X \) be an \( f \)-bounded set, \( \lambda \in \mathbb{R} \). Then
   \[ F_{\lambda K}(\lambda A) = \inf_{k \in K} \sup_{a \in A} |f(\lambda a - \lambda k)| = |\lambda| \inf_{k \in K} \sup_{a \in A} |f(a - k)| = |\lambda| F_K(A). \]
If \( \lambda = 0 \), we are done. If \( \lambda \neq 0 \) and \( k_0 \in P_{\lambda K}(\lambda A) \), then \( k_0 \in \lambda K \) and
\[
\sup_{a \in A} |f(\lambda a - k_0)| = \inf_{k \in K} \sup_{a \in A} |f(\lambda a - \lambda k)|.
\]
This implies that
\[
\sup_{a \in A} \left| f(a - \frac{1}{\lambda} k_0) \right| = F_K(A),
\]
which implies that \( \frac{1}{\lambda} k_0 \in P_{\lambda K}^f(A) \).

(6) This follows immediately from part 5. \( \square \)

**Theorem 2.**

Let \( f \) be an absolutely homogeneous real valued function on \( X \) and \( M \) be a subspace of \( X \). Then,

1. \( F_M(\lambda A) = |\lambda| F_M(A) \), for all \( f \)-bounded sets \( A \subset X \) and \( \lambda \in \mathbb{R} - \{0\} \).
2. \( P_M^f(\lambda A) = \lambda P_M^f(A) \), for all \( f \)-bounded sets \( A \subset X \) and \( \lambda \in \mathbb{R} - \{0\} \).

**Proof:**

(1) Let \( A \subset X \) be an \( f \)-bounded set and \( \lambda \neq 0 \in \mathbb{R} \). Then,
\[
F_M(\lambda A) = \inf_{m \in M} \sup_{a \in A} |f(\lambda a - m)| = |\lambda| \inf_{m' \in M} \sup_{a \in A} \left| f(a - m') \right| = |\lambda| F_M(A).
\]

(2) Let \( m_0 \in P_M^f(\lambda A) \). Then,
\[
\sup_{a \in A} \left| \lambda \left| f(a - \frac{1}{\lambda} m_0) \right| \right| = \sup_{a \in A} \left| f(\lambda a - m_0) \right|
= \inf_{m \in M} \sup_{a \in A} |f(\lambda a - m)|
= \inf_{m' \in M} \sup_{a \in A} |\lambda| \left| f(a - m') \right|.
\]
Therefore,
\[
\sup_{a \in A} \left| f(a - \frac{1}{\lambda} m_0) \right| = \inf_{m' \in M} \sup_{a \in A} \left| f(a - m') \right| = F_M(A),
\]
for all \( \lambda \in \mathbb{R} - \{0\} \), which implies that \( \frac{1}{\lambda} m_0 \in P_M^f(A) \), and so \( m_0 \in \lambda P_M^f(A) \). \( \square \)

For a subset \( K \) of \( X \), let us define \( \hat{K}_F \) such that
\[
\hat{K}_F = \left\{ A \subset X : F_K(A) = \sup_{a \in A} f(a) \right\}.
\]
Using this we prove the following theorem characterizing \( f \)-simultaneously proximinal sets.

**Theorem 3.**

Let \( K \) be a subspace of \( X \). Then \( K \) is \( f \)-simultaneously proximinal in \( X \) if and only if every \( f \)-bounded subset \( A \) of \( X \) can be written as \( B + k \) for some \( k \in K \) and \( B \in \hat{K}_F \).
Proof:
Suppose the condition hold. Let $A \subset X$ be an $f-$bounded subset of $X$. By assumption there exists $k_0 \in K$ and $B \in \widehat{K}_F$ such that $A = B + k_0$. Hence $A - k_0 \in \widehat{K}_F$. Therefore,

$$
\sup_{a \in A} |f(a - k_0)| = F_K(A - k_0)
= \inf_{k \in K} \sup_{a \in A} |f(a - k)|
= \inf_{k' \in K} \sup_{a \in A} |f(a - k'| = F_K(A).
$$

Hence, $K$ is $f-$simultaneously proximinal.

