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A Numerical Scheme for Generalized Fractional Optimal Control Problems

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Abstract

This paper introduces a generalization of the Fractional Optimal Control Problem (GFOCP). Proposed generalizations differ in terms of explaining the constraint involved in the dynamical system of the control problem. We assume the constraint as an arbitrary function of fractional derivatives and fractional integrals. By this assumption the restriction on constraint, to be of some prescribed function of fractional operators, is removed. Deduction of necessary optimality conditions followed by particular cases and examples has been provided. Additionally, we construct a solution scheme for the suggested class of (GFOCP)'s. The formulation of this scheme is done by implementing the Adomian decomposition method on necessary optimality conditions. An example is presented to demonstrate the application of solution scheme. Fractional operators used throughout the paper are either Riemann-Liouville or Caputo's fractional operators.

Keywords: Adomian decomposition method; Fractional derivative/integrals; Fractional optimal control problems

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1. Introduction

Fractional calculus is an extension of classical calculus, where the order of derivative (or integral) is not restricted to set of integers. This admissible innovation from an integer to a non-integer leads to boundless applications in various areas like viscoelasticity, control problems, image and signal processing, and others. The idea of the fractional derivative originated in 1695 during a conversation between L'Hospital and Leibniz about $\frac{d^n x}{dy^n}$ for $n = \frac{1}{2}$. Their primarily discussion has been described as the foundation of fractional calculus (see Oldham and Spanier (1974)).

Fractional optimal control problem (FOCP) refers to an optimal control problem in which the objective function or the differential equations governing the constraints are comprised of fractional operators. The existence of (FOCP) was given by Bhatt in (1973). Early work in the area of fractional optimal control is acceptably documented (see Manabe (2003)).

Let us consider the problem to find the optimal control that minimizes the objective function,

$$J[u] = \int_0^1 F(t, x(t), u(t))dt,$$

subject to the constraint

$${}_0^c D_t^\alpha x = G(t, x, u),$$

and the initial condition

$$x(0) = x_0,$$

where $x(t)$ is the state variable, t represents the time, and F and G are two arbitrary functions. One may observe that the constraint involves the fractional derivative of x denoted by ${}_0^c D_t^\alpha x$. This is a well known problem in fractional optimal control (see Agrawal (2004)).

In Agrawal (1989), a general formulation for the numerical solution of optimal control problem is presented. Earlier, solving differential equations containing fractional order derivative had not been studied in the literature. His work opened the area to do further research in finding a solution of Euler-Lagrange equations occurring in solving (FOCP)'s. During (2001-2008) Agrawal has given formulation of Euler-Lagrange equations containing fractional derivative in Riemann-Liouvile and Caputo sense (see Agrawal (2002), Agrawal (2004)). Moreover, Agrawal and Baleanu have formulated different types of solution schemes to solve (FOCP)'s as given in Agrawal (2007), Agrawal (2008), and Baleanu (2007).

This paper introduces the class of generalized fractional optimal control problem (GFOCP) where the constraint is an arbitrary function of fractional derivative and fractional integral of the state variable. The proposed generalization has advantages compared to previously defined problems of fractional optimal control. For instance, on giving different functions to the constraint, all possible previous classes of optimal control problems are covered.

We prove the necessary conditions for the class of (GFOCP)'s in Section 3. In order to obtain other classes (already existing in the literature), sufficient numbers of remarks and examples are given. In Section 4, we formulate a solution scheme for (GFOCP)'s by employing the Adomian Decomposition Method (ADM). The purpose is to implement (ADM) and construct a generalized solution scheme for Euler-Lagrange equations occurring in (GFOCP)'s. The method presented yields a solution which is free from rounding-off errors since it does not involve discretization and is computationally inexpensive. As a special case, this formulation is used to solve the control equations for a quadratic linear fractional optimal control problem.

2. Preliminaries

We shall now give some fundamental definitions of fractional order operators together with a brief description of the Adomian decomposition method used all through the paper.

