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Generalized Fractional Integral of the Product of Two Aleph-Functions

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Abstract

This paper is devoted to the study and develops the generalized fractional integral operators for a new special function, which is called Aleph-function. The considered generalized fractional integration operators contain the Appell hypergeometric function F_3 as a kernel. We establish two results of the product of two Aleph-functions involving Saigo-Maeda operators. On account of the general nature of the Saigo-Maeda operators and the Aleph-function, some results involving Saigo, Riemann-Liouville and Erdélyi-Kober integral operators are obtained as special cases of the main result.

Keywords: Aleph function, generalized fractional integral operators, fractional calculus, Mellin-Barnes type integrals, Appell function F_3, H-function, I-function

AMS-MSC 2010 No.: 26A33, 33E20, 33C20, 33C45

1. Introduction and Preliminaries

The object of this paper is to study the generalized fractional integration operators associated with the Appell function F_3 as a kernel, introduced by Saigo-Maeda (1996). The Aleph-function is an extension of the I-function, which itself is a generalization of the well-known and familiar G- and H-functions in one variable.

The Aleph-function is defined by means of a Mellin-Barnes type integral in the following manner cf. Südland et al. (1998, 2001), (see also Saxena and Pogány (2010, 2011)):

$$\aleph[z] = \aleph_{p_i,q_i,\tau_i;r}^{m,n}\left[z \left| \begin{matrix} (a_j,A_j)_{1,n},\ldots,[\tau_j(a_j,A_j)]_{n+1,p_i} \\ (b_j,B_j)_{1,m},\ldots,[\tau_j(b_j,B_j)]_{m+1,q_i} \end{matrix} \right. \right] := \frac{1}{2\pi i}\int_L \Omega_{p_i,q_i,\tau_i;r}^{m,n}(s)\, z^{-s}\, ds, \tag{1.1}$$

where $z \neq 0$, $i = \sqrt{-1}$ and

$$\Omega_{p_i,q_i,\tau_i;r}^{m,n}(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j + B_j s)\prod_{j=1}^{n}\Gamma(1-a_j-A_j s)}{\sum_{i=1}^{r}\tau_i \prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}s)}. \tag{1.2}$$

The integration path $L = L_{i\gamma\infty}$, $(\gamma \in \Re)$ extends from $\gamma - i\infty$ to $\gamma + i\infty$, and is such that the poles of $\Gamma(1-a_j-A_j s)$, $j = (\overline{1,n})$ (the symbol $(\overline{1,n})$ is used for 1,2,…,n) do not coincide with the poles of $\Gamma(b_j + B_j s)$, $j = (\overline{1,m})$. The parameters p_i, q_i are non-negative integers satisfying the condition $0 \le n \le p_i$, $1 \le m \le q_i$, $\tau_i > 0$ for $i = \overline{1,r}$. The parameters $A_j, B_j, A_{ji}, B_{ji} > 0$ and $a_j, b_j, a_{ji}, b_{ji} \in C$. The empty product in (1.2) is interpreted as unity. The existence conditions for the defining integral (1.1) are given below:

$$\varphi_l > 0, \quad |\arg(z)| < \frac{\pi}{2}\varphi_l, \; l = \overline{1,r}; \tag{1.3}$$

$$\varphi_l \ge 0, \quad |\arg(z)| < \frac{\pi}{2}\varphi_l, \quad \Re\{\zeta_l\} + 1 < 0, \tag{1.4}$$

$$\varphi_l = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \tau_l \left(\sum_{j=n+1}^{p_l} A_{jl} + \sum_{j=m+1}^{q_l} B_{jl} \right) \tag{1.5}$$

$$\zeta_l = \sum_{j=1}^{m} b_j - \sum_{j=1}^{n} a_j + \tau_l \left(\sum_{j=m+1}^{q_l} b_{jl} - \sum_{j=n+1}^{p_l} a_{jl} \right) + \frac{1}{2}(p_l - q_l), \quad (l = \overline{1,r}). \tag{1.6}$$

Fractional integration of Aleph-function is discussed by Saxena and Pogány (2011), and Ram and Kumar (2011A).

***Remark* 1.**

For $\tau_i = 1,\ i = \overline{1,r}$ in (1.1), we get the I -function due to V.P. Saxena (1982), defined in the following manner:

$$I^{m,n}_{p_i,q_i;r}[z] = \aleph^{m,n}_{p_i,q_i,1;r}[z] = \aleph^{m,n}_{p_i,q_i,1;r}\left[z \left| \begin{matrix} (a_j,A_j)_{1,n},\ldots,(a_j,A_j)_{n+1,p_i} \\ (b_j,B_j)_{1,m},\ldots,(b_j,B_j)_{m+1,q_i} \end{matrix} \right. \right] := \frac{1}{2\pi i} \int_L \Omega^{m,n}_{p_i,q_i,1;r}(s)\, z^{-s}\, ds, \tag{1.7}$$

where the kernel $\Omega^{m,n}_{p_i,q_i,1;r}(s)$ is given in (1.2). The existence conditions for the integral in (1.7) are the same as given in (1.3) - (1.6) with $\tau_i = 1,\ i = \overline{1,r}$.

If we further set $r = 1$, then (1.7) reduces to the familiar H - function cf. Fox (1961), Mathai et al. (2010), and Srivastava et al. (1982):

$$H^{m,n}_{p,q}[z] = \aleph^{m,n}_{p_i,q_i,1;1}[z] = \aleph^{m,n}_{p_i,q_i,1;1}\left[z \left| \begin{matrix} (a_p,A_p) \\ (b_q,B_q) \end{matrix} \right. \right] := \frac{1}{2\pi i} \int_L \Omega^{m,n}_{p_i,q_i,1;1}(s)\, z^{-s}\, ds, \tag{1.8}$$

where the kernel $\Omega^{m,n}_{p_i,q_i,1;1}(s)$ can be obtained from (1.2).

