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Heat source thermoelastic problem in a hollow elliptic cylinder under time-reversal principle

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Abstract

The article investigates the time-reversal thermoelasticity of a hollow elliptical cylinder for determining the temperature distribution and its associated thermal stresses at a certain point using integral transform techniques by unifying classical orthogonal polynomials as the kernel. Furthermore, by considering a circle as a special kind of ellipse, it is seen that the temperature distribution and the comparative study of a circular cylinder can be derived as a special case from the present mathematical solution. The numerical results obtained are accurate enough for practical purposes.

Keywords: Hollow elliptic cylinder; circular cylinder; temperature distribution; thermal stresses; integral transform

AMS-MSC 2010 No.: 35B07, 35G30, 35K05, 44A10

1. Introduction

The determination of initial temperature distribution from a known physical distribution of temperature at any instant is known as the time-reversal problem. This has aided in finding the temperature distribution of a prior state when it can be determined at any position and at any instant. It is not surprising that a considerable amount of work has been done over the past decades of the time-reversal heat conduction challenge due to technical interest.

During the past years, time-reversal heat conduction problems with circular boundary have been worked out by many authors. Masket (1965) considered Green's functions or Influence functions and discussed a class of heat conduction problem which he has termed as 'Time-reversal problem'. Sabherwal (1965) has considered the time-reversal problems in heat conduction for (i) semi-infinite medium, (ii) rectangular plate. Choubey (1969) has studied a time-reversal heat conduction problem of heat conduction for a solid elliptical cylinder by applying a finite Mathieu transform. Mehta (1976) tackled some time-reversal heat conduction with the help of integral transform for evaluating (i) heat flow in a cylindrical shell of infinite height with heat generation and radiation, (ii) heat flow in a truncated wedge of finite height, (iii) heat flow on a semi-infinite solid containing an exterior plane crack with circular boundary and an infinitely long cylindrical cavity. Under the title of 'time-reversal problem', Patel (1978) investigated time-reversal heat conduction problem of the circular cylinder with radiation type boundary conditions applying the unconventional finite integral transform.

Recently, Bagde (2013) investigated the time-reversal inverse heat conduction problem of an elliptical plate for determining the temperature distribution and unknown temperature gradient at a particular point at any time using Mathieu transform and unconventional finite integral transform. However, the aforementioned researchers have not considered any thermoelastic problem using the above principle, particularly in elliptical coordinates system.

Although the aforementioned time-reversal concept can be applicable to the field of underwater acoustics (Fink 1996, 2006), Thermo and photo-acoustic tomography (Kawakatsu 2008), Non-Destructive Evaluation (Reyes-Rodríguez 2014), Electromagnetic Fields (Mora 2012) and Ultrasonic Fields (Fink 1992) etc., few authors (Fink 1997, Larmat 2010, Montagner 2012) have proposed an analytical analysis of the method, especially in the case of an elastic medium and for a finite body such as the Earth. Similarly, time-reversal of wave propagation (Fouque 2007) analysis in a randomly layered medium using the asymptotic theory of ordinary differential equations can also be beneficial to source estimation of layered media.

In solid mechanics, there is an ample number of cases in which heat production in solids has led to various technical problems of mechanical applications wherein heat produced is rapidly sought to be transferred or dissipated. Reviewing the previous studies, it was observed that no analytical procedure has been established in the realm of solid mechanics, considering internal heat source generated within the body. The main objective here is to theoretically treat the time-reversal thermoelastic problem of heat transfer in the region where heat is generated within the system.

To establish the time-reversal formulation, the following assumptions need to be made.

1. The material of the cylinder is elastic, homogeneous, and isotropic.
2. Thin walled cylinder has been considered during the investigation with a ratio of length to the thickness greater than 8.
3. The deflection (the normal component of the displacement vector) of the mid-plane is small as compared to the thickness of the plate.
4. The stress perpendicular to the middle plane is small compared to the other stress components and may be neglected in the stress-strain relations.

In this paper, extended repeated integral transformation involving ordinary and modified Mathieu functions of first and second kind of order *n* and Jacobi transform are defined. Inversion formula has been established and some properties are mentioned. The transform has been used to investigate thermoelasticity problem of the hollow elliptical cylinder occupying the space

$$D = \{(\xi,\eta,z) \in R^3 : \xi_i \le \xi \le \xi_o,\ 0 \le \eta \le 2\pi,\ -1 \le z \le 1\},$$

having compounded effect with known surrounding temperature at time τ and internal heat source generated within the material.

2. Statement of the problem

The thermoelastic problem of an elliptical cylinder can be rigorously analysed by introducing the elliptical coordinates (ξ,η,z), which are related to the rectangular coordinates (x,y,z). For convenience, we transform to an elliptic cylindrical coordinate system i.e., $x = c\cosh\xi\cos\eta,\quad y = c\sinh\xi\sin\eta$ and $z = z$. The curves $\eta =$constant represent a family of confocal hyperbolas while the curves $\xi =$constant represent a family of confocal ellipses. The length 2*c* is the distance between their common foci. Both sets of curves intersect each other orthogonally at every point in space. The parameter ξ defines the interfocal line taking the range $\xi \in (\xi_i,\ \xi_o)$, the coordinate η is an angular coordinate taking the range $\eta \in [0, 2\pi)$, and thickness as $z \in (-1,1)$.

2.1. Heat conduction problem

The governing differential equation for heat conduction can be defined as

$$\kappa[h^2(T_{,\xi\xi} + T_{,\eta\eta}) + (1-z^2)T_{,zz}] + \Phi(\xi,\eta,z,t) = T_{,t} \tag{1}$$

and has to be solved with the conditions

$$T(\xi,\eta,z,\tau) = f(\xi,\eta,z) \text{ (known) for } \tau > 0, \tag{2}$$

$$T(\xi,\eta,z,t) = 0 \quad \text{at} \quad \xi = \xi_o, \tag{3}$$

$$T_{,\xi}(\xi,\eta,z,t) = 0 \quad \text{at} \quad \xi = \xi_i. \tag{4}$$

We consider the hollow elliptical cylinder occupying the space *D* which ends at *z* = ±1 and whose lateral surface is insulated. Thus, the thermal conductivity vanishes at the ends; it follows that the ends are also insulated. We assume the initial conditions as

$$T(\xi,\eta,z,t) = g(\xi,\eta,z) \text{ at } t = 0 \text{ (unknown) for all } -1 \le z \le 1, \tag{5}$$

where $g(\xi,\eta,z)$ is a suitable function so that the Mathieu-Jacobi transform of $g(\xi,\eta,z)$ exists, $\Phi(\xi,\eta,z,t)$ is the heat source function for the problem, $\kappa = \lambda/\rho C$ represents the thermal diffusivity in which λ is the thermal conductivity of the material, ρ is the density, τ is any time greater than zero, C is the calorific capacity which is assumed to be constant.