2.1. Fractional Derivatives/Integrals

Recently, several definitions of the fractional derivative are available in the literature. Some of the famous and widely adopted fractional order derivatives are Riemann-Liouville, Grunwald-Letnikov, Weyl, Caputo, and Riesz fractional derivatives (see Miller (1993), Oldham (1974), Podlubny (1999), Butzer (2000)). For definitions and properties of fractional derivatives/integrals, we refer to Ross (1973). These are essential to carry out the computations in present work.

Let $f \in C[a,b]$, where $C[a,b]$ is the space of all continuous functions defined over $[a,b]$.

Definition 1.

For all $t \in [a,b]$ and $\alpha > 0$, the Left Riemann-Liouville Fractional Integral (LRLFI) of order α is defined as

$$ {}_aI_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad t > a. $$

Definition 2.

For all $t \in [a,b]$ and $\alpha > 0$, the Right Riemann-Liouville Fractional Integral (RRLFI) of order α is defined as

$$ {}_tI_b^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau) d\tau, \quad t < b. $$

Let us consider $f \in C^n[a,b]$, where $C^n[a,b]$ is the space of n times continuously differentiable functions defined over $[a,b]$.

Definition 3.

For all $t \in [a,b]$, $n-1 \leq \alpha < n$, the Left Riemann-Liouville Fractional Derivative (LRLFD) of order α is defined as

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau. $$

Definition 4.

For all $t \in [a,b]$, $n-1 \leq \alpha < n$, the Right Riemann-Liouville Fractional Derivative (RRLFD) of order α is defined as

$$ {}_tD_b^\alpha f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_t^b (\tau-t)^{n-\alpha-1} f(\tau) d\tau. $$

Definition 5.

For all $t \in [a,b]$, $n-1 \leq \alpha < n$, the Left Caputo Fractional Derivative (LCFD) of order α is defined as

$$ {}_a^cD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^n(\tau) d\tau. $$

Definition 6.

For all $t \in [a,b]$, $n-1 \leq \alpha < n$, the Right Caputo Fractional Derivative (RCFD) of order α is defined as

$$ {}_t^cD_b^\alpha f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (\tau-t)^{n-\alpha-1} f^n(\tau) d\tau. $$

Definition 7.

If f, g and the fractional derivatives ${}_aD_t^\alpha g$ and ${}_tD_b^\alpha f$ are continuous at every point $t \in [a,b]$, then

$$ \int_a^b f(t)\, {}_aD_t^\alpha g\, dt = \int_a^b g(t)\, {}_tD_b^\alpha f\, dt, $$

for any $0 < \alpha < 1$.

Definition 8.

Let $f \in C^k[a,b]$. Further

$$ \int_a^b f(x)\, h(x)\, dx = 0, $$

for every function $h \in C^k[a,b]$ with $h(a) = 0 = h(b)$. Then the fundamental lemma of the calculus of variations (Gelfand, 1973) states that $f(x)$ is identically zero on $[a,b]$.

We next describe the Adomian decomposition method in short, which is implemented in this paper to construct a generalized solution scheme for (GFOCP)'s.

2.2. Adomian Decomposition Method (ADM)

(ADM) was introduced by G. Adomian in the beginning of 1980s (refer to Adomian (1981), Adomian (1989), Adomian (1990), Adomian (1994)). It has been used to solve functional equations of the form

$$u = f + L(u) + N(u). \tag{1}$$

Here L and N are, respectively, the linear and non-linear part of u. Equation (1) represents a variety of equations such as non-linear ordinary differential equations, partial differential equations, integral equations, fractional differential equations, and system of equations containing linear and non-linear functions.

In (ADM), the solution for the equation (1) is expressed in the form of infinite series

$$u = \sum_{n=0}^{\infty} u_n,$$

with $u_0 = f$. Further, it is assumed that the non-linear term $N(u)$ can be expressed as $\sum_{n=0}^{\infty} A_n$, where $A_n' s$ are referred as Adomian polynomials.