***Remark* 2.**

Inayat-Hussain (1987) introduced a generalization of the H -function in the form

$$\overline{H}^{m,n}_{p,q}[z] = \overline{H}^{m,n}_{p,q}\left[z \left| \begin{matrix} (a_j,\alpha_j;A_j)_1^n,(a_j,\alpha_j)_{n+1}^p \\ (b_j,\beta_j)_1^m,(b_j,\beta_j;B_j)_{m+1}^q \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_L \overline{\theta}(\xi)\, z^{-\xi}\, d\xi, \quad (z \neq 0), \tag{1.9}$$

where $i = \sqrt{-1}$ and

$$\overline{\theta}(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j \xi) \prod_{j=1}^{n} \left\{ \Gamma(1 - a_j - \alpha_j \xi) \right\}^{A_j}}{\prod_{j=m+1}^{q} \left\{ \Gamma(1 - b_j - \beta_j \xi) \right\}^{B_j} \prod_{j=n+1}^{p} \Gamma(a_j + \alpha_j \xi)}, \tag{1.10}$$

which contains fractional power of some of the gamma functions. The parameters $\alpha_i, \beta_j > 0\ (i = \overline{1,p},\ j = \overline{1,q})$ and the exponents $A_i,\ B_j\ (i = \overline{1,p},\ j = \overline{1,q})$ can take non-integer values; $L = L_{i\gamma\infty}$ remains the same contour as for $\aleph$ -function (1.1). From the above definition of the $\overline{H}$ -function and the definition (1.1) of the $\aleph$ -function, it is evident that $\overline{H}$ -function contains fractional powers of some of the gamma functions, whereas $\aleph$-function does not contain any fractional power of the gamma function, hence it is not possible to obtain a connection between $\aleph$-function and the $\overline{H}$-function; both special functions generalize Fox's H-function (cf. Fox (1961)), but in diverse directions.

The $\aleph$- function of two variables occurring in the present paper is defined and represented by in the following manner:

$$\aleph[x,y] = \aleph^{0,n:m_1,n_1;m_2,n_2}_{p,q:p_i,q_i,\tau_i;p_i',q_i',\tau_i';r}\left[\begin{array}{c|c} x & (a_j;\alpha_j,A_j)_{1,p}:(c_j,C_j)_{1,n_1},\ldots,[\tau_j(c_j,C_j)]_{n_1+1,p_i};(e_j,E_j)_{1,n_2},\ldots,[\tau_j'(e_j,E_j)]_{n_2+1,p_i'} \\ y & (b_j;\beta_j,B_j)_{1,q}:(d_j,D_j)_{1,m_1},\ldots,[\tau_j(d_j,D_j)]_{m_1+1,q_i};(f_j,F_j)_{1,m_2},\ldots,[\tau_j'(f_j,F_j)]_{m_2+1,q_i'} \end{array}\right]$$

$$= \frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\phi(s,\xi)\theta_1(s)\theta_2(\xi)\, x^{-s}y^{-\xi}\,ds\,d\xi\,, \tag{1.11}$$

$$\phi(s,\xi) = \frac{\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s-A_j\xi)}{\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s+A_j\xi)\prod_{j=1}^{q}\Gamma(1-b_j-\beta_j s-B_j\xi)}, \tag{1.12}$$

$$\theta_1(s) = \Omega^{m_1,n_1}_{p_i,q_i,\tau_i;r}(s) = \frac{\prod_{j=1}^{m_1}\Gamma(d_j+D_j s)\prod_{j=1}^{n_1}\Gamma(1-c_j-C_j s)}{\sum_{i=1}^{r}\tau_i\prod_{j=m_1+1}^{q_i}\Gamma(1-d_{ji}-D_{ji}s)\prod_{j=n_1+1}^{p_i}\Gamma(c_{ji}+C_{ji}s)}\ , \tag{1.13}$$

$$\theta_2(\xi) = \Omega^{m_2,n_2}_{p_i',q_i',\tau_i';r}(\xi) = \frac{\prod_{j=1}^{m_2}\Gamma(f_j+F_j\xi)\prod_{j=1}^{n_2}\Gamma(1-e_j-E_j\xi)}{\sum_{i=1}^{r}\tau_i'\prod_{j=m_2+1}^{q_i'}\Gamma(1-f_{ji}-F_{ji}\xi)\prod_{j=n_2+1}^{p_i'}\Gamma(e_{ji}+E_{ji}\xi)}, \tag{1.14}$$

2. Generalized Fractional Integral Operators

Saigo and Maeda (1996) introduced the seven-parameter generalized fractional integral operators $I^{\alpha,\alpha',\beta,\beta',\gamma}_{0,x}$, $I^{\alpha,\alpha',\beta,\beta',\gamma}_{x,\infty}$ associated with the Appell F_3 function as a kernel function, calling it left-sided and right-sided.

Let $\alpha,\alpha',\beta,\beta',\gamma\in C$, $\mathrm{Re}(\gamma)>0$, and $x\in R_+$. Then the generalized fractional integral operators involving Appell function F_3 as a kernel are defined by Saigo and Maeda (1996) as:

$$\left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x)=\frac{x^{-\alpha}}{\Gamma(\gamma)}\int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-t/x,1-x/t\right) f(t)\,dt\,, \tag{2.1}$$

and

$$\left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x)=\frac{x^{-\alpha'}}{\Gamma(\gamma)}\int_x^\infty (t-x)^{\gamma-1} t^{-\alpha} F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-x/t,1-t/x\right) f(t)\,dt\,, \tag{2.2}$$

where, $\operatorname{Re}(\gamma)$ denotes the real part of γ, and $F_3\left(\alpha,\alpha',\beta,\beta';\gamma;z,\xi\right)$ is the familiar Appell hypergeometric function of two variables defined by

$$F_3\left(\alpha,\alpha',\beta,\beta';\gamma;z,\xi\right)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}}\frac{z^m}{m!}\frac{\xi^n}{n!} \qquad \left(|z|<1,|\xi|<1\right). \tag{2.3}$$

Lemma 1. Saigo and Maeda (1996), p. 394, equations (4.18) and (4.19)

Let $\alpha,\alpha',\beta,\beta',\gamma\in C$. Then, there holds the following power function formulae:

i. If $\operatorname{Re}(\gamma)>0$, $\operatorname{Re}(\rho)>\max\left[0,\operatorname{Re}(\alpha+\alpha'+\beta-\gamma),\operatorname{Re}(\alpha'-\beta')\right]$, then

$$I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1}=\Gamma\left[\begin{matrix}\rho,\rho+\gamma-\alpha-\alpha'-\beta,\rho+\beta'-\alpha'\\ \rho+\gamma-\alpha-\alpha',\rho+\gamma-\alpha'-\beta,\rho+\beta'\end{matrix}\right]x^{\rho-\alpha-\alpha'+\gamma-1}, \tag{2.4}$$

ii. If $\operatorname{Re}(\gamma)>0$, $\operatorname{Re}(\rho)<1+\min\left[\operatorname{Re}(-\beta),\operatorname{Re}(\alpha+\alpha'-\gamma),\operatorname{Re}(\alpha+\beta'-\gamma)\right]$, then

$$I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1}=\Gamma\left[\begin{matrix}1+\alpha+\alpha'-\gamma-\rho,1+\alpha+\beta'-\gamma-\rho,1-\beta-\rho\\ 1-\rho,1+\alpha+\alpha'+\beta'-\gamma-\rho,1+\alpha-\beta-\rho\end{matrix}\right]x^{\rho-\alpha-\alpha'+\gamma-1}. \tag{2.5}$$

Here, we have used the symbol $\Gamma\left[\begin{matrix}\cdots\\ \cdots\end{matrix}\right]$ representing the fraction of many Gamma functions.