For the sake of brevity, we consider heat generated in the solid per unit volume as

$$\Phi(\xi,\eta,z,t)=[(\beta-\alpha)-(\beta+\alpha+2)\,z]\,T,_z\,, \tag{6}$$

and

$$h^{-2}=(c^2/2)(\cosh 2\xi-\cos 2\eta)\,.$$

2.2. Associated thermal stress problem

The medium is defined by

$$z=\ell(=2)\,,\ \xi_i\le\xi\le\xi_o\,,\ 0\le\eta\le 2\pi\,.$$

Compiling various boundary conditions, elliptical coordinates are defined to determine the influence of thermal boundary conditions on the thermal stresses. Since we have assumed that the cylinder is sufficiently thin, we can assume that the plane, initially normal to the middle or neutral plane (z = 0) before bending, remains straight and normal to the middle surface during the deformation. The length of such elements is not altered. This means that the axial stress is negligible compared to the other stress components. This can be neglected in the stress-strain relations. According to the aforesaid assumption, the potential function ϕ for such a system satisfies the equation

$$h^2(\phi,_{\xi\xi}+\phi,_{\eta\eta})=\frac{1+\upsilon}{1-\upsilon}\alpha_t T\,, \tag{7}$$

where υ denotes the Poisson's ratio, α_t the coefficient of linear expansion.

The components of the stresses given by Misra (1971) are represented by the use of the stress function ϕ and is illustrated as

$$\left.\begin{aligned}(1/h^4)\bar{\sigma}_{\xi\xi} &= -2G(c^2/2)[(\cosh 2\xi-\cos 2\eta)\,\phi,_{\eta\eta}+\sinh 2\xi\ \phi,_\xi-\ \sin 2\eta\ \phi,_\eta\,],\\ (1/h^4)\bar{\sigma}_{\eta\eta} &= -2G(c^2/2)[(\cosh 2\xi-\cos 2\eta)\phi,_{\xi\xi}-\sinh 2\xi\ \phi,_\xi+\sin 2\eta\ \phi,_\eta\,],\\ (1/h^4)\bar{\sigma}_{\xi\eta} &= -2G(c^2/2)[-(\cosh 2\xi-\cos 2\eta)\phi,_{\xi\eta}+\ \sin 2\xi\ \phi,_\eta+\sinh 2\eta\ \phi,_\xi\,].\end{aligned}\right\} \tag{8}$$

It is to be noted that the condition on the boundary of the plate should be stress-free and is yet to be satisfied. To this end, we find the complementary stresses $\bar{\bar{\sigma}}_{ij}$ satisfying the following relations

$$\bar{\sigma}_{\xi\xi}+\bar{\bar{\sigma}}_{\xi\xi}=0,\ \ \bar{\sigma}_{\xi\eta}+\bar{\bar{\sigma}}_{\xi\eta}=0\quad \text{on}\quad \xi=\xi_0\,. \tag{9}$$

To solve the isothermal elastic problem, let us make use of the Airy stress function which satisfies the bilaplacian equation

$$[h^2(\chi_{,\xi\xi}+\chi_{,\eta\eta})]^2=0\,. \tag{10}$$

Then, the complementary stresses are given by

$$\left.\begin{aligned}(1/h^4)\bar{\bar{\sigma}}_{\xi\xi} &= (c^2/2)[(\cosh 2\xi-\cos 2\eta)\,\chi_{,\eta\eta}+\sinh 2\xi\;\chi_{,\xi}-\sin 2\eta\;\chi_{,\eta}],\\ (1/h^4)\bar{\bar{\sigma}}_{\eta\eta} &= (c^2/2)[(\cosh 2\xi-\cos 2\eta)\chi_{,\xi\xi}-\sinh 2\xi\;\chi_{,\xi}+\sin 2\eta\;\chi_{,\eta}],\\ (1/h^4)\bar{\bar{\sigma}}_{\xi\eta} &= (c^2/2)[-(\cosh 2\xi-\cos 2\eta)\chi_{,\xi\eta}+\sin 2\xi\;\chi_{,\eta}+\sinh 2\eta\;\chi_{,\xi}].\end{aligned}\right\} \tag{11}$$

Thus, the final stresses can be represented as

$$\left.\begin{aligned}\sigma_{\xi\xi} &= \bar{\sigma}_{\xi\xi}+\bar{\bar{\sigma}}_{\xi\xi}\,,\\ \sigma_{\eta\eta} &= \bar{\sigma}_{\eta\eta}+\bar{\bar{\sigma}}_{\eta\eta}\,,\\ \sigma_{\xi\eta} &= \bar{\sigma}_{\xi\eta}+\bar{\bar{\sigma}}_{\xi\eta}\,.\end{aligned}\right\} \tag{12}$$

Equations (1) to (12) constitute the mathematical formulation of the problem under consideration.

3. Solution for the Problem

3.1. Solution of the heat conduction problem

In order to solve fundamental differential Equation (1), first we introduce the extended transformation (refer Appendix) over the variables (ξ,η,z) as

$$\begin{aligned}\overline{M}(\alpha,\beta,\gamma,\xi_i,\xi_o,q_{2n,m}) = \int_{-1}^{1}\int_{\xi_i}^{\xi_o}\int_0^{2\pi} & M(\xi,\eta,z)(\cosh 2\xi-\cos 2\eta)\\ & \times B_{2n}(\xi,q_{2n,m})\; ce_{2n}(\eta,q_{2n,m})\\ & \times(1-z)^{\alpha}(1+z)^{\beta}P_{\gamma}^{(\alpha,\beta)}(z)\,d\xi\, d\eta\, dz.\end{aligned} \tag{13}$$

In this way, we may define the inversion theorem of (13) in the form

$$\begin{aligned}M(\xi,\eta,z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{\gamma=0}^{\infty} & \overline{M}(\alpha,\beta,\gamma,\xi_i,\xi_o,q_{2n,m})Be_{2n}(\xi,q_{2n,m})\\ & \times ce_{2n}(\eta,q_{2n,m})P_{\gamma}^{(\alpha,\beta)}(z)/C_{n,m}\delta_{\gamma}.\end{aligned} \tag{14}$$