$A_n' s$ are defined by the expression

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} N\left(\sum_{n=0}^{\infty} u_n(x) \lambda^n \right)_{\lambda=0} .$$

Since L is linear we have

$$\sum_{n=0}^{\infty} u_n = f + \sum_{n=0}^{\infty} L(u_n) + \sum_{n=0}^{\infty} A_n \, .$$

The recursive relation for u_n is as follows:

$$\begin{aligned} u_0 &= f, \\ u_1 &= L(u_0) + A_0, \\ u_2 &= L(u_1) + A_1, \\ &. \\ &. \\ &. \\ u_n &= L(u_{n-1}) + A_{n-1}. \end{aligned}$$

The solution u is in form of k-term approximation

$$u = \sum_{n=0}^{k-1} u_n \,,$$

for a suitable integer k.

3. Generalized Fractional Optimal Control Problem (GFOCP)

Fractional optimal control problems have already been studied by many researchers (refer to Agrawal (2004), Baleanu (2007), Agrawal (2007)). The generalized class of (FOCP) is described by considering the constraint as an arbitrary function of fractional derivative and fractional integral of the state variable $x(t)$.

(P): For $0 < \alpha, \beta < 1$, find an optimal control $u(t)$ that minimizes the performance index

$$J[u] = \int_0^1 F(t, x(t), u(t))dt,$$

subject to the constraint

$$H({}_0D_t^{\alpha}x(t)\,,\,{}_0I_t^{1-\beta}x(t)) = G(t, x, u),$$

and initial condition

$$x(0) = x_0.$$

Here $x(t)$ is a state variable at time t. F and G are the arbitrary functions of the state and the control variable $x(t)$ and $u(t)$ respectively. The limits of integration have been taken as 0 and 1 for simplicity of the problem.

Note:

- We allow the function H to be an arbitrary function of ${}_0D_t^\alpha x(t)$ as well as ${}_0I_t^{1-\beta}x(t)$.
- Formerly introduced FOCP's present in literature assumes a prescribed function as a constraint involving fractional derivative. For example, ${}_0D_t^\alpha x(t) = G(t,x,u)$ (here $H = {}_0D_t^\alpha x(t)$).

To find the optimal control $u(t)$ for the problem (P), we define a modified performance index as

$$\bar{J}(u) = \int_0^1 F(x,u,t) + \lambda[G(x,u,t) - H({}_0D_t^\alpha x(t)\,,\,{}_0I_t^{1-\beta}x(t))]\,dt,$$

where λ is a Lagrange multiplier also known as costate or adjoint variable.

Taking variation of the modified index $\bar{J}(u)$, we obtain

$$\begin{aligned}\delta\bar{J}(u) \;=\; &\int_0^1 \left[\frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial u}\delta u + \delta\lambda(G-H)\right.\\ &\left. + \lambda\left(\frac{\partial G}{\partial x}\delta x + \frac{\partial G}{\partial u}\delta u - \frac{\partial H}{\partial({}_0D_t^\alpha x)}\delta({}_0D_t^\alpha x) - \frac{\partial H}{\partial({}_0I_t^{1-\beta}x)}\delta({}_0D_t^\alpha x)\right)\right] dt,\end{aligned}$$

where $\delta x, \delta u$ and $\delta\lambda$ are the variations of x, u and λ, respectively, with the specified terminal conditions.

Using integration by parts

$$\int_0^1 \lambda\,\delta({}_0D_t^\alpha x)\,dt = \int_0^1 \delta x\,({}_tD_1^\alpha\lambda)\,dt,$$

and

$$\int_0^1 \lambda\,\delta({}_0I_t^{1-\beta}x)\,dt = \int_0^1 \delta x\,({}_tI_1^{1-\beta}\lambda)\,dt,$$

provided $\delta x(0) = 0$ or $\lambda(0) = 0$, and $\delta x(1) = 0$ or $\lambda(1) = 0$. As $x(0)$ is specified, we take $\delta x(0) = 0$ and since $x(1)$ is not specified, we require $\lambda(1) = 0$.