Conversely, suppose $K$ is $f-$simultaneously proximinal and $A \subset X$ be an $f-$bounded subset of $X$. Then there exists $k_0 \in K$ such that

$$
\sup_{a \in A} |f(a - k_0)| = \inf_{k \in K} \sup_{a \in A} |f(a - k)| = \inf_{k' \in K} \sup_{a \in A} |f(a - (k' + k_0))|
$$

where $k = k' + k_0$. Hence,

$$
\sup_{a \in A} |f(a - k_0)| = F_K(A - k_0).
$$

Consequently, $A - k_0 \in \widehat{K}_F$. So there exists $B \in \widehat{K}_F$ such that $A - k_0 = B$ or $A = B + k_0$. □

Theorem 4.
Let $f$ be a real valued continuous function on $X$ such that $x = 0$ if and only if $f(x) = 0$. If $K$ is $f-$simultaneously proximinal, then $K$ is $f-$closed.

Proof:
Let $\{k_m\}$ be a sequence of $K$ and $x \in X$, such that $f(x - k_m) \to 0$. Taking $A = \{x\}$, we have

$$
F_K(A) = \inf_{k \in K} \sup_{a \in A} |f(a - k)| \leq |f(x - k_m)| \to 0.
$$

Since $K$ is $f-$simultaneously proximinal, there exists $k_0 \in K$ such that

$$
F_K(A) = |f(x - k_0)| = 0.
$$

Hence, $f(x - k_0) = 0$. Using assumption it follows that $x - k_0 = 0$. Therefore, $x = k_0 \in K$ and $K$ is $f-$closed. □

3. $f-$Simultaneous Approximation in Quotient Space

Let $M$ be a closed subspace of $X$. Then a function $\tilde{f} : (X/M) \to \mathbb{R}$ can be defined as follows:

$$
\tilde{f}(x + M) = \inf_{y \in M} |f(x + y)|.
$$

Proposition 1.
Let $M$ be a closed subspace of $X$. If $A$ is $f-$bounded in $X$, then $A/M$ is $\tilde{f}-$bounded in $X/M$. 
Proof:
Let $A$ be an $f$–bounded subset in $X$. Since $M$ is a subspace, for $x + M \in A/M$
$$\left| \tilde{f}(x + M) \right| = \inf_{y \in M} |f(x + y)| \leq |f(x)|.$$ Consequently since $A$ is an $f$–bounded subset of $X$, it follows that $A/M$ is $\tilde{f}$–bounded in $X/M$. □

Theorem 5.
Let $M$ a closed subspace of $X$. If $B$ is $\tilde{f}$–bounded in $X/M$, then there exists an $f$–bounded subset $A$ of $X$ such that $B = A/M$.

Proof:
Let $B$ be a nonempty $\tilde{f}$–bounded in $X/M$. Let $C = \bigcup_{b \in B} b$. Claim: $B = \{ \overline{x} = x + M : x \in C \}$. Indeed if $b \in B$, then $b = x_b + M$ for some $x_b \in X$. But $M$ is a subspace. Thus $x_b = x_b + 0 \in x_b + M \subseteq C$. Hence $b = x_b + M \in \{ \overline{x} = x + M : x \in C \}$ and $B \subseteq \{ \overline{x} = x + M : x \in C \}$. Similarly if $x \in C$, then $x \in b_x + M$ for some $b_x + M \in B$. This implies that $x = b_x + m_x$ for some $m_x \in M$. Hence $x + M = b_x + m_x + M = b_x + M \in B$. Therefore, $\{ \overline{x} = x + F : x \in C \} \subseteq B$.

Now clearly $C$ is not bounded unless $M$ is trivial. Note that $B$ is $\tilde{f}$–bounded. So there exists $K > 0$ such that $|\tilde{f}(b)| \leq K$ for all $b \in B$. Consider the set $A = \{ x \in C : |f(x)| \leq K + 1 \} \subseteq C$.