Performing above integral transformation under the conditions (3) to (4), we obtain

$$\overline{T}_{,t}=-[k\vartheta_{2n,m}^2+\gamma(\alpha+\beta+\gamma+1)]\;\overline{T}(\alpha,\beta,\gamma,\xi_i,\xi_o,q_{2n,m},t), \tag{15}$$

in which

$$\vartheta_{2n,m}^{2} = 4q_{2n,m} / c^{2}.$$

Simplifying Equation (15) and using condition (2), one obtains

$$\begin{aligned}\overline{T}(\alpha,\beta,\gamma,\xi_i,\xi_o,q_{2n,m},t) &= \overline{f}(\alpha,\beta,\gamma,\xi_i,\xi_o,q_{2n,m}) \\ &\quad \times \exp\{-[k\vartheta_{2n,m}^{2} + \gamma(\alpha+\beta+\gamma+1)](t-\tau)\},\end{aligned} \tag{16}$$

and then, using the inversion theorems of the transform rules defined by Equation (14) on Equation (16), yields

$$\begin{aligned}T(\xi,\eta,z,t) &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{\gamma=0}^{\infty} P_{\gamma}^{(\alpha,\beta)}(z)\overline{f}(\alpha,\beta,\gamma,\xi_i,\xi_o,q_{2n,m}) \\ &\quad \times Be_{2n}(\xi,q_{2n,m})\, ce_{2n}(\eta,q_{2n,m}) \\ &\quad \times \exp\{-[k\vartheta_{2n,m}^{2} + \gamma(\alpha+\beta+\gamma+1)](\tau-t)\} / Cn,m,\end{aligned} \tag{17}$$

and the required unknown temperature using condition (5) is derived as

$$\begin{aligned}g(\xi,\eta,z) &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{\gamma=0}^{\infty} P_{\gamma}^{(\alpha,\beta)}(z)\overline{f}(\alpha,\beta,\gamma,\xi_i,\xi_o,q_{2n,m}) \\ &\quad \times Be_{2n}(\xi,q_{2n,m})\, ce_{2n}(\eta,q_{2n,m}) \\ &\quad \times \exp\{-[k\vartheta_{2n,m}^{2} + \gamma(\alpha+\beta+\gamma+1)]\tau\} / C_{n,m}.\end{aligned} \tag{18}$$

3.2. Solution of the thermal stress problem

Referring to the fundamental Equation (1) and its solution (17) for the heat conduction problem, the solution to the displacement function is represented by the Goodier's thermoelastic displacement potential ϕ, which is given by Equation (7) as

$$\begin{aligned}\phi(\xi,\eta,\ell) &= -\alpha_t\left(\frac{1+\upsilon}{1-\upsilon}\right)c^2h^2 \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{r=0}^{\infty} P_{\gamma}^{(\alpha,\beta)}(z)\,\overline{f}(\alpha,\beta,\gamma,\xi_i,\xi_o,q_{2n,m}) \\ &\quad \times Be_{2n}(\xi,q_{2n,m})\, ce_{2n}(\eta,q_{2n,m}) \\ &\quad \times \exp[-(k\vartheta_{2n,m}^{2} + \gamma(\alpha+\beta+\gamma+1))(\tau-t)] / 4q_{2n,m}C_{n,m}.\end{aligned} \tag{19}$$

Now assume Airy's stress function which satisfies Equation (10) as,

$$\chi(\xi,\eta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{r=0}^{\infty} q_{2n,m} P_{\gamma}^{(\alpha,\beta)}(z)\bar{f}(\alpha,\beta,\gamma,\xi_i,\xi_o,q_{2n,m}) \times [X_{n,m}(\cosh 2\xi - \cos 2\eta) + Y_{n,m}(\cosh 2\xi + \cos 2\eta)] \times Be_{2n}(\xi,q_{2n,m}) \times ce_{2n}(\eta,q_{2n,m}) \times \exp\{-[k\vartheta_{2n,m}^2 + \gamma(\alpha+\beta+\gamma+1)](\tau - t)\}/C_{n,m}, \tag{20}$$

in which $X_{n,m}$ and $Y_{n,m}$ are the arbitrary functions that can be determined finally by using condition (9).

$$\left.\begin{aligned} X_{n,m} &= \frac{G\alpha_t}{8q_{2n,m}C_{n,m}}\left(\frac{1+\upsilon}{1-\upsilon}\right)c^4h^2[1-3\cos 2\eta \sec h\, 2\xi_0 \\ &\quad + \cos ec\, 2\eta \sinh 2\xi_0 + \cot 2\eta \tanh 2\xi_0], \\ Y_{n,m} &= \frac{G\alpha_t}{8q_{2n,m}C_{n,m}}\left(\frac{1+\upsilon}{1-\upsilon}\right)c^4h^2[-3\sec h\, 2\xi_0 + \cos ec\, 2\eta \tanh 2\xi_0]. \end{aligned}\right\} \tag{21}$$

Substituting $X_{n,m}$ and $Y_{n,m}$ in Equations (20), (8), and (11), we get

$$\chi(\xi,\eta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{r=0}^{\infty} \frac{G\alpha_t}{8C_{2n,m}^2}\left(\frac{1+\upsilon}{1-\upsilon}\right)c^4h^2 P_{\gamma}^{(\alpha,\beta)}(z)\bar{f}(\alpha,\beta,\gamma,\xi_i,\xi_o,q_{2n,m}) \times\{[(1-3\cos 2\eta \sec h\, 2\xi_0 + \cos ec\, 2\eta \sinh 2\xi_0 + \cot 2\eta \tanh 2\xi_0)(\cosh 2\xi - \cos 2\eta) + (-3\sec h\, 2\xi_0 + \cos ec\, 2\eta \tanh 2\xi_0) \times (\cosh 2\xi + \cos 2\eta)] Be_{2n}(\xi,q_{2n,m})\, ce_{2n}(\eta,q_{2n,m})\} \times \exp\{-[k\vartheta_{2n,m}^2 + \gamma(\alpha+\beta+\gamma+1)](\tau - t)\}. \tag{22}$$

Using Equations (8), (11), (19) and (22), one obtains the solution for thermal stresses as

$$\bar{\sigma}_{\xi\xi} = G\alpha_t\left(\frac{1+\upsilon}{1-\upsilon}\right)c^4h^6 \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{r=0}^{\infty} \frac{P_{\gamma}^{(\alpha,\beta)}(z)}{4q_{2n,m}C_{2n,m}} \bar{f}(\alpha,\beta,\gamma,\xi_i,\xi_o,q_{2n,m}) \times\{(\cosh 2\xi - \cos 2\eta) Be_{2n}(\xi,q_{2n,m})\, ce''_{2n}(\eta,q_{2n,m}) + \sinh 2\xi\, Be'_{2n}(\xi,q_{2n,m})\, ce_{2n}(\eta,q_{2n,m}) - \sin 2\eta\, Be_{2n}(\xi,q_{2n,m} ce'_{2n}(\eta,q_{2n,m})\} \times \exp\{-[k\vartheta_{2n,m}^2 + \gamma(\alpha+\beta+\gamma+1)](\tau - t)\}, \tag{23}$$

$$\bar{\sigma}_{\eta\eta} = \left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t G c^4 h^6 \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{r=0}^{\infty} \frac{P_\gamma^{(\alpha,\beta)}(z)}{4q_{2n,m}C_{2n,m}} \bar{f}(\alpha,\beta,\gamma,\xi_i,\xi_o,q_{2n,m})$$
$$\times\{(\cosh 2\xi - \cos 2\eta)\, Be''_{2n}(\xi,q_{2n,m}) ce_{2n}(\eta,q_{2n,m})$$
$$-\sinh 2\xi\, Be'_{2n}(\xi,q_{2n,m}) ce_{2n}(\eta,q_{2n,m}) \tag{24}$$
$$+\sin 2\eta\, Be_{2n}(\xi,q_{2n,m}) ce'_{2n}(\eta,q_{2n,m})\}$$
$$\times\exp\{-[k\vartheta_{2n,m}^2 + \gamma(\alpha+\beta+\gamma+1)](\tau - t)\},$$