Thus, we obtain

$$\begin{aligned}\delta\bar{J}(u) = \int_0^1 &\left[(G-H)\delta\lambda + \left(\frac{\partial F}{\partial x} + \lambda\frac{\partial G}{\partial x} - \frac{\partial H}{\partial({}_0D_t^\alpha x)}\,{}_tD_1^\alpha\lambda - \frac{\partial H}{\partial({}_0I_t^{1-\beta}x)}\,{}_tI_1^{1-\beta}\lambda\right)\delta x\right.\\ &\left. + \left(\frac{\partial F}{\partial u} + \lambda\frac{\partial G}{\partial u}\right)\delta u\right]\,dt.\end{aligned} \tag{2}$$

To minimize $\bar{J}(u)$ (same as minimizing $J(u)$), we need the variation $\delta\bar{J}(u)$ to be zero. It further requires that the coefficients of $\delta x, \delta u$ and $\delta\lambda$ in equation (2) to be zero. Thus,

$$\begin{aligned} H(\cdot\,,\,\cdot) &= G(x,u,t), &(3)\\ \frac{\partial H}{\partial({}_0D_t^\alpha x)}\,{}_tD_1^\alpha\lambda + \frac{\partial H}{\partial({}_0I_t^{1-\beta}x)}\,{}_tI_1^{1-\beta}\lambda &= \frac{\partial F}{\partial x} + \lambda\frac{\partial G}{\partial x}, &(4)\\ \frac{\partial F}{\partial u} + \lambda\frac{\partial G}{\partial u} &= 0, &(5) \end{aligned}$$

with terminal conditions

$$x(0) = x_0 \quad \text{and} \qquad \lambda(1) = 0. \tag{6}$$

Equations (3) to (5) represents necessary optimality conditions for the posed fractional optimal control problem (P) together with the terminal conditions given by equation (6).

Note: Once the costate variable λ is obtained by solving (3) and (4), the control variable $u(t)$ can easily be obtained by (5). Thus, we arrive at the following theorem.

Theorem 1.

If u is an optimal control of problem (P), then u satisfies the necessary conditions given by

$$\begin{aligned} H({}_0D_t^\alpha x(t)\,,\,{}_0I_t^{1-\beta}x(t)) &= G(x,u,t)\\ \frac{\partial H}{\partial({}_0D_t^\alpha x)}\,{}_tD_1^\alpha\lambda + \frac{\partial H}{\partial({}_0I_t^{1-\beta}x)}\,{}_tI_1^{1-\beta}\lambda &= \frac{\partial F}{\partial x} + \lambda\frac{\partial G}{\partial x},\\ \frac{\partial F}{\partial u} + \lambda\frac{\partial G}{\partial u} &= 0, \end{aligned}$$

with terminal conditions

$$x(0) = x_0 \quad \text{and} \qquad \lambda(1) = 0.$$

Here $x(t)$, $\lambda(t)$ are the state and costate variables respectively at time t.

Remark 1.

For $H(\cdot\,,\,\cdot) = {}_0D_t^\alpha x$, the problem (P) reduces to find an optimal control $u(t)$ for minimizing the performance index

$$J[u] = \int_0^1 F(t, x(t), u(t))dt,$$

subject to

$$ {}_0D_t^\alpha x = G(x,u,t), $$

and initial condition

$$ x(0) = x_0. $$

This is the simplest fractional optimal control problem given by Agrawal (1989).

One may note that the function H considered in problem (P) involves left Riemann-Liouville fractional operators. Similar types of results can be obtained by taking (RRLFD) and (RRLFI) or Caputo's fractional derivatives in both right and left fractional derivatives sense.

The next theorem presents the necessary optimality conditions when Caputo's left fractional order derivative are applied in place of (LRLFD).

(P*)**: The function $H({}_0D_t^\alpha x(t)\,,\,{}_0I_t^{1-\beta}x(t))$ is replaced by $H({}_0^cD_t^\alpha x(t)\,,\,{}_0I_t^{1-\beta}x(t))$ in the constraint of the problem (P).**

Theorem 2.