3. Left-Sided Generalized Fractional Integral of the Product of two $\aleph$ -Functions

In this section, we study the left-sided generalized fractional integration $I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ as defined by (2.1). Here we establish the product of two $\aleph$ -functions involving left-sided Saigo-Maeda integral operator. Generalized fractional integration formulas for the product of special functions are discussed by Ram and Kumar (2011B), and Gupta et al. (2010).

Theorem 3.1.

Let $\alpha,\alpha',\beta,\beta',\gamma,\sigma,\lambda,\omega\in C,\ \mathrm{Re}(\gamma)>0,\ (\mu,\upsilon>0)$, and

$$\mathrm{Re}(\sigma)+\mu\min_{1\le j\le m_1}\mathrm{Re}\left(\frac{b_j}{B_j}\right)+\upsilon\min_{1\le j\le m_2}\mathrm{Re}\left(\frac{d_j}{D_j}\right)>\max\left[0,\mathrm{Re}(\alpha'-\beta'),\mathrm{Re}(\alpha+\alpha'+\beta'-\gamma)\right].$$

Further, let the constants

$$a_j,b_j,a_{ji},b_{ji}\in C,\quad A_j,B_j,A_{ji},B_{ji}\in R_+\ \ (i=1,\dots,p_i;\ j=1,\dots,q_i);\ c_j,d_j,c_{ji},d_{ji}\in C,$$
$$C_j,D_j,C_{ji},D_{ji}\in R_+\ \ (i=1,\dots,p_i';\ j=1,\dots,q_i'),\ \tau_i,\tau_i'>0\ \text{ for }\ i=\overline{1,r}$$

also satisfy the conditions are given (1.3) - (1.6). Then, the left-sided generalized fractional integration $I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the product of two $\aleph$-functions exists and the following relation holds:

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\sigma-1}\,\aleph_{p_i,q_i,\tau_i;r}^{m_1,n_1}\left[\lambda t^{\mu}\left|\begin{matrix}(a_j,A_j)_{1,n_1},\dots,[\tau_j(a_j,A_j)]_{n_1+1,p_i}\\(b_j,B_j)_{1,m_1},\dots,[\tau_j(b_j,B_j)]_{m_1+1,q_i}\end{matrix}\right.\right]\right.\right.$$
$$\left.\left.\cdot\aleph_{p_i',q_i',\tau_i';r}^{m_2,n_2}\left[\omega t^{\upsilon}\left|\begin{matrix}(c_j,C_j)_{1,n_2},\dots,[\tau_j'(c_j,C_j)]_{n_2+1,p_i'}\\(d_j,D_j)_{1,m_2},\dots,[\tau_j'(d_j,D_j)]_{m_2+1,q_i'}\end{matrix}\right.\right]\right)\right\}(x)$$

$$=\ x^{\sigma-\alpha-\alpha'+\gamma-1}\,\aleph_{3,3:p_i,q_i,\tau_i;p_i',q_i',\tau_i';r}^{0,3:m_1,n_1;m_2,n_2}\left[\begin{matrix}\lambda x^{\mu}\\ \omega x^{\upsilon}\end{matrix}\left|\begin{matrix}(1-\sigma;\mu,\upsilon),(1-\sigma-\gamma+\alpha+\alpha'+\beta;\mu,\upsilon),\\(1-\sigma-\gamma+\alpha+\alpha';\mu,\upsilon),\ (1-\sigma-\beta';\mu,\upsilon),\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(1-\sigma-\beta'+\alpha';\mu,\upsilon):(a_j,A_j)_{1,n_1},\dots,[\tau_j(a_j,A_j)]_{n_1+1,p_i};(c_j,C_j)_{1,n_2},\dots,[\tau_j'(c_j,C_j)]_{n_2+1,p_i'}\\(1-\sigma-\gamma+\alpha'+\beta;\mu,\upsilon):(b_j,B_j)_{1,m_1},\dots,[\tau_j(b_j,B_j)]_{m_1+1,q_i};\ (d_j,D_j)_{1,m_2},\dots,[\tau_j'(d_j,D_j)]_{m_2+1,q_i'}\end{matrix}\right].$$

(3.1)

Proof:

In order to prove (3.1), we first express the product of two Aleph functions occurring on the left hand side of (3.1) in terms of Mellin-Barnes contour integral with the help of equation (1.1) and interchanging the order of integration, we obtain (say I):

$$I=\frac{1}{(2\pi i)^2}\int_{L_1}\Omega^{m_1,n_1}_{p_i,q_i,\tau_i;r}(s)\,\lambda^{-s}ds\int_{L_2}\Omega^{m_2,n_2}_{p_i',q_i',\tau_i';r}(\xi)\,\omega^{-\xi}\,d\xi\,\left(I^{\alpha,\alpha',\beta,\beta',\gamma}_{0+}\,t^{\sigma-\mu s-\upsilon\xi-1}\right)(x)$$

$$=\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\Omega^{m_1,n_1}_{p_i,q_i,\tau_1;r}(s)\Omega^{m_2,n_2}_{p_i',q_i',\tau_i';r}(\xi)\,\lambda^{-s}\omega^{-\xi}\,\left(I^{\alpha,\alpha',\beta,\beta',\gamma}_{0+}\,t^{\sigma-\mu s-\upsilon\xi-1}\right)(x)\,ds\,d\xi,$$

from (2.4), we arrive at

$$=\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\frac{\Gamma(\sigma-\mu s-\upsilon\xi)\,\Gamma(\sigma-\mu s-\upsilon\xi+\gamma-\alpha-\alpha'-\beta)\,\Gamma(\sigma-\mu s-\upsilon\xi+\beta'-\alpha)}{\Gamma(\sigma-\mu s-\upsilon\xi+\gamma-\alpha-\alpha')\,\Gamma(\sigma-\mu s-\upsilon\xi+\gamma-\alpha'-\beta)\,\Gamma(\sigma-\mu s-\upsilon\xi+\beta')}$$
$$\times\frac{\prod_{j=1}^{m_1}\Gamma(b_j+B_js)\prod_{j=1}^{n_1}\Gamma(1-a_j-A_js)}{\sum_{i=1}^{r}\tau_i\prod_{j=m_1+1}^{q_i}\Gamma(1-b_{ji}-B_{ji}s)\prod_{j=n_1+1}^{p_i}\Gamma(a_{ji}+A_{ji}s)}\;\frac{\prod_{j=1}^{m_2}\Gamma(d_j+D_j\xi)\prod_{j=1}^{n_2}\Gamma(1-c_j-C_j\xi)}{\sum_{i=1}^{r}\tau_i'\prod_{j=m_2+1}^{q_i'}\Gamma(1-d_{ji}-D_{ji}\xi)\prod_{j=n_2+1}^{p_i'}\Gamma(c_{ji}+C_{ji}\xi)}$$
$$\times\;x^{\sigma-\mu s-\upsilon\xi-\alpha-\alpha'+\gamma-1}\,\lambda^{-s}\omega^{-\xi}\,ds\,d\xi\,. \tag{3.2}$$

By interpreting the Mellin-Barnes counter integral thus obtained in terms of the $\aleph$-function of two variables as given in (1.11), we obtain the result (3.1). This completes the proof of Theorem 3.1.