$$\bar{\sigma}_{\xi\eta} = \left(\frac{1+\upsilon}{1-\upsilon}\right) G\alpha_t c^4 h^6 \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{r=0}^{\infty} \frac{P_\gamma^{(\alpha,\beta)}(z)}{4q_{2n,m}C_{2n,m}} \bar{f}(\alpha,\beta,\gamma,\xi_i,\xi_o,q_{2n,m})$$
$$\times\{-(\cosh 2\xi - \cos 2\eta)\, Be'_{2n}(\xi,q_{2n,m}) ce'_{2n}(\eta,q_{2n,m})$$
$$+\sin 2\xi\, Be_{2n}(\xi,q_{2n,m}) ce'_{2n}(\eta,q_{2n,m}) \tag{25}$$
$$+\sinh 2\eta\, Be'_{2n}(\xi,q_{2n,m})\, ce_{2n}(\eta,q_{2n,m})\}$$
$$\times\exp\{-[k\vartheta_{2n,m}^2 + \gamma(\alpha+\beta+\gamma+1)](\tau - t)\},$$

$$\bar{\bar{\sigma}}_{\xi\xi} = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{r=0}^{\infty} \frac{G\alpha_t}{8C_{2n,m}^2}\left(\frac{1+\upsilon}{1-\upsilon}\right) c^4 h^2 P_\gamma^{(\alpha,\beta)}(z)\bar{f}(\alpha,\beta,\gamma,\xi_i,\xi_o,q_{2n,m})$$
$$\times\{\operatorname{cosec} 2\eta\, \operatorname{sech} 2\xi_o\, ce_{2n}(\eta,q_{2n,m})(2\cosh 2\xi_o \sin 2\eta - 3\sin 4\eta$$
$$+2\cos 2\eta \sinh 2\xi_0 + \sinh 4\xi_0)\sinh 2\xi(2\sinh 2\xi Ce_{2n}(\xi,q_{2n,m})$$
$$+\cosh 2\xi\, Ce'_{2n}(\xi,q_{2n,m})) - 2\cosh 2\xi Ce_{2n}(\xi,q_{2n,m})$$
$$\times(ce_{2n}(\eta,q_{2n,m})(6\operatorname{sech} 2\xi_o\, \sin^2 2\eta - 2(\cot 2\eta \sinh 2\xi_0$$
$$+\operatorname{cosec} 2\eta \tanh 2\xi_0)) + (1/2)(-3\operatorname{sech} 2\xi_0 \sin 4\eta + 2(\sin 2\eta + \sinh 2\xi_0$$
$$+\cos 2\eta \tanh 2\xi_0))c'e_{2n}(\eta,q_{2n,m})) - (\cosh 2\xi - \cos 2\eta)\cosh 2\xi$$
$$\times Ce_{2n}(\xi,q_{2n,m})(4ce_{2n}(\eta,q_{2n,m})(3\cos 2\eta \operatorname{sech} 2\xi_o + \operatorname{cosec} 2\eta$$
$$\times((\cot^2 2\eta + \operatorname{cosec}^2 2\eta)\, \sinh 2\xi_0 + 2\cot 2\eta \operatorname{cosec} 2\eta \tanh 2\xi_0))$$
$$+12\operatorname{sech} 2\xi_o\, \sin 2\eta c'e_{2n}(\eta,q_{2n,m}) - 4\operatorname{cosec} 2\eta \cot 2\eta \sinh 2\xi_0$$
$$\times c'e_{2n}(\eta,q_{2n,m}) - 4\operatorname{cosec}^2 2\eta \tanh 2\xi_0 c'e_{2n}(\eta,q_{2n,m}) + (1 - 3\cos 2\eta \tag{26}$$
$$\times \operatorname{sech} 2\xi_o + \operatorname{cosec} 2\eta \sinh 2\xi_0 \tanh 2\xi_0 c''e_{2n}(\eta,q_{2n,m}))]\}$$
$$\times\exp\{-[k\vartheta_{2n,m}^2 + \gamma(\alpha+\beta+\gamma+1)](\tau - t)\},$$

$$\bar{\bar{\sigma}}_{\eta\eta} = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{r=0}^{\infty} \frac{G\alpha_t}{8C_{2n,m}^2}\left(\frac{1+\upsilon}{1-\upsilon}\right)c^4 h^2 P_\gamma^{(\alpha,\beta)}(z)\bar{f}(\alpha,\beta,\gamma,\xi_i,\xi_o,q_{2n,m})$$
$$\times\{(-\cos ec\, 2\eta \sec h 2\xi_o\, ce_{2n}(\eta,q_{2n,m}))(2\cosh 2\xi_o \sin 2\eta - 3\sin 4\eta$$
$$+2\cos 2\eta \sinh 2\xi_0 + \sinh 4\xi_0)\sinh 2\xi(2\sinh 2\xi Ce_{2n}(\xi,q_{2n,m})$$
$$+\cosh 2\xi\, Ce'_{2n}(\xi,q_{2n,m})) + 2\cosh 2\xi Ce_{2n}(\xi,q_{2n,m})$$
$$\times(ce_{2n}(\eta,q_{2n,m})(6\sec h 2\xi_o\, \sin^2 2\eta - 2(\cot 2\eta \sin h 2\xi_0$$
$$+\cos ec\, 2\eta \tanh 2\xi_0)) + (1/2)(-3\sec h 2\xi_0 \sin 4\eta + 2(\sin 2\eta + \sinh 2\xi_0$$
$$+\cos 2\eta \tanh 2\xi_0))c'e_{2n}(\eta,q_{2n,m})) - (\cosh 2\xi - \cos 2\eta)$$
$$\times ce_{2n}(\eta,q_{2n,m})\cos ec\, 2\eta \sec h 2\xi_o\, (-2\cosh 2\xi_0 \sin 2\eta + 3\sin 4\eta$$
$$-2\cos 2\eta \sinh 2\xi_o - \sinh 4\xi_o)(4\sinh 2\xi C'e_{2n}(\xi,q_{2n,m})$$
$$+\cosh 2\xi(4Ce_{2n}(\xi,q_{2n,m}) + C''e_{2n}(\xi,q_{2n,m})))\} \tag{27}$$
$$\times\exp\{-[k\vartheta_{2n,m}^2 + \gamma(\alpha+\beta+\gamma+1)](\tau - t)\},$$