If u is an optimal control of problem (P^*), then u satisfies the necessary conditions given by

$$ \begin{aligned} H({}_0^cD_t^\alpha x(t)\,,\,{}_0I_t^{1-\beta}x(t)) &= G(x,u,t) \\ \frac{\partial H}{\partial({}_0^cD_t^\alpha x)}\,{}_tD_1^\alpha\lambda + \frac{\partial H}{\partial({}_0I_t^{1-\beta}x)}\,{}_tI_1^{1-\beta}\lambda &= \frac{\partial F}{\partial x} + \lambda\frac{\partial G}{\partial x}, \\ \frac{\partial F}{\partial u} + \lambda\frac{\partial G}{\partial u} &= 0, \end{aligned} $$

with terminal conditions

$$ x(0) = x_0 \quad \text{and} \qquad \lambda(1) = 0. $$

Example 1.

For $0 < \alpha < 1$, $\beta = 1$, we assume H to be a linear function in its argument. We consider the fractional optimal control problem (P^*) where the performance index is an integral of quadratic forms in state and control variables,

$$ J(u) = \frac{1}{2}\int_0^1 [q(t)x^2(t) + r(t)u^2]\,dt, \tag{7} $$

where $q(t) \geq 0$ and $r(t) > 0$, thus dynamics of the system is described by the following linear fractional differential equation,

$$ k\,{}_0^cD_t^\alpha x + l\,x(t) = a(t)x + b(t)u, $$

with initial condition

$$ x(0) = x_0, $$

and here k, l are fixed real numbers.

The necessary optimality conditions are then obtained as

$$\begin{aligned} k\,{}_0^cD_t^\alpha x + l\,x(t) &= a(t)x + b(t)u, && (8)\\ k\,{}_tD_1^\alpha \lambda + l\,\lambda &= q(t)x + a(t)\lambda, && (9)\\ r(t)u + b(t)\lambda &= 0. && (10)\end{aligned}$$

with terminal conditions

$$x(0) = x_0 \quad \text{and} \quad \lambda(1) = 0.$$

From (8) and (10), we get

$$k\,{}_0^cD_t^\alpha x = (a(t) - l)x(t) - r^{-1}b^2(t)\lambda. \tag{11}$$

The state $x(t)$ and costate $\lambda(t)$ are obtained by solving the fractional differential equations (9) and (11) subject to the given terminal conditions. The control variable $u(t)$ then clearly can be obtained by (10).

Remark 2.

For $l = 0$ and $k = 1$ in Example 1, the problem reduces to find an optimal control for the performance index

$$J(u) = \frac{1}{2}\int_0^1 [q(t)x^2(t) + r(t)u^2]\,dt,$$

where $q(t) \geq 0$ and $r(t) > 0$, and the dynamics of the system is described by the following linear fractional differential equation,

$${}_0^cD_t^\alpha x = a(t)x + b(t)u,$$

with initial condition

$$x(0) = x_0.$$

The necessary optimality conditions for this problem, given in Agrawal (2004), are as follows

$$\begin{aligned} {}_0^cD_t^\alpha x &= a(t)x + b(t)u,\\ {}_tD_1^\alpha \lambda &= q(t)x + a(t)\lambda,\\ r(t)u + b(t)\lambda &= 0,\end{aligned}$$

with terminal conditions

$$x(0) = x_0 \quad \text{and} \quad \lambda(1) = 0.$$

Example 2.

For $0 < \beta < 1$, $\alpha = 1$, assume H to be a linear function in its argument. We consider the fractional optimal control problem (P) where the performance index is an integral of quadratic forms in state and control variables,

$$J(u) = \frac{1}{2}\int_0^1 [q(t)x^2(t) + r(t)u^2]\, dt,$$

where $q(t) \geq 0$ and $r(t) > 0$, and the dynamics of the system is described by the following linear fractional differential equation,

$$k\, x'(t) + l\, {}_0I_t^{1-\beta} x = a(t)x + b(t)u,$$

with initial condition

$$x(0) = x_0,$$

and here k, l are fixed real numbers.