Special Cases of Theorem 3.1:

Corollary 3.1.

If we put $\tau_i=1,\tau_i'=1\;(i=1,2,...,r)$ in (3.1) and take (1.7) into account, then we arrive at the following result in the term of I-function cf. Saxena (1982):

$$\left\{I^{\alpha,\alpha',\beta,\beta',\gamma}_{0+}\left(t^{\sigma-1}\;I^{m_1,n_1}_{p_i,q_i;r}\left[\lambda\,t^{\mu}\left|\begin{matrix}(a_j,A_j)_{1,n_1},...,(a_j,A_j)_{n_1+1,p_i}\\(b_j,B_j)_{1,m_1},...,(b_j,B_j)_{m_1+1,q_i}\end{matrix}\right.\right]\right.\right.$$
$$\left.\left..I^{m_2,n_2}_{p_i',q_i';r}\left[\omega t^{\upsilon}\left|\begin{matrix}(c_j,C_j)_{1,n_2},...,(c_j,C_j)_{n_2+1,p_i'}\\(d_j,D_j)_{1,m_2},...,(d_j,D_j)_{m_2+1,q_i'}\end{matrix}\right.\right]\right)\right\}(x)$$
$$=x^{\sigma-\alpha-\alpha'+\gamma-1}\,I^{0,3:m_1,n_1;m_2,n_2}_{3,3:p_i,q_i;p_i',q_i';r}\left[\begin{matrix}\lambda x^{\mu}\\\omega x^{\upsilon}\end{matrix}\left|\begin{matrix}(1-\sigma;\mu,\upsilon),(1-\sigma-\gamma+\alpha+\alpha'+\beta;\mu,\upsilon),\\(1-\sigma-\gamma+\alpha+\alpha';\mu,\upsilon),\,(1-\sigma-\beta';\mu,\upsilon),\end{matrix}\right.\right.$$
$$\left.\begin{matrix}(1-\sigma-\beta'+\alpha';\mu,\upsilon):(a_j,A_j)_{1,n_1},...,(a_j,A_j)_{n_1+1,p_i};(c_j,C_j)_{1,n_2},...,(c_j,C_j)_{n_2+1,p_i'}\\(1-\sigma-\gamma+\alpha'+\beta;\mu,\upsilon):(b_j,B_j)_{1,m_1},...,(b_j,B_j)_{m_1+1,q_i};(d_j,D_j)_{1,m_2},...,(d_j,D_j)_{m_2+1,q_i'}\end{matrix}\right]. \tag{3.3}$$

The existence conditions for (3.3) are the same as given in Theorem 3.1.

Corollary 3.2.

If we set $\tau_i = 1, \tau_i' = 1$ $\left(i = \overline{1, r}\right)$ and set $r = 1$ in (3.1) and take (1.8) into account, then we arrive at the following result in the term of product of two H-functions given by Ram and Kumar (2011B, Equation (17), p. 36).

$$\left\{ I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} \left(t^{\sigma-1} \; H_{p,q}^{m_1,n_1}\left[\lambda t^{\mu} \left| \begin{matrix} (a_j, A_j)_{1,p} \\ (b_j, B_j)_{1,q} \end{matrix} \right. \right] \; H_{p',q'}^{m_2,n_2}\left[\omega t^{\upsilon} \left| \begin{matrix} (c_j, C_j)_{1,p'} \\ (d_j, D_j)_{1,q'} \end{matrix} \right. \right] \right) \right\}(x)$$
$$= x^{\sigma-\alpha-\alpha'+\gamma-1} H_{3,3:p,q;p',q'}^{0,3:m_1,n_1;m_2,n_2} \left[\begin{matrix} \lambda x^{\mu} \\ \omega x^{\upsilon} \end{matrix} \left| \begin{matrix} (1-\sigma;\mu,\upsilon),(1-\sigma-\gamma+\alpha+\alpha'+\beta;\mu,\upsilon),(1-\sigma-\beta'+\alpha';\mu,\upsilon): \\ (1-\sigma-\gamma+\alpha+\alpha';\mu,\upsilon),\ (1-\sigma-\beta';\mu,\upsilon),(1-\sigma-\gamma+\alpha'+\beta;\mu,\upsilon): \end{matrix} \right. \right.$$
$$\left. \begin{matrix} (a_j, A_j)_{1,p}; (c_j, C_j)_{1,p'} \\ (b_j, B_j)_{1,q}; (d_j, D_j)_{1,q'} \end{matrix} \right]. \tag{3.4}$$

The existence conditions for (3.4) are the same as given in Theorem 1.

Now, if we follow Theorem 1 in respective case $\alpha' = \beta' = 0,\ \beta = -\eta, \alpha = \alpha + \beta,\ \gamma = \alpha$. Then we arrive at the following corollary concerning left-sided Saigo fractional integral operator cf. Saigo (1978).

Corollary 3.3. Let $\alpha, \beta, \eta, \sigma, \lambda, \omega \in C,\ \mathrm{Re}(\alpha) > 0,\ \mu, \upsilon > 0$ and let the constants

$$a_j, b_j, a_{ji}, b_{ji} \in C,\ A_j, B_j, A_{ji}, B_{ji} \in R_+\ \left(i = 1, ..., p_i;\ j = 1, ..., q_i\right);\ c_j, d_j, c_{ji}, d_{ji} \in C,$$
$$C_j, D_j, C_{ji}, D_{ji} \in R_+\ \left(i = 1, ..., p_i';\ j = 1, ..., q_i'\right),\ \ \tau_i, \tau_i' > 0 \ \text{ for } i = \overline{1, r}.$$

Further, satisfy the condition

$$\mathrm{Re}(\sigma) + \mu \min_{1 \le j \le m_1} \mathrm{Re}\left(\frac{b_j}{B_j} \right) + \upsilon \min_{1 \le j \le m_2} \mathrm{Re}\left(\frac{d_j}{D_j} \right) > \max\left[0, \mathrm{Re}(\beta - \eta)\right].$$