Now we claim that for all $x \in C$,
$$\overline{x} \cap A = (x + M) \cap A \neq \phi.$$ Given $x \in C$. Since
$$\left| \tilde{f}(x + M) \right| = \inf_{m \in M} |f(x + m)| \leq K,$$
there exists $m_x \in M$ such that $|f(x + m_x)| < K + 1. But x + m_x \in x + M \subseteq C$. Hence $x + m_x \in (x + M) \cap A \neq \phi$. Claim $B = A/M$. Since $A \subseteq C$, we have $A/F \subseteq \{ \overline{x} = x + F : x \in C \} = B$. To show the other inclusion, let $b \in B = \{ \overline{x} = x + M : x \in C \}$. Then $b = x_b + M$ for some $x_b \in C$. But $(x_b + M) \cap A \neq \phi$. Thus there exists $a \in A$ such that $a = x_b + m_a \in x_b + M$. Therefore, $b = x_b + M = (x_b + m_a) + M = a + M \in A/M$. Hence $B \subseteq A/M$. Consequently $A/M = B$. □

Theorem 6.
Let $K$ be a subspace of $X$ and $M$ be a closed $f$–proximinal subspace of $K$. If $k_0$ is a point of $f$–best simultaneous approximation to $A \subseteq X$ in $K$, then $k_0 + M$ is an $\tilde{f}$–best simultaneous approximation to $A/M$ in $K/M$. 

344 Sh. Al-Sharif and Kh. Qaraman
**Proof:**

Suppose $k_0 + M$ is not $\tilde{f}$—best simultaneous approximation to $A/M$ in $K/M$. Then, for at least $k \in K$, say $k_1 \in K$, we have

$$\sup_{a \in A} \tilde{f}(a - k + M) < \sup_{a \in A} \tilde{f}(a - k_0 + M).$$

Since

$$\sup_{a \in A} \tilde{f}(a - k_0 + M) = \sup \inf_{a \in A} |f(a - k_0 + m)| \leq \sup_{a \in A} |f(a - k_0)|,$$

we have

$$\sup_{a \in A} \tilde{f}(a - k_1 + M) = \sup \inf_{a \in A} |f(a - k_1 + m)| < \sup_{a \in A} |f(a - k_0)|.$$

But $M$ is $f$—proximinal, so for some $m_0 \in M$ we have

$$\sup_{a \in A} |f(a - k_1 + m_0)| = \sup \inf_{a \in A} |f(a - k_1 + m)| < \sup_{a \in A} |f(a - k_0)|.$$

Since $M \subset K$, it follows that $k_1 - m_0 \in K$. Therefore, $k_0$ not $f$—best simultaneous approximation to $A$ in $K$, which is a contradiction. □

**Corollary 1.**

Let $K$ be a subspace of $X$ and $M$ is a closed $f$—proximinal subspace of $K$. If $K$ is $f$—simultaneously proximinal in $X$, then $K/M$ is $\tilde{f}$—simultaneously proximinal in $X/M$.

**Proof:**

Let $B$ be an $\tilde{f}$—bounded subset of $X/M$. Then, by Theorem 5, there exists $f$—bounded subset $A \subset X$ such that $B = A/M$. If $K$ is $f$—simultaneously proximinal in $X$, then there exists at least $k_0 \in K$ such that $k_0$ is $f$—best simultaneous approximation to $A$ in $K$. By Theorem 6, $k_0 + M$ is an $\tilde{f}$—best simultaneous approximation to $A/M$ in $K/M$, so $K/M$ is $\tilde{f}$—simultaneously proximinal in $X/M$. □

**Theorem 7.**

Let $K$ be a subspace of $X$ and $M$ is a closed $f$—proximinal subspace of $K$. If $K/M$ is $\tilde{f}$—simultaneously proximinal in $X/M$, then $K$ is $f$—simultaneously proximinal in $X$.

**Proof:**

Let $A$ be an $f$— bounded subset of $X$. By Proposition 1, $A/M$ is $\tilde{f}$—bounded in $X/M$. Since $K/M$ is $\tilde{f}$—simultaneously proximinal in $X/M$, then there exists $k_0 + M \in K/M$ such that $k_0 + M$ is $\tilde{f}$—best simultaneous approximation to $A/M$ from $K/M$, so

$$\sup_{a \in A} \tilde{f}(a - k_0 + M) = \inf_{k \in K} \sup_{a \in A} \tilde{f}(a - k + M)$$

$$= \inf_{k \in K} \sup_{a \in A} |f(a - k + M)|$$

$$\leq \inf_{k \in K} \sup_{a \in A} |f(a - k + m)|$$

$$= \inf_{k \in K} \sup_{a \in A} |f(a - k')|,$$

\(1\)
where, \( k' = k - m \in K \). Since \( M \) is \( f \)-proximinal, there exists \( m_0 \in M \) such that
\[
\sup_{a \in A} |f(a - k_0 - m_0)| = \sup_{a \in A} \inf_{m \in M} |f(a - k_0 + m)| = \sup_{a \in A} f(a - k_0 + M).
\] (2)