Necessary optimality conditions are then obtained as

$$\begin{aligned} k\, x'(t) + l\, {}_0I_t^{1-\beta} x &= a(t)x + b(t)u, \\ l\, {}_tI_1^{1-\beta}\lambda - k\,\frac{d\lambda}{dt} &= q(t)x + a(t)\lambda, \\ r(t)u + b(t)\lambda &= 0. \end{aligned}$$

with

$$x(0) = x_0 \quad \text{and} \quad \lambda(1) = 0.$$

Clearly,

$$l\, {}_0I_t^{1-\beta} x = a(t)x(t) - kx'(t) - r^{-1}b^2(t)\lambda.$$

We may note that the state $x(t)$ and costate $\lambda(t)$ are obtained by solving the above fractional differential equations subject to the given terminal conditions. The control variable $u(t)$ then clearly can be obtained by $r(t)u + b(t)\lambda = 0$.

4. Solution Scheme

In this section, we formulate a solution scheme for the problem (P^*). This scheme is constructed by implementing Adomian Decomposition Method on the necessary optimality conditions of problem (P^*).

Note:

- The same solution scheme can also be applied to problem (P). For simplicity of problem, we consider (P^*) involving Caputo's fractional derivative.
- For $0 < \alpha = \beta < 1$, we assume H involved in the constraint of (P^*) to be a linear function in its argument, i.e.,

$$H({}_0^cD_t^\alpha x\,,\,{}_0I_t^{1-\alpha}x) = k\,{}_0^cD_t^{1-\alpha}x + l\,{}_0I_t^{1-\alpha}x,$$

  where k, l are fixed real numbers.

The necessary optimality can now be written as

$$\begin{aligned} k\,{}_0^cD_t^\alpha x(t) + l\,{}_0I_t^{1-\alpha}x(t) &= G(x,u,t), && (12)\\ k\,{}_tD_1^\alpha\lambda + l\,{}_tI_1^{1-\alpha}\lambda &= \frac{\partial F}{\partial x} + \lambda\frac{\partial G}{\partial x}, && (13)\\ \frac{\partial F}{\partial u} + \lambda\frac{\partial G}{\partial u} &= 0, && (14) \end{aligned}$$

with terminal conditions

$$x(0) = A \quad \text{and} \quad \lambda(1) = 0.$$

Formulation of Solution scheme:

Observe that (14) represents an equation in the control variable u and the costate variable λ; that is, one can write u in terms of λ,

$$u \equiv P(\lambda). \tag{15}$$

Using equations (15), (12) and (13) becomes

$$\begin{aligned} k\,{}_0^cD_t^\alpha x(t) + l\,{}_0I_t^{1-\alpha}x(t) &= G^*(t,x,\lambda), && (16)\\ k\,{}_tD_1^\alpha\lambda + l\,{}_tI_1^{1-\alpha}\lambda &= Q(x,\lambda), && (17) \end{aligned}$$

where G^* is obtained by replacing u in terms of λ. $Q(x,\lambda) = \frac{\partial F}{\partial x} + \lambda\frac{\partial G}{\partial x}$.

Operating fractional integral operator ${}_0I_t^\alpha$ and ${}_tI_1^\alpha$ on both sides of (16) and (17)

$$\begin{aligned} x(t) &= x(0) + \frac{1}{k}\left[{}_0I_t^\alpha\, G^*(t,x,\lambda) - l\int_0^t x(t)dt\right];\ k \neq 0\\ \text{and}\quad \lambda(t) &= \lambda(1) + \frac{1}{k}\left[{}_tI_1^\alpha\, G^*(t,x,\lambda) - l\int_t^1 \lambda(t)dt\right];\ k \neq 0. \end{aligned}$$

Denoting the integral operators $L_1[\cdot] = {}_0I_t^\alpha[\cdot],\ L_2[\cdot] = \int_0^t[\cdot]dt,\ L_3[\cdot] = {}_tI_1^\alpha[\cdot]$ and $L_4[\cdot] = \int_t^1[\cdot]dt$, we get

$$\begin{aligned} x(t) &= x(0) + \frac{1}{k}\left\{L_1[G^*(t,x,\lambda)] - l\,L_2[x(t)]\right\};\ k \neq 0 && (18)\\ \text{and}\quad \lambda(t) &= \lambda(1) + \frac{1}{k}\left\{L_3[G^*(t,x,\lambda)] - l\,L_4[\lambda(t)]\right\};\ k \neq 0, && (19) \end{aligned}$$

as L_n, $n = 1,2,3,4,$ are linear operators, Adomian decomposition method can be applied.