Then the left-sided Saigo fractional integral $I_{0+}^{\alpha,\beta,\eta}$ of the product of two $\aleph$-functions exists and the following relation holds:

$$\left\{ I_{0+}^{\alpha,\beta,\eta} \left(t^{\sigma-1} \; \aleph_{p_i,q_i,\tau_i;r}^{m_1,n_1}\left[\lambda t^{\mu} \left| \begin{matrix} (a_j, A_j)_{1,n_1}, ..., \left[\tau_j (a_j, A_j)\right]_{n_1+1,p_i} \\ (b_j, B_j)_{1,m_1}, ..., \left[\tau_j (b_j, B_j)\right]_{m_1+1,q_i} \end{matrix} \right. \right] \right. \right.$$

$$
.\aleph^{m_2,n_2}_{p'_i,q'_i,\tau'_i;r}\left[\omega t^{\upsilon}\left|\begin{matrix}\left(c_j,C_j\right)_{1,n_2},\ldots,\left[\tau'_j\left(c_j,C_j\right)\right]_{n_2+1,p'_i}\\ \left(d_j,D_j\right)_{1,m_2},\ldots,\left[\tau'_j\left(d_j,D_j\right)\right]_{m_2+1,q'_i}\end{matrix}\right.\right]\Bigg)\Bigg\}(x)
$$

$$
=x^{\sigma-\beta-1}\aleph^{0,2:m_1,n_1;m_2,n_2}_{2,2:p_i,q_i,\tau_i;p'_i,q'_i,\tau'_i;r}\left[\begin{matrix}\lambda x^{\mu}\\ \omega x^{\upsilon}\end{matrix}\left|\begin{matrix}(1-\sigma;\mu,\upsilon),(1-\sigma-\eta+\beta;\mu,\upsilon):\\ (1-\sigma+\beta;\mu,\upsilon),(1-\sigma-\alpha-\eta;\mu,\upsilon):\end{matrix}\right.\right.
$$

$$
\left.\begin{matrix}\left(a_j,A_j\right)_{1,n_1},\ldots,\left[\tau_j\left(a_j,A_j\right)\right]_{n_1+1,p_i};\left(c_j,C_j\right)_{1,n_2},\ldots,\left[\tau'_j\left(c_j,C_j\right)\right]_{n_2+1,p'_i}\\ \left(b_j,B_j\right)_{1,m_1},\ldots,\left[\tau_j\left(b_j,B_j\right)\right]_{m_1+1,q_i};\left(d_j,D_j\right)_{1,m_2},\ldots,\left[\tau'_j\left(d_j,D_j\right)\right]_{m_2+1,q'_i}\end{matrix}\right]. \tag{3.5}
$$

For $\beta=-\alpha$ in Corollary 3.3, the Saigo operator reduces to Riemann-Liouville operator cf. Srivastava and Saxena (2000), and we obtain the following result:

Corollary 3.4.

$$
\Bigg\{I^{\alpha}_{0+}\Bigg(t^{\sigma-1}\aleph^{m_1,n_1}_{p_i,q_i,\tau_i;r}\left[\lambda t^{\mu}\left|\begin{matrix}\left(a_j,A_j\right)_{1,n_1},\ldots,\left[\tau_j\left(a_j,A_j\right)\right]_{n_1+1,p_i}\\ \left(b_j,B_j\right)_{1,m_1},\ldots,\left[\tau_j\left(b_j,B_j\right)\right]_{m_1+1,q_i}\end{matrix}\right.\right]
$$

$$
.\aleph^{m_2,n_2}_{p'_i,q'_i,\tau'_i;r}\left[\omega t^{\upsilon}\left|\begin{matrix}\left(c_j,C_j\right)_{1,n_2},\ldots,\left[\tau'_j\left(c_j,C_j\right)\right]_{n_2+1,p'_i}\\ \left(d_j,D_j\right)_{1,m_2},\ldots,\left[\tau'_j\left(d_j,D_j\right)\right]_{m_2+1,q'_i}\end{matrix}\right.\right]\Bigg)\Bigg\}(x)
$$

$$
=x^{\sigma+\alpha-1}\aleph^{0,1:m_1,n_1;m_2,n_2}_{1,1:p_i,q_i,\tau_i;p'_i,q'_i,\tau'_i;r}\left[\begin{matrix}\lambda x^{\mu}\\ \omega x^{\upsilon}\end{matrix}\left|\begin{matrix}(1-\sigma;\mu,\upsilon):\left(a_j,A_j\right)_{1,n_1},\ldots,\left[\tau_j\left(a_j,A_j\right)\right]_{n_1+1,p_i};\\ (1-\sigma-\alpha;\mu,\upsilon):\left(b_j,B_j\right)_{1,m_1},\ldots,\left[\tau_j\left(b_j,B_j\right)\right]_{m_1+1,q_i};\end{matrix}\right.\right.
$$

$$
\left.\begin{matrix}\left(c_j,C_j\right)_{1,n_2},\ldots,\left[\tau'_j\left(c_j,C_j\right)\right]_{n_2+1,p'_i}\\ \left(d_j,D_j\right)_{1,m_2},\ldots,\left[\tau'_j\left(d_j,D_j\right)\right]_{m_2+1,q'_i}\end{matrix}\right]. \tag{3.6}
$$

Now, if we set $\beta=0$ in Corollary 1.4, the Riemann-Liouville operator reduces to Erdélyi-Kober operator cf. Srivastava and Saxena (2000), and we obtain the following result:

Corollary 3.5.

$$\left\{I_{\eta,\alpha}^{+}\left(t^{\sigma-1}\,\aleph_{p_i,q_i,\tau_i;r}^{m_1,n_1}\left[\lambda t^{\mu}\left|\begin{matrix}(a_j,A_j)_{1,n_1},\ldots,\left[\tau_j(a_j,A_j)\right]_{n_1+1,p_i}\\(b_j,B_j)_{1,m_1},\ldots,\left[\tau_j(b_j,B_j)\right]_{m_1+1,q_i}\end{matrix}\right.\right]\right.\right.$$

$$\left.\left..\aleph_{p_i',q_i',\tau_i';r}^{m_2,n_2}\left[\omega t^{\upsilon}\left|\begin{matrix}(c_j,C_j)_{1,n_2},\ldots,\left[\tau_j'(c_j,C_j)\right]_{n_2+1,p_i'}\\(d_j,D_j)_{1,m_2},\ldots,\left[\tau_j'(d_j,D_j)\right]_{m_2+1,q_i'}\end{matrix}\right.\right]\right)\right\}(x)$$