$$\bar{\bar{\sigma}}_{\xi\eta} = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{r=0}^{\infty} \frac{G\alpha_t}{8C_{2n,m}^2}\left(\frac{1+\upsilon}{1-\upsilon}\right)c^4 h^2 P_\gamma^{(\alpha,\beta)}(z)\bar{f}(\alpha,\beta,\gamma,\xi_i,\xi_o,q_{2n,m})$$
$$\times\{\sec h 2\xi_o\, ce_{2n}(\eta,q_{2n,m})(2\cosh 2\xi_o \sin 2\eta - 3\sin 4\eta$$
$$+2\cos 2\eta \sinh 2\xi_0 + \sinh 4\xi_0)(2\sinh 2\xi\, Ce_{2n}(\xi,q_{2n,m})$$
$$+\cosh 2\xi\, Ce'_{2n}(\xi,q_{2n,m})) - (1/2)(\cos 2\eta - \cosh 2\xi)\cos ec^2\, 2\eta$$
$$\times\sec h 2\xi_o\, (2\sinh 2\xi\, Ce_{2n}(\xi,q_{2n,m}) + \cosh 2\xi\, Ce'_{2n}(\xi,q_{2n,m}))$$
$$\times(ce_{2n}(\eta,q_{2n,m})(-24\sin^3 2\eta + 8\sinh 2\xi_0 + 4\cos 2\eta \sinh 2\xi_0$$
$$+\sinh 4\xi_0) - 2\sin 2\eta(2\cosh 2\xi_o \sin 2\eta - 3\sin 4\eta + 2\cos 2\eta \sinh 2\xi_0$$
$$+\sinh 4\xi_0)ce_{2n}(\eta,q_{2n,m})) + \sinh 2\xi \cosh 2\xi\, Ce_{2n}(\xi,q_{2n,m}) \tag{28}$$
$$\times(6\sec h 2\xi_o\, \sin 2\eta - 2\cos ec\, 2\eta(\cot 2\eta \sinh 2\xi_0 + \cos ec 2\eta \tanh 2\xi_0))$$
$$+(1 - 3\cos 2\eta \sec h 2\xi_o + 2\cos ec\, 2\eta \sinh 2\xi_0 + \cot 2\eta \tan h 2\xi_0)$$
$$\times ce'_{2n}(\eta,q_{2n,m}))\}\exp\{-[k\vartheta_{2n,m}^2 + \gamma(\alpha+\beta+\gamma+1)](\tau - t)\}.$$

4. Transition to hollow circular cylinder

When the elliptical cylinder tends to a circular cylinder of inner radius a and outer radius b is considered then, the semi-focal $c \to 0$. We can say that λ_m is the root of the transcendental equation $J_0(\alpha_m) = 0$. Also,

$$e \to 0 \ [\text{as } \xi \to \infty],\ \cosh 2\xi\, d\xi \to 2\cosh 2\xi \sinh 2\xi\, d\xi \to 2r dr / c^2,$$
$$\sinh\xi \to \cosh\xi,\ h\cosh\xi \to r\ [\text{as } h \to 0],\ \cosh\xi\, d\xi \to r\, dr,\ h\sinh\xi\, d\xi \to dr.$$

Using results from McLachlan (1947, p.330),

$$
\begin{aligned}
&Be_0(\xi, q_{0,m}) \to p_0' J_0(\lambda_m r),\ Be_0'(\xi, q_{0,m}) \to p_0' J_0'(\lambda_m r),\\
&Be_0''(\xi, q_{0,m}) \to p_0' J_0''(\lambda_m r),\ ce_0(\eta, q_m) \to 1/\sqrt{2},\ A_0^{(0)} \to 1/\sqrt{2}, A_2^{(0)} \to 0,\\
&\Theta_{2m} \to 0,\ \lambda_{0,m}^2 = \alpha_{0,m}^2 / a^2 = \alpha_m^2 / a^2 = \lambda_m^2, C_{n,m} \to C_m,\\
&p_0' = Be_0(0, q_{0,m}) ce_0(2\pi, q_{0,m}) / A_0^{(0)},\\
&\overline{f}(\alpha, \beta, \gamma, \xi_i, \xi_o, q_{2n,m}) \to \overline{f}(\alpha, \beta, \gamma, a, b, \alpha_m).
\end{aligned}
$$

Taking into account the aforesaid parameters, the temperature distribution in cylindrical coordinate is finally represented by

$$
\begin{aligned}
T(r,z,t) = &\sum_{m=1}^{\infty} \sum_{\gamma=0}^{\infty} P_\gamma^{(\alpha,\beta)}(z)\, \overline{f}(\alpha, \beta, \gamma, a, b, \alpha_m)\\
&\times p_0' J_0(\lambda_m r) \exp\{-[k\lambda_m^2 + \gamma(\alpha+\beta+\gamma+1)](\tau - t)\} / (\sqrt{2}\, Cm),
\end{aligned}
\tag{29}
$$

in which

$$
C_m = \int_a^b r[p_0' J_0(\lambda_m r)]^2 dr.
$$

The aforementioned degenerated results of Equation (29) agree with the results of Varghese and Khobragade (2007).

5. Numerical Results, Discussion and Remarks

For the sake of simplicity of calculation, we set

$$
f(\xi, \eta, z) = \delta(\xi - \xi_0)\, \delta(\eta - \eta_0)\, z^\gamma. \tag{30}
$$

The numerical computations have been carried out for Aluminum metal with parameter a = 0.73 cm, b = 0.93 cm, ℓ = 2 cm, Modulus of Elasticity $E = 6.9 \times 10^6$ N/cm^2, Shear modulus $G = 2.7 \times 10^6$ N/cm^2, Poisson's ratio υ = 0.281, Thermal expansion coefficient $\alpha = 25.5 \times 10^{-6}$ cm/cm-^{0}C, Thermal diffusivity κ = 0.86 cm^2/sec, Thermal conductivity λ = 0.48 cal sec^{-1}/cm ^{0}C with $q_{n,m}$ = 0.0986, 0.3947, 0.8882, 1.5791, 2.4674, 3.5530, 4.8361, 6.3165, 7.9943, 9.8696, 11.9422, 14.2122, 16.6796, 19.3444, 22.2066, 25.2661, 28.5231, 31.9775, 35.6292 etc., are the positive & real roots of the transcendental equation (A3).

In order to examine the influence of heating on the plate, the numerical calculation for all variables was performed. The same has been depicted in the following figures applying MATHEMATICA software. Figures 1–3 illustrates the numerical results of temperature and stresses on the elliptical plate due to interior heat generated within the solid, under thermal boundary condition subjected to the known temperature at any instant time τ.

As shown in Figure 1(a), the temperature increases as the time proceeds along the radial direction and is the greatest at the other edge of the plate due to the known initial temperature. The variation of normal stresses $\sigma_{\xi\xi}$, $\sigma_{\eta\eta}$, and $\sigma_{\xi\eta}$ is shown in Figures 1(b), 1(c) and 1(d), respectively. From Figure 1(b), the large compressive stress occurs on the inner heated surface and the tensile stress occurs on the inner surface which drops along the radial direction for radial stress satisfying Equation (9). From Figure 1(c), it is observed that the maximum tensile stress occurs while heating inside the core of the plate due to the combined energy of internal heat generation and the known temperature at τ. From Figure 1(d), it is seen that the negative shear stress profile further deepens the value in the mid-core due to the compressive stress, which attains the absolute value zero satisfying the traction free property as declared in Equation (9) at $\xi = \xi_0$.