Consequently, combining (1) and (2) since \( M \subset K \), it follows that
\[
\sup_{a \in A} |f(a - k_0 - m_0)| \leq \inf_{k' \in K} \sup_{a \in A} |f(a - k')| \\
\leq \sup_{a \in A} |f(a - k_0 - m_0)|
\]

Hence,
\[
\sup_{a \in A} |f(a - k_0 + m_0)| = \inf_{k' \in K} \sup_{a \in A} |f(a - k')|
\]

So \( k_0 + m_0 \) is an \( f \)-best simultaneous approximation to \( A \) from \( K \) and \( K \) is \( f \)-simultaneously proximinal in \( X \). \( \square \)

**Theorem 8.**

Let \( W \) and \( M \) be two subspaces of \( X \). If \( M \) is a closed \( f \)-proximinal subspace of \( X \), then the following assertions are equivalent:
1. \( W/M \) is \( \bar{f} \)-simultaneously proximinal in \( X/M \),
2. \( W + M \) is \( f \)-simultaneously proximinal in \( X \).

**Proof:**

(1) \( \Rightarrow \) (2). Since \( (W + M)/M = W/M \) and \( M \) are \( f \)-simultaneously proximinal, using Theorem 7, it follows that \( W + M \) is \( f \)-simultaneously proximinal in \( X \).

(2) \( \Rightarrow \) (1). Since \( W + M \) is \( f \)-simultaneously proximinal and \( M \subseteq W + M \), by Corollary 1, \( (W + M)/M = W/M \) is simultaneously \( f \)-proximinal. \( \square \)

**Theorem 9.**

Let \( K, M \) be two subspaces of \( X \) such that, \( M \subset K \). If \( M \) is closed \( f \)-simultaneously proximinal in \( X \) and \( K \) is \( f \)-simultaneously Chebyshev in \( X \), then, \( K/M \) is \( f \)-simultaneously Chebyshev in \( X/M \).

**Proof:**

Suppose not. Then there exists \( A, f \)-bounded subset of \( X \) such that \( A/M \in X/M \) is \( \bar{f} \)-bounded and \( k_1 + M, k_2 + M \in P_{K/M}^f(A/M) \) such that \( k_1 + M \neq k_2 + M \). Thus \( k_1 - k_2 \notin M \). Since \( M \) is an \( f \)-simultaneously proximinal in \( X \), then
\[
P_{M}^f(A - k_1) \neq \phi \quad \text{and} \quad P_{M}^f(A - k_2) \neq \phi.
\]

Let \( m_1 \in P_{M}^f(A - k_1) \), and \( m_2 \in P_{M}^f(A - k_2) \). By Theorem 7, \( k_1 + m_1 \) and \( k_2 + m_2 \) are \( f \)-best simultaneous approximations to \( A \) from \( K \). Since \( K \) is \( f \)-simultaneously Chebyshev in \( X \), then \( k_1 + m_1 = k_2 + m_2 \) and hence \( k_1 - k_2 = m_1 - m_2 \in M \), which is a contradiction. \( \square \)
Definition 5.

A subset $K$ of $X$ is called $f$–quasi-simultaneously Chebyshev if $P^f_K(A)$ is nonempty and $f$-compact set in $X$ for all $f$–bounded subsets of $X$.

Theorem 10.

Let $M$ be a closed $f$–simultaneously proximinal subspace of $X$ and $K$ is $f$–quasi-simultaneously Chebyshev of $X$ such that $M \subset K$. Then, $K/M$ is $\tilde{f}$–quasi-simultaneously Chebyshev in $X/M$.