$$=x^{\sigma-1}\,\aleph_{1,1:p_i,q_i,\tau_i;p_i',q_i',\tau_i';r}^{0,1:m_1,n_1;m_2,n_2}\left[\begin{matrix}\lambda x^{\mu}\\\omega x^{\upsilon}\end{matrix}\left|\begin{matrix}(1-\sigma-\eta;\mu,\upsilon):(a_j,A_j)_{1,n_1},\ldots,\left[\tau_j(a_j,A_j)\right]_{n_1+1,p_i};\\(1-\sigma-\alpha-\eta;\mu,\upsilon):(b_j,B_j)_{1,m_1},\ldots,\left[\tau_j(b_j,B_j)\right]_{m_1+1,q_i};\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(c_j,C_j)_{1,n_2},\ldots,\left[\tau_j'(c_j,C_j)\right]_{n_2+1,p_i'}\\(d_j,D_j)_{1,m_2},\ldots,\left[\tau_j'(d_j,D_j)\right]_{m_2+1,q_i'}\end{matrix}\right]. \tag{3.7}$$

We can also obtain results of I -function and H -function for the corollaries 1.3, 1.4 and 1.5 by following the same method as done in corollaries 1.1 and 1.2.

4. Right-Sided Generalized Fractional Integral of the Product of two $\aleph$ -Functions:

In this section, we study the right-sided generalized fractional integration $I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$ defined in (2.2). Here we establish the product of two $\aleph$ -functions involving right-sided Saigo-Maeda integral operator.

Theorem 4.1.

Let $\alpha,\alpha',\beta,\beta',\gamma,\sigma,\lambda,\omega\in C$, $\mathrm{Re}(\gamma)>0$, $(\mu,\upsilon>0)$ and

$$\mathrm{Re}(\sigma)-\mu\min_{1\le j\le m_1}\mathrm{Re}\left(\frac{b_j}{B_j}\right)-\upsilon\min_{1\le j\le m_2}\mathrm{Re}\left(\frac{d_j}{D_j}\right)<1+\min\left[\mathrm{Re}(-\beta),\mathrm{Re}(\alpha+\alpha'-\gamma),\mathrm{Re}(\alpha+\beta'-\gamma)\right].$$

Further, let the constants $a_j,b_j,a_{ji},b_{ji}\in C$, $A_j,B_j,A_{ji},B_{ji}\in R_{+}$ $(i=1,\ldots,p_i;\,j=1,\ldots,q_i)$;

$c_j, d_j, c_{ji}, d_{ji} \in C,\;\; C_j, D_j, C_{ji}, D_{ji} \in R_+ \;\; \left(i = 1,\dots,p_i';\; j = 1,\dots,q_i'\right),\;\; \tau_i, \tau_i' > 0$, for $i = \overline{1,r}$ also satisfy the conditions as given (1.3) - (1.6). Then the right-sided generalized fractional integration $I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the product of two $\aleph$-functions exists and the following relation holds:

$$\left\{ I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} \left(t^{\sigma-1}\, \aleph_{p_i,q_i,\tau_i;r}^{m_1,n_1} \left[\lambda t^{-\mu} \left| \begin{matrix} \left(a_j, A_j\right)_{1,n_1}, \dots, \left[\tau_j\left(a_j, A_j\right)\right]_{n_1+1,p_i} \\ \left(b_j, B_j\right)_{1,m_1}, \dots, \left[\tau_j\left(b_j, B_j\right)\right]_{m_1+1,q_i} \end{matrix} \right. \right] \right.\right.$$

$$\left.\left. .\aleph_{p_i',q_i',\tau_i';r}^{m_2,n_2} \left[\omega t^{-\upsilon} \left| \begin{matrix} \left(c_j, C_j\right)_{1,n_2}, \dots, \left[\tau_j'\left(c_j, C_j\right)\right]_{n_2+1,p_i'} \\ \left(d_j, D_j\right)_{1,m_2}, \dots, \left[\tau_j'\left(d_j, D_j\right)\right]_{m_2+1,q_i'} \end{matrix} \right. \right] \right) \right\}(x)$$

$$= x^{\sigma-\alpha-\alpha'+\gamma-1}\, \aleph_{3,3:p_i,q_i,\tau_i;p_i',q_i',\tau_i';r}^{0,3:m_1,n_1;m_2,n_2} \left[\begin{matrix} \lambda x^{-\mu} \\ \omega x^{-\upsilon} \end{matrix} \left| \begin{matrix} (\sigma+\gamma-\alpha-\alpha';\mu,\upsilon), (\sigma+\gamma-\alpha-\beta';\mu,\upsilon), \\ (\sigma;\mu,\upsilon), (\sigma+\gamma-\alpha-\alpha'-\beta';\mu,\upsilon), \end{matrix} \right. \right.$$

$$\left. \begin{matrix} (\sigma+\beta;\mu,\upsilon) : \left(a_j, A_j\right)_{1,n_1}, \dots, \left[\tau_j\left(a_j, A_j\right)\right]_{n_1+1,p_i}; \left(c_j, C_j\right)_{1,n_2}, \dots, \left[\tau_j'\left(c_j, C_j\right)\right]_{n_2+1,p_i'} \\ (\sigma-\alpha+\beta;\mu,\upsilon) : \left(b_j, B_j\right)_{1,m_1}, \dots, \left[\tau_j\left(b_j, B_j\right)\right]_{m_1+1,q_i}; \left(d_j, D_j\right)_{1,m_2}, \dots, \left[\tau_j'\left(d_j, D_j\right)\right]_{m_2+1,q_i'} \end{matrix} \right]. \quad (4.1)$$

Special Cases of Theorem 4.1:

Corollary 4.1.