Write x and λ in the form of an infinite series

$$x = \sum_{i=0}^{\infty} x_i \quad \text{and} \quad \lambda = \sum_{j=0}^{\infty} \lambda_j \, .$$

For $k \neq 0$ (ADM) gives, $x_0 = x(0) = A$, $\lambda_0 = \lambda(1) = 0$ and the values of x_i and λ_j (for $i, j = 1,2,3,...$) are given as

$$\begin{aligned}
x_1 &= \frac{1}{k}\left\{L_1[G^*(t, x_0, \lambda_0)] - l\, L_2[x_0]\right\},\\
\lambda_1 &= \frac{1}{k}\left\{L_3[G^*(t, x_0, \lambda_0)] - l\, L_4[\lambda_0]\right\},\\
x_2 &= \frac{1}{k}\left\{L_1[G^*(t, x_1, \lambda_1)] - l\, L_2[x_1]\right\},\\
\lambda_2 &= \frac{1}{k}\left\{L_3[G^*(t, x_1, \lambda_1)] - l\, L_4[\lambda_1]\right\},\\
&.\\
&.\\
&.\\
x_n &= \frac{1}{k}\left\{L_1[G^*(t, x_{n-1}, \lambda_{n-1})] - l\, L_2[x_{n-1}]\right\},\\
\lambda_n &= \frac{1}{k}\left\{L_3[G^*(t, x_{n-1}, \lambda_{n-1})] - l\, L_4[\lambda_{n-1}]\right\}.
\end{aligned}$$

For a suitable integer n, the n-term approximations of x and λ are

$$x = \sum_{i=0}^{n-1} x_i \quad \text{and} \quad \lambda = \sum_{j=0}^{n-1} \lambda_j .$$

Once λ is known, the control variable u can be obtained from (15).

Remark 3.

We note that for every $i = 0,1,2,3,...,$ in order to obtain x_{i+1} and λ_{i+1}, storing a set of two values $\{x_i, \lambda_i\}$ is required.

We shall now consider an example as a particular case where the performance index is an integral of quadratic forms in the state and control variable:

$$J(u) = \frac{1}{2}\int_0^1 [q(t)x^2(t) + r(t)u^2]dt, \tag{20}$$

where $q(t) \geq 0$ and $r(t) > 0$, the dynamics of the system is described by

$$ {}_0^cD_t^\alpha x = a(t)x + b(t)u. $$

with initial condition

$$ x(0) = x_0. $$

This problem was studied by Agarwal (2004).

Example 3.

Consider the following (FOCP) to find the control $u(t)$ that minimizes the quadratic performance index

$$ J[u] = \frac{1}{2}\int_0^1 [x^2(t) + u^2(t)]dt, \tag{21} $$

subject to the constraints

$$ {}^cD_{0+}^\alpha x(t) = -x + u, \quad 0 < \alpha < 1 \tag{22} $$

with initial condition

$$ x(0) = 1. \tag{23} $$

The necessary Euler-Lagrange equations and terminal conditions for this problem are

$$ {}^cD_{0+}^\alpha x(t) = -x + u, \tag{24} $$

$$ {}^cD_{1-}^\alpha u(t) = -x - u, \tag{25} $$

with terminal conditions

$$ x(0) = 1 \qquad \text{and} \qquad u(1) = 0. $$

In 2007, Agrawal and Baleanu gave a numerical scheme specifically for this problem. In Baleanu (2007), fractional derivatives are approximated using the Grunwald-Letnikov definition. Here, we shall use the formulation (stated in main result) to solve this problem.