Proof:

Since $K$ is $f$–simultaneously proximinal in $X$, By Corollary 1, $K/M$ is $\tilde{f}$–simultaneously proximinal in $X/M$. Let $B$ be an $\tilde{f}$–bounded subset of $X/M$. Then, by Theorem 5, $B = A/M$ for an $f$–bounded subset $A$ of $X$. If $(k_n + M)$ a sequence in $P^f_{K/M}(A/M)$, by the proof of Theorem 7, for every $n$, there exists $m_n \in M$ such that $k_n + m_n = k'_n \in P^f_K(A)$. But since $M$ is a subspace, we have

$$k'_n + M = k_n + m_n + M = k_n + M.$$ 

Since $K$ is $f$–quasi-simultaneously Chebyshev in $X$, the sequence $\{k_n\}$ has a subsequence $\{k_{n_i}\}$ such that $f(k_{n_i} - k_0) \to 0$ for some $k_0 \in P^f_K(A)$. But

$$\tilde{f}(k_{n_i} - k_0 + M) \leq |f(k_{n_i} - k_0)| \to 0.$$ 

Therefore,

$$\tilde{f}(k_{n_i} - k_0 + M) \to 0$$

and

$$\tilde{f}((k_{n_i} + M) - (k_0 + M)) \to 0.$$ 

Hence, $P^f_{K/M}(A/M)$ is $\tilde{f}$-compact and $K/M$ is $\tilde{f}$–quasi-simultaneously Chebyshev. This complete the proof. □

Definition 6.

A topological vector space $X$ is said to have the $f$– property if every $f$–bounded sequence in $X$ has an $f$– convergent subsequence, where $f$ is a real valued continuous function on $X$.

Note that the space $X = l^2$ has the $f$–property for every projection $f : X \to \mathbb{R}$, and if $f(x) = \|x\|$, then every finite dimensional Banach space has the $f$–property.

Proposition 2.

Let $f$ be an absolutely homogeneous subadditive continuous real valued function on a topological vector space $X$ and $K$ be an $f$–closed subspace of $X$. Then, for any $f$–bounded subset $A$ of $X$, $P^f_K(A)$ is $f$–closed.

Proof:

Let $K$ be an $f$–closed subspace of $X$ and $A$ be an $f$–bounded subset of $X$. If $\{k_m\}$ is a sequence in $P^f_K(A)$ and $x \in X$ such that $f(k_m - x) \to 0$, then $x \in K$ since $K$ is $f$–closed.
Further,
\[
\inf_{k \in K} \sup_{a \in A} |f(a - k)| = \sup_{a \in A} |f(a - k_m)| = \sup_{a \in A} |f((a - x) - (k_m - x))| \geq \sup_{a \in A} |f(a - x) - f(k_m - x)|.
\]

Taking the limit as \(m \to \infty\), we get
\[
\inf_{k \in K} \sup_{a \in A} |f(a - k)| \geq \sup_{a \in A} |f(a - x)|.
\]

Consequently,
\[
\inf_{k \in K} \sup_{a \in A} |f(a - k)| = \sup_{a \in A} |f(a - x)|.
\]

Then, \(x \in P^f_K(A)\) and \(P^f_K(A)\) is \(f\)-closed. \(\square\)

**Theorem 11.**

Let \(f\) be a real valued sub-additive continuous function on a topological vector space \(X\) that has the \(f\)-property and \(M\) be a closed subspace of \(X\). If \(W\) is a subspace of \(X\) such that \(W + M\) is \(f\)-closed, then the following assertions are equivalent:

1. \(W/M\) is \(\tilde{f}\)–simultaneously quasi-Chebyshev in \(X/M\).
2. \(W + M\) is \(f\)–simultaneously quasi-Chebyshev in \(X\).

**Proof:**

(1) \(\Rightarrow\) (2) Since \(M\) is \(f\)–simultaneously proximinal by Theorem 8, \(W + M\) is \(f\)–simultaneously proximinal in \(X\). Let \(A\) be an arbitrary \(f\)–bounded set in \(X\). Then \(P^f_{W+M}(A) \neq \emptyset\). Now to show that \(P^f_{W+M}(A)\) is \(f\)–compact, we need to show that every sequence in \(P^f_{W+M}(A)\) has an \(f\)–convergent subsequence. Let \(\{g_n\}_{n=1}^{\infty}\) be an arbitrary sequence in \(P^f_{W+M}(A)\). Then by Theorem 6, for each \(n > 1\), \(g_n + M \in P^f_{(W+M)/M}(A/M)\). Since \(P^f_{(W+M)/M}(A/M)\) is \(\tilde{f}\)–compact, one can choose \(g_0 \in W + M\) with \(g_0 + M \in P^f_{(W+M)/M}(A/M)\) and \(\{g_{n_k} + M\}_{k=1}^{\infty}\) is \(\tilde{f}\)–convergent to \(g_0 + M\) for some subsequence \(\{g_{n_k} + M\}_{k=1}^{\infty}\) of \(\{g_n + M\}_{n=1}^{\infty}\). That means,
\[
\tilde{f}(g_0 - g_{n_k} + M) = \inf_{m \in M} |f(g_0 - g_{n_k} - m)| \to 0.
\]