If we put $\tau_i = 1, \tau_i' = 1 \; (i = 1, 2, \dots, r)$ in (4.1) and take (1.7) into account, then we arrive at the following result in the term of I-function cf. Saxena (1982).

$$\left\{ I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} \left(t^{\sigma-1}\, I_{p_i,q_i;r}^{m_1,n_1} \left[\lambda t^{-\mu} \left| \begin{matrix} \left(a_j, A_j\right)_{1,n_1}, \dots, \left(a_j, A_j\right)_{n_1+1,p_i} \\ \left(b_j, B_j\right)_{1,m_1}, \dots, \left(b_j, B_j\right)_{m_1+1,q_i} \end{matrix} \right. \right] \right.\right.$$

$$\left.\left. .I_{p_i',q_i';r}^{m_2,n_2} \left[\omega t^{-\upsilon} \left| \begin{matrix} \left(c_j, C_j\right)_{1,n_2}, \dots, \left(c_j, C_j\right)_{n_2+1,p_i'} \\ \left(d_j, D_j\right)_{1,m_2}, \dots, \left(d_j, D_j\right)_{m_2+1,q_i'} \end{matrix} \right. \right] \right) \right\}(x)$$

$$= x^{\sigma-\alpha-\alpha'+\gamma-1}\, I_{3,3:p_i,q_i;p_i',q_i';r}^{0,3:m_1,n_1;m_2,n_2} \left[\begin{matrix} \lambda x^{-\mu} \\ \omega x^{-\upsilon} \end{matrix} \left| \begin{matrix} (\sigma+\gamma-\alpha-\alpha';\mu,\upsilon), (\sigma+\gamma-\alpha-\beta';\mu,\upsilon), (\sigma+\beta;\mu,\upsilon) : \\ (\sigma;\mu,\upsilon), (\sigma+\gamma-\alpha-\alpha'-\beta';\mu,\upsilon), (\sigma-\alpha+\beta;\mu,\upsilon) : \end{matrix} \right. \right.$$

$$\left. \begin{matrix} \left(a_j, A_j\right)_{1,n_1}, \dots, \left(a_j, A_j\right)_{n_1+1,p_i}; \left(c_j, C_j\right)_{1,n_2}, \dots, \left(c_j, C_j\right)_{n_2+1,p_i'} \\ \left(b_j, B_j\right)_{1,m_1}, \dots, \left(b_j, B_j\right)_{m_1+1,q_i}; \left(d_j, D_j\right)_{1,m_2}, \dots, \left(d_j, D_j\right)_{m_2+1,q_i'} \end{matrix} \right]. \quad (4.2)$$

The existence conditions for (4.2) are the same as given in Theorem 4.1.

Corollary 4.2.

If we set $\tau_i = 1, \tau'_i = 1 \; \left(i = \overline{1, r}\right)$ and set $r = 1$ in (4.1) and take (1.8) into account, then we arrive at the following result in the term of product of two H-functions given by Ram and Kumar (2011B, Eqn. (20), p. 39).

$$\left\{ I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} \left(t^{\sigma-1} \; H_{p,q}^{m_1,n_1} \left[\lambda t^{-\mu} \left| \begin{matrix} (a_j, A_j)_{1,p} \\ (b_j, B_j)_{1,q} \end{matrix} \right. \right] \; H_{p',q'}^{m_2,n_2} \left[\omega t^{-\upsilon} \left| \begin{matrix} (c_j, C_j)_{1,p'} \\ (d_j, D_j)_{1,q'} \end{matrix} \right. \right] \right) \right\} (x)$$
$$= x^{\sigma-\alpha-\alpha'+\gamma-1} \, H_{3,3:p,q;p',q'}^{0,3:m_1,n_1;m_2,n_2} \left[\begin{matrix} \lambda x^{-\mu} \\ \omega x^{-\upsilon} \end{matrix} \left| \begin{matrix} (\sigma+\gamma-\alpha-\alpha'; \mu, \upsilon), (\sigma+\gamma-\alpha-\beta'; \mu, \upsilon), (\sigma+\beta; \mu, \upsilon): \\ (\sigma; \mu, \upsilon), \; (\sigma+\gamma-\alpha-\alpha'-\beta'; \mu, \upsilon), (\sigma+\beta-\alpha; \mu, \upsilon): \end{matrix} \right. \right.$$
$$\left. \begin{matrix} (a_j, A_j)_{1,p} ; (c_j, C_j)_{1,p'} \\ (b_j, B_j)_{1,q} ; \; (d_j, D_j)_{1,q'} \end{matrix} \right]. \tag{4.3}$$

The existence conditions for (4.3) are the same as given in Theorem 2.

Now, if we follow Theorem 2 in respective case $\alpha' = \beta' = 0, \; \beta = -\eta, \alpha = \alpha + \beta, \; \gamma = \alpha$. Then we arrive at the following corollary concerning right-sided Saigo fractional integration operator cf. Saigo (1978).

Corollary 4.3. Let $\alpha, \beta, \eta, \sigma, \lambda, \omega \in C, \;\; \mathrm{Re}(\alpha) > 0, \;\; \mu, \upsilon > 0$ and let the constants

$$a_j, b_j, a_{ji}, b_{ji} \in C, \;\; A_j, B_j, A_{ji}, B_{ji} \in R_+ \; \left(i = 1, \ldots, p_i; \; j = 1, \ldots, q_i\right); \; c_j, d_j, c_{ji}, d_{ji} \in C,$$
$$C_j, D_j, C_{ji}, D_{ji} \in R_+ \;\; \left(i = 1, \ldots, p'_i; \; j = 1, \ldots, q'_i\right), \;\; \tau_i, \tau'_i > 0 \;\text{ for }\; i = \overline{1, r}.$$

Further, satisfy the condition

$$\mathrm{Re}(\sigma) - \mu \min_{1 \le j \le m_1} \mathrm{Re}\left(\frac{b_j}{B_j} \right) - \upsilon \min_{1 \le j \le m_2} \mathrm{Re}\left(\frac{d_j}{D_j} \right) < 1 + \min\left[\mathrm{Re}(\beta), \mathrm{Re}(\eta) \right].$$

Then, the right-sided Saigo fractional integral $I_{-}^{\alpha,\beta,\eta}$ of the product of two $\aleph$-functions exists and the following relation holds:

$$\left\{I_{-}^{\alpha,\beta,\eta}\left(t^{\sigma-1}\,\aleph_{p_i,q_i,\tau_i;r}^{m_1,n_1}\left[\lambda t^{-\mu}\left|\begin{matrix}(a_j,A_j)_{1,n_1},\ldots,[\tau_j(a_j,A_j)]_{n_1+1,p_i}\\(b_j,B_j)_{1,m_1},\ldots,[\tau_j(b_j,B_j)]_{m_1+1,q_i}\end{matrix}\right.\right]\right.\right.$$

$$\left.\left..\aleph_{p_i',q_i',\tau_i';r}^{m_2,n_2}\left[\omega t^{-\upsilon}\left|\begin{matrix}(c_j,C_j)_{1,n_2},\ldots,[\tau_j'(c_j,C_j)]_{n_2+1,p_i'}\\(d_j,D_j)_{1,m_2},\ldots,[\tau_j'(d_j,D_j)]_{m_2+1,q_i'}\end{matrix}\right.\right]\right)\right\}(x)$$

$$=x^{\sigma-\beta-1}\,\aleph_{2,2:p_i,q_i,\tau_i;p_i',q_i',\tau_i';r}^{0,2:m_1,n_1;m_2,n_2}\left[\begin{matrix}\lambda x^{-\mu}\\ \omega x^{-\upsilon}\end{matrix}\left|\begin{matrix}(\sigma-\beta;\mu,\upsilon),(\sigma-\eta;\mu,\upsilon):(a_j,A_j)_{1,n_1},\ldots,[\tau_j(a_j,A_j)]_{n_1+1,p_i};\\(\sigma;\mu,\upsilon),(\sigma-\alpha-\beta-\eta;\mu,\upsilon):(b_j,B_j)_{1,m_1},\ldots,[\tau_j(b_j,B_j)]_{m_1+1,q_i};\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(c_j,C_j)_{1,n_2},\ldots,[\tau_j'(c_j,C_j)]_{n_2+1,p_i'}\\(d_j,D_j)_{1,m_2},\ldots,[\tau_j'(d_j,D_j)]_{m_2+1,q_i'}\end{matrix}\right]. \tag{4.4}$$