Figure 2 illustrates the temperature and thermal stresses along the axial direction. Figure 2(a) indicates the time variation of temperature distribution along z direction of the plate for different values of ξ. The maximum value of temperature magnitude occurs at the outer edge due to available internal heat energy throughout the body. The distribution of temperature gradient at each time decreases in the unheated area of the central part of ellipse boundary tending below zero in one direction. Figures 2(c) and 2(d) indicate that the stresses $\sigma_{\eta\eta}$ and $\sigma_{\xi\eta}$ have maximum tensile force on the outer surface due to maximum expansion at the outer part of the plate and its absolute value increases with radius. Figure 2(b) depicts that the radial stress $\sigma_{\xi\xi}$ attains minimum at the outer core due to the compressive stress occurring at the outer region.

Figure 3(a) shows that the temperature distribution along the time series for different values of known surrounding temperature at any instant which maximizes its magnitude towards outer edge may be due to energized heat supply. Figure 3(b) depicts that the temperature distribution along η direction attains maximum expansion at its central core.

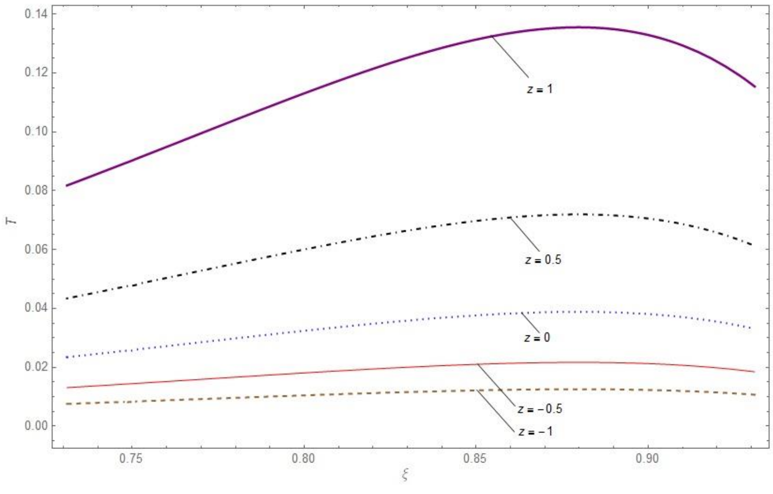

Figure 1(a). Temperature distribution along ξ for different values of z