Now, since \(M\) is \(f\)–proximinal in \(X\), there exists \(m_{n_k} \in M\) such that \(m_{n_k} \in P^f_M(g_0 - g_{n_k})\), for every \(k \geq 1\), and hence
\[
|f(g_0 - g_{n_k} - m_{n_k})| = \inf_{m \in M} |f(g_0 - g_{n_k} - m)|.
\]

Therefore,
\[
\lim_{k \to \infty} f(g_0 - g_{n_k} - m_{n_k}) = 0.
\]

On the other hand, \(\{g_{n_k}\}_{k=1}^{\infty}\) is an \(f\)–bounded sequence because \(g_n \in P^f_{W+M}(A)\). In fact \(|f(g_n)| \leq 2\sup_{a \in A} |f(a)|\). Since \(M\) has the \(f\)-property, with out loss of generality, we may assume
that for some \( m_0 \in M \), \( f (m_{nk} - m_0) \rightarrow 0 \). Let \( g' = g_0 - m_0 \). Then, \( g' \in W + M \) and
\[
f (g' - g_{nk}) = f (g_0 - m_0 - g_{nk}) \\
\leq f (g_0 - g_{nk} - m_{nk}) + f (m_{nk} - m_0),
\]
\( \forall \ k \geq 1 \). Thus, \( \lim_{k \to \infty} f (g' - g_{nk}) = 0 \). Since \( \{g_{nk}\}_{k=1}^{\infty} \in P_{W+M}^f (A) \), for every \( k \geq 1 \), and \( P_{W+M}^f (A) \) is \( f \)-closed, since \( W + M \) is \( f \)-closed by Proposition 19, we conclude that \( g' \in P_{W+M}^f (A) \). Hence, \( P_{W+M}^f (A) \) is \( f \)-compact.

\[(2) \Rightarrow (1) \] Since \( M \) and \( W + M \) are \( f \)-simultaneously proximinal and \( M \subseteq W + M \), then \( (W + M)/M = W/M \) is \( \tilde{f} \)-simultaneously proximinal in \( X/M \).

Now, let \( A \) be an arbitrary \( f \)-bounded set in \( X \). Then, \( P_{W/M}^f (A/M) \) is non-empty. So from the hypothesis we have \( W + M \) is \( f \)-simultaneously quasi-Chebyshev in \( X \), and hence \( P_{W+M}^f (A) \) is \( f \)-compact in \( X \). Using Theorem 6, we conclude that
\[
P_{(W+M)/M}^f (A/M) = \pi \left( P_{W+M}^f (A) \right),
\]
where \( \pi : X \to X/M, \pi(x) = x + M \), is continuous. Consequently \( P_{W/M}^f (A/M) \) is \( \tilde{f} \)-compact. Therefore, \( W/M \) is \( f \)-simultaneously quasi-Chebyshev in \( X \). \( \square \)

Note that Theorem 11 is still true if the restriction \( W + M \) is \( f \)-closed is replaced by the condition that the function \( f(x) = 0 \) if and only if \( x = 0 \) and use Theorem 4 to prove that \( W + M \) is \( f \)-closed.

4. Conclusions

In this paper we introduce and study the concept of \( f \)-simultaneous approximation of a nonempty subset \( K \) of Hausdorff topological vector space \( X \), existence and uniqueness as a generalization to the problem of simultaneous approximation in the sense that if the function \( f \) is taken to be the usual norm, the problem is turned out to be precisely the problem of best approximation in the usual sense. Further, we obtain some results regarding \( f \)-simultaneous approximation in the quotient space.

REFERENCES