***Remark* 4.1.**

We can also obtain results concerning Riemann-Liouville and Erdélyi-Kober operators by putting $\beta=-\alpha$ and $\beta=0$ respectively in Corollary 4.3.

5. Further Applications:

(i) If we specialize the first H-function in Corollary 1.2 to the exponential function by taking $\mu=1$, we obtain

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\sigma-1}e^{-\lambda t}\,H_{p',q'}^{m_2,n_2}\left[\omega t^{\upsilon}\left|\begin{matrix}(c_j,C_j)_{1,p'}\\(d_j,D_j)_{1,q'}\end{matrix}\right.\right]\right)\right\}(x)$$

$$=x^{\sigma-\alpha-\alpha'+\gamma-1}\,H_{3,3:0,1;p',q'}^{0,3:1,0;m_2,n_2}\left[\begin{matrix}\lambda x\\ \omega x^{\upsilon}\end{matrix}\left|\begin{matrix}(1-\sigma;1,\upsilon),(1-\sigma-\gamma+\alpha+\alpha'+\beta;1,\upsilon),(1-\sigma-\beta'+\alpha;1,\upsilon):\\(1-\sigma-\gamma+\alpha+\alpha';1,\upsilon),(1-\sigma-\gamma+\alpha'+\beta;1,\upsilon),(1-\sigma-\beta';1,\upsilon):\end{matrix}\right.\right.$$

$$\left.\begin{matrix}-\;;(c_j,C_j)_{1,p'}\\(0,1);(d_j,D_j)_{1,q'}\end{matrix}\right]. \tag{5.1}$$

The condition of validity of the above result can be easily obtained from Corollary 1.2.

Further on letting $\lambda \to 0$ in the above result, it reduces in the following form:

$$\left\{ I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma} \left(t^{\sigma-1} \ H_{p',q'}^{m_2,n_2} \left[\omega t^{\upsilon} \left| \begin{matrix} (c_j, C_j)_{1,p'} \\ (d_j, D_j)_{1,q'} \end{matrix} \right. \right] \right) \right\} (x)$$

$$= x^{\sigma-\alpha-\alpha'+\gamma-1} H_{p'+3,q'+3}^{m_2,n_2+3} \left[\omega x^{\upsilon} \left| \begin{matrix} (c_j, C_j)_{1,p'}, (1-\sigma,\upsilon), (1-\sigma-\gamma+\alpha+\alpha'+\beta,\upsilon), (1-\sigma-\beta'+\alpha,\upsilon), (c_j, C_j)_{n_2+1,p'} \\ (d_j, D_j)_{1,q'}, (1-\sigma-\gamma+\alpha+\alpha',\upsilon), (1-\sigma-\gamma+\alpha'+\beta,\upsilon), (1-\sigma-\beta',\upsilon) \end{matrix} \right. \right]. \tag{5.2}$$

(ii) If we reduce the H-function to the generalized Wright hypergeometric function [cf. Srivastava et al. (1982), p.19, eq. (2.6.11)] in the result given by (3.4), we arrive at

$$\left\{ I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} \left(t^{\sigma-1} \ {}_p\psi_q \left[\omega t^{\upsilon} \left| \begin{matrix} (c_j, C_j)_{1,p} \\ (d_j, D_j)_{1,q} \end{matrix} \right. \right] \right) \right\} (x)$$

$$= x^{\sigma-\alpha-\alpha'+\gamma-1} \ {}_{p+3}\psi_{q+3} \left[\omega x^{\upsilon} \left| \begin{matrix} (c_j, C_j)_{1,p}, (\sigma,\upsilon), (\sigma+\gamma-\alpha-\alpha'-\beta,\upsilon), (\sigma+\beta'-\alpha,\upsilon) \\ (d_j, D_j)_{1,q}, (\sigma+\gamma-\alpha-\alpha',\upsilon), (\sigma+\gamma-\alpha'-\beta,\upsilon), (\sigma+\beta',\upsilon) \end{matrix} \right. \right]. \tag{5.3}$$

The conditions of validity of (5.3) can be easily defined from Corollary 1.2.

Further we can also obtain its companion result by setting

$$\alpha' = \beta' = 0,\ \beta = -\eta, \alpha = \alpha + \beta,\ \gamma = \alpha$$

in (5.3) [cf. Gupta et al. (2010), p. 210, Equation (27)]:

$$\left\{ I_{0+}^{\alpha,\beta,\eta} \left(t^{\sigma-1} \ {}_p\psi_q \left[\omega t^{\upsilon} \left| \begin{matrix} (c_j, C_j)_{1,p} \\ (d_j, D_j)_{1,q} \end{matrix} \right. \right] \right) \right\} (x)$$

$$= x^{\sigma-\beta-1} \ {}_{p+2}\psi_{q+2} \left[\omega x^{\upsilon} \left| \begin{matrix} (c_j, C_j)_{1,p}, (\sigma,\upsilon), \ (\sigma+\eta-\beta,\upsilon) \\ (d_j, D_j)_{1,q}, (\sigma-\beta,\upsilon), (\sigma+\alpha+\eta,\upsilon) \end{matrix} \right. \right]. \tag{5.4}$$

A number of several special cases of Image 2 can also be obtained but we do not mention them here on account of lack of space.

5. Conclusions

In this research article we have studied and developed the generalized fractional integral operators for $\aleph$-function. We have established two results of the product of two $\aleph$-functions involving Saigo-Maeda operators which are believed to be new. A large number of new and known results involving Saigo, Riemann-Liouville and Erdélyi-Kober integral operators are the special cases of our main results. The obtained results provide extension of the results given by Ram and Kumar (2011B) for the generalized fractional integration of the product of two H-functions; given results also provide extension of the generalized fractional integration of the product of two I-functions. A number of several special cases as Mittag-Leffler function, Whittaker function and Bessel function of the first kind can be developed for Corollary 1.2 and 2.2.

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