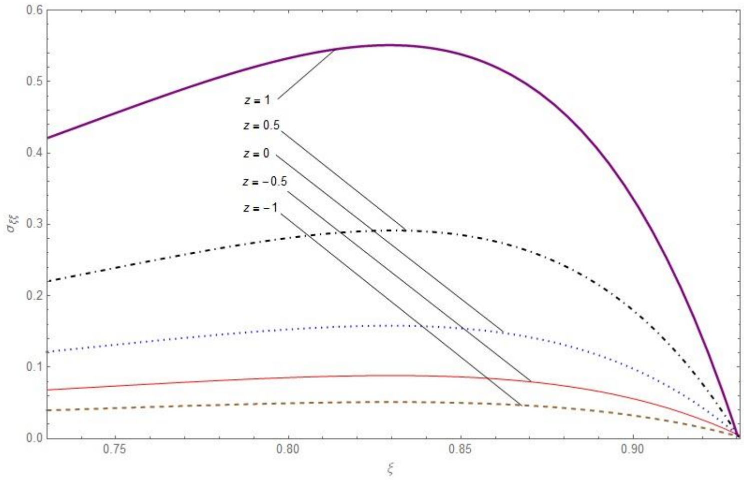

Figure 1(b). $\sigma_{\xi\xi}$ along ξ for different values of z

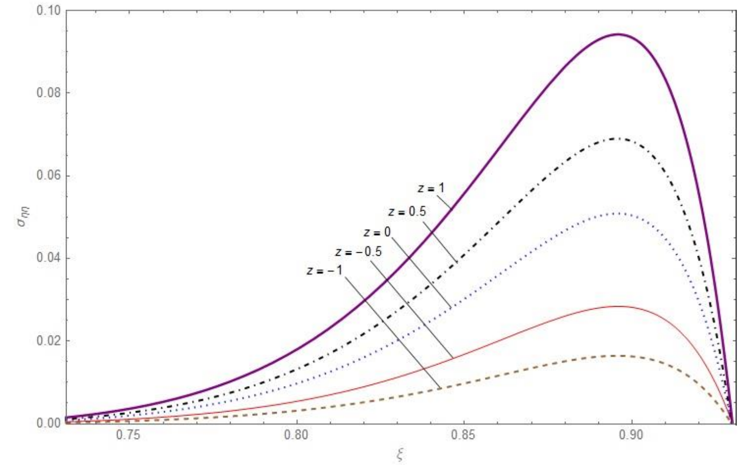

Figure 1(c). $\sigma_{\eta\eta}$ along ξ for different values of z

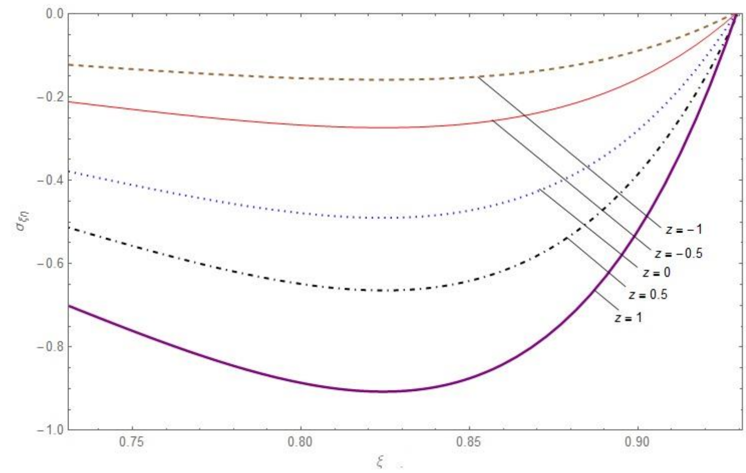

Figure 1(d). $\sigma_{\xi\eta}$ along ξ for different values of z

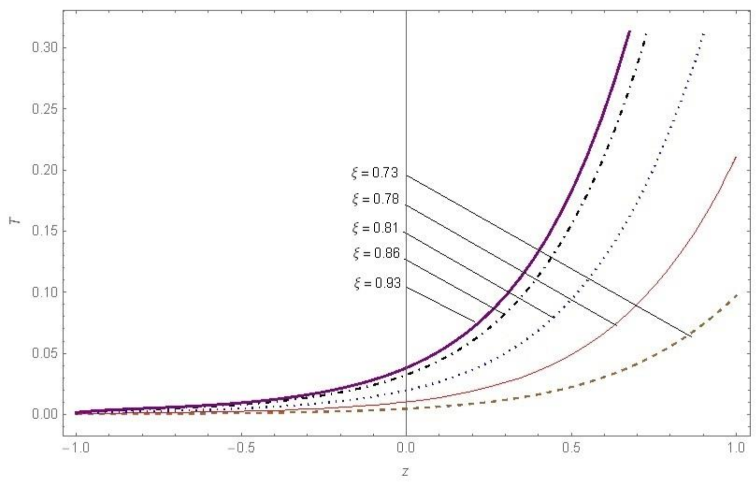

Figure 2(a). Temperature distribution along *z* for different values of ξ

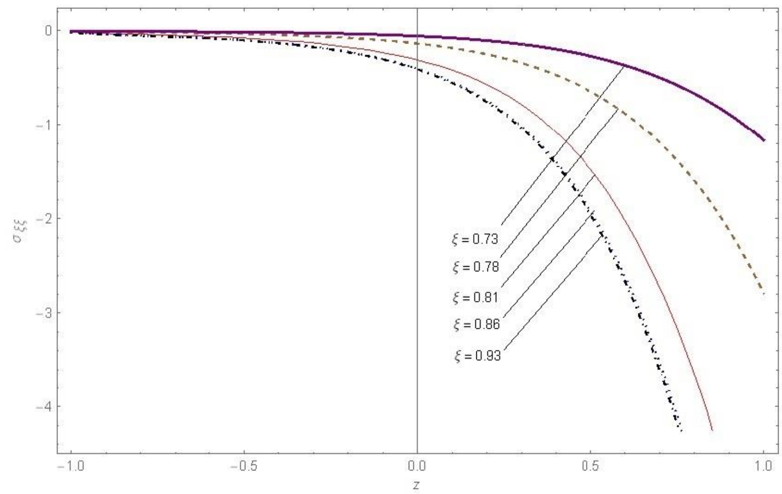

Figure 2(b). $\sigma_{\xi\xi}$ along *z* for different values of ξ

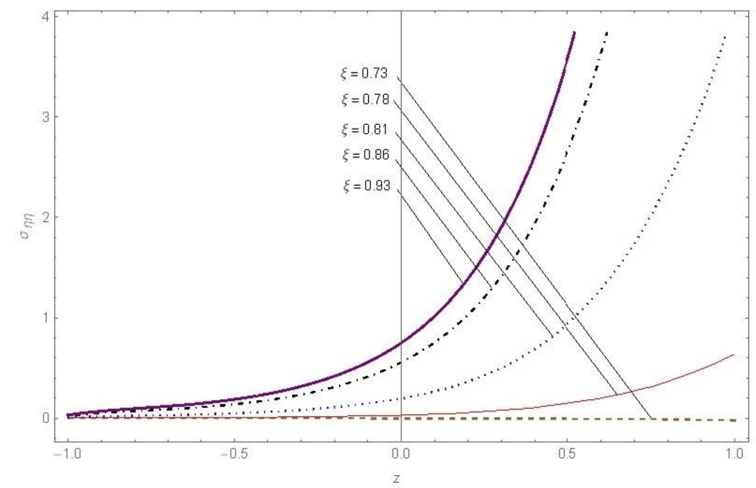

Figure 2(c). $\sigma_{\eta\eta}$ along *z* for different values of ξ

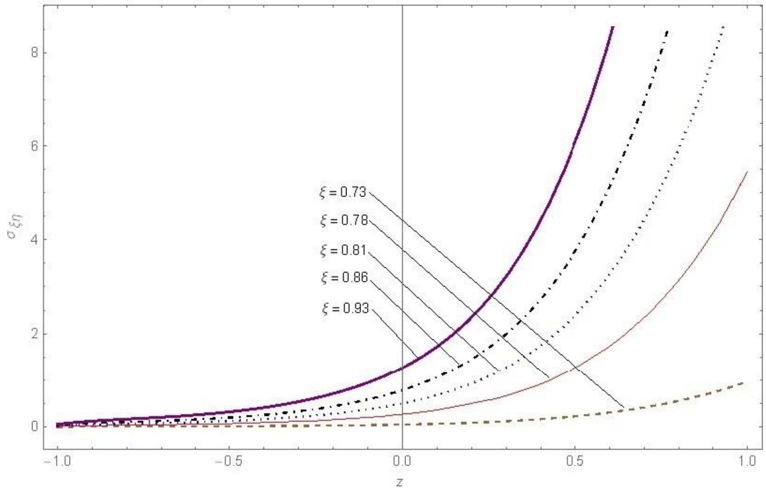

Figure 2(d). $\sigma_{\xi\eta}$ along *z* for different values of ξ

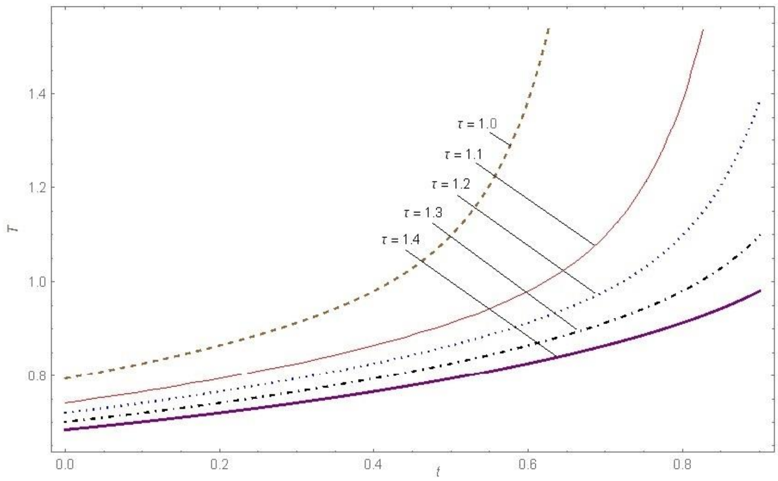

Figure 3 (a). Temperature distribution along *t* for different values of τ

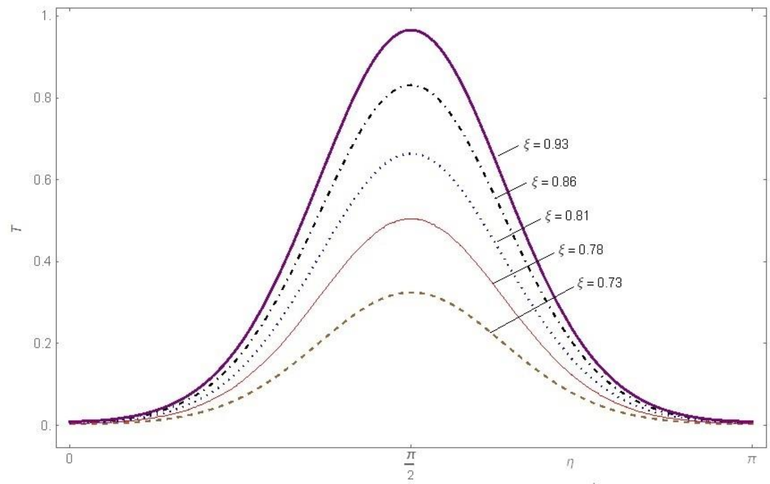

Figure 3(b). Temperature distribution along η for different values of ξ

6. Conclusion

The proposed operational method explains retrospective (i.e. time-reversal ones) heat conduction and its associated thermal stresses with heat generation in the elliptical coordinate system. In this paper, initially we considered the known temperature distribution at a given time τ and we needed to determine the initial temperature distribution.