Solution:

Given the necessary Euler-Lagrange conditions together with the terminal conditions, by equation (24), we have

$$ {}^cD_{0+}^\alpha x(t) = -x + u, $$

and applying the left integral operator I_{0+}^α on both sides, we get

$$ x(t) = x(0) - I_{0+}^\alpha x(t) + I_{0+}^\alpha u(t). $$

Similarly, by equation (25)

$$u(t) = u(1) - I^{\alpha}_{1-}x(t) - I^{\alpha}_{1-}u(t).$$

Thus,

$$x(t) = 1 - I^{\alpha}_{0+}x(t) + I^{\alpha}_{0+}u(t), \tag{26}$$

$$u(t) = -I^{\alpha}_{1-}x(t) - I^{\alpha}_{1-}u(t). \tag{27}$$

Let

$$L_1 \equiv I^{\alpha}_{0+} \quad \text{and} \quad L_2 \equiv I^{\alpha}_{1-}\,.$$

In order to apply (ADM), we express u and x in the form of an infinite series

$$u = \sum_{n=0}^{\infty} u_n \quad \text{and} \quad x = \sum_{n=0}^{\infty} x_n\,.$$

Since L_1 and L_2 are linear, we can rewrite (26) and (27) as

$$\begin{aligned}
\sum_{n=0}^{\infty} x_n &= 1 - \sum_{n=0}^{\infty} L_1(u_n) + \sum_{n=0}^{\infty} L_1(x_n), \\
\sum_{n=0}^{\infty} u_n &= -\sum_{n=0}^{\infty} L_2(u_n) - \sum_{n=0}^{\infty} L_2(x_n),
\end{aligned}$$

with $x_0 = 1$ and $u_0 = 0$.

Thus the recursive relation for u_n and x_n is obtained as follows:

$$\begin{aligned}
x_0 &= 1, \\
u_0 &= 0, \\
x_1 &= -L_1(x_0) + L_1(u_0), \\
u_1 &= -L_2(x_0) - L_2(u_0), \\
x_2 &= -L_1(x_1) + L_1(u_1), \\
u_2 &= -L_2(x_1) - L_2(u_1), \\
&. \\
&. \\
&. \\
x_n &= -L_1(x_{n-1}) + L_1(u_{n-1}), \\
u_n &= -L_2(x_{n-1}) - L_2(u_{n-1}).
\end{aligned}$$

For a suitable k, the solution is given as $k-$term approximation

$$u = u_0 + u_1 + u_2 + ... + u_{k-1},$$

and

$$x = x_0 + x_1 + x_2 + ... + x_{k-1}.$$

Note: We mark that for every $i = 0, 1, 2...$, we need to store two values (x_i, u_i) in order to get another set of values (x_{i+1}, u_{i+1}). The consequent values of (x_{i+1}, u_{i+1}) can be obtained by simply evaluating the fractional order integrals of (x_i, u_i) as $(x_0, u_0) = (1, 0)$, $(x_1, u_1) = (-\frac{t^\alpha}{\Gamma(\alpha+1)}, -\frac{(1-t)^\alpha}{\Gamma(\alpha+1)})$,... etc.

In particular, for $\alpha = \frac{1}{2}$: $(x_0, u_0) = (1, 0)$, $(x_1, u_1) = \left(-\frac{2}{\sqrt{\pi}}\sqrt{t}, -\frac{2}{\sqrt{\pi}}\sqrt{1-t}\right)$ and $(x_2, u_2) = \left(\frac{2}{\pi}\left(-\sqrt{t} + \frac{\pi t}{2} + (-1+t)\arctan h[\sqrt{t}]\right), \frac{2}{\pi}\left(\sqrt{1-t} + \frac{\pi}{2} + t\log(1+\sqrt{1-t}) - \frac{1}{2}t(\pi + \log t)\right)\right)$.

5. Conclusion

A new generalization of fractional optimal control problems has been introduced. This generalization addresses almost all the possible classes of (FOCP)'s and classical optimal control problems. Remarks and examples presented illustrate the bridge between various classes of optimal control problems. Additionally, a solution scheme has been formulated for a class of fractional optimal control problems. The fractional derivatives involved are in terms of Caputo's derivative. The formulation utilized composition formula for Caputo derivatives and the Adomian decomposition method. Furthermore, implementation of a quadratic performance index as a special case is presented. It is expected that the proposed work will further initiate research in generalizing the existing numerical scheme to solve fractional optimal control problems.

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