The results obtained while carrying out our research can be generalized as follows,

- The advantage of this method is its generality and its mathematical power to handle different types of mechanical and thermal boundary conditions during time-reversal process.
- The maximum tensile stress shifts from the outer surface due to maximum expansion at the outer part of the plate and its absolute value increases with radius. This could be due to heat, stress, concentration or available internal heat sources under the known temperature field.
- Finally, the maximum tensile stress occurs in the circular core on the major axis compared to elliptical central part indicating the distribution of weak heating. It might be due to insufficient penetration of heat through the elliptical inner surface.

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REFERENCES

Bhonsle B.R., Prasad R.D. and Thakare N.K. (1976). A unified presentation of integral transforms with classical orthogonal polynomials as kernels, Ind. J. of Pure Appl. Maths. 7(1), 717-727.

Bagde S.D. and Khobragade N.W. (2013). Time reversal problem of heat conduction for elliptic cylindrical shell with internal heat source, International Journal of Scientific & Engineering Research. 4(2), 1-3.

Choubey N.K. (1969). Time reversal problem of heat conduction for a solid elliptical cylinder, Journal of Science and Engineering Research. 13, 258-261.

Debnath L. (1963). On Jacobi transforms, Bull. Cal. Math. Soc. 55(3), 113-120.

Fujiwara I. (1966). A unified presentation of classical orthogonal polynomials, Math. Japan. 11, 133-148.

Fink M. (1992). Time reversal of Ultrasonic fields-Part I: basic principles, IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control, 39(5), 555-566.

Fink M. (1996). Time reversal in acoustics, Contemp. Phys., 37(2), 95–109.

Fink M. (1997). Time reversed acoustics, Phys. Today, 50(3), 34–40.

Fink M. (2006). Time reversal acoustics in geophysics, Geophysics, 71(3), S151–S164.

Fouque J.-P., Garnier J., Papanicolaou G. and Solna K. (2007). Wave propagation and time reversal in randomly layered media, Springer-Verlag New York.

Gupta R.K. (1964). A finite transform involving Mathieu functions and its application, Proc. Net. Inst. Sc., India, Part A, 30(6), 779-795.

Larmat C. and Guyer R. (2010). Time-reversal methods in geophysics, Physics Today, 63(8), 31-35.

Masket A.V. (1965). Time Reversal in Heat Conduction, American Journal of Physics. 33(3), 196-201.

McLachlan N.W. (1947). Theory and Application of Mathieu function, Clarendon press, Oxford.

Mehta B.S. (1976). Time-reversal problem of heat conduction in solids of spherical symmetry, Def. Sci. J. 26(2), 81-86.

Misra J.C. (1971). A note on the thermal stresses in an elliptic plate due to a source of heat at the centre of the plate, the edge being stress-free and insulated, Bull. Cal. Math. Soc. 63(2), 135-141.

Montagner J.-P., Larmat C., Capdeville Y., Fink M., Phung H., Romanowicz B., Clevede E. and Kawakatsu H. (2012). Time-reversal method and cross-correlation techniques by normal mode theory: a three-point problem, Geophys. J. Int. 1-16.

Mora N., Rachidi F. and Rubinstein M. (2012). Application of the time reversal of electromagnetic fields to locate lightning discharges, Atmospheric Research, 117, 78-85.

Patel S.R. (1978). Time-reversal problems in transient heat conduction with radiation as boundary conditions, Labdev J. Sci. Tech. Part A, 13(3-4), 232-240.

Reyes-Rodríguez S., Lei N., Crowgey B. and Udpa S.S. (2014) Time reversal and microwave techniques for solving inverse problem in Non-Destructive Evaluation, NDT & E International, 62, 106–114.

Sabharwal K.C. (1965). Some time-reversal problems of heat conduction, Indian J. pure and applied physics. 3, 449-450.

Varghese V. and Khobragade N. W. (2007). Alternative solution of transient heat conduction in a circular plate with radiation, Int. J. Appl. Math., 20 (8), 1133-1140.

Appendix

The required integral transforms

(I) Gupta (1964) has defined an integral transform analogous to finite Hankel transform for a continuous and single valued function $f(\xi,\eta)$ in the region $\xi_i \le \xi \le \xi_o,\ 0 \le \eta \le 2\pi$ which vanishing on both the boundaries $\xi = \xi_i$ and $\xi = \xi_o$, as

$$\begin{aligned}\bar{f}(q_{n,m}) = \int_{\xi_i}^{\xi_o} \int_0^{2\pi} & (\cosh 2\xi - \cos 2\eta)\, B_{2n}(\xi, q_{2n,m}) \\ & \times ce_{2n}(\eta, q_{2n,m})\, f(\xi,\eta)\, d\xi\, d\eta,\end{aligned} \tag{A1}$$

in which

$$\begin{aligned}B_{2n}(\xi, q_{2n,m}) = [&\{Fey_{2n}(\xi_i, q_{2n,m}) - Fey_{2n}(\xi_o, q_{2n,m})\}Ce_{2n}(\xi, q_{2n,m}) \\ &- \{Ce_{2n}(\xi_i, q_{2n,m}) - Ce_{2n}(\xi_o, q_{2n,m})\}Fey_{2n}(\xi, q_{2n,m})],\end{aligned} \tag{A2}$$

and $q_{2n,m}$ is the root of the transcendental equation

$$Ce_{2n}(\xi_o, q_{2n,m}) Fey_{2n}(\xi_i, q_{2n,m}) - Fey_{2n}(\xi_o, q_{2n,m}) Ce_{2n}(\xi_i, q_{2n,m}) = 0 . \quad \text{(A3)}$$

Here $Ce_{2n}(\xi_0, q) = 0$, $ce_n(\eta, q)$ [McLachlan (1947, pp.21)] is a Mathieu function of the first kind of order *n*, $Ce_n(\xi, q)$ [McLachlan (1947, pp.27)] is a modified Mathieu function of the first kind of order *n*. Hence the inversion formula for aforementioned transform is given as

$$f(\xi, \eta) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \bar{f}(q_{2n,m}) Be_{2n}(\xi, q_{2n,m}) ce_{2n}(\eta, q_{2n,m}) / C_{n,m}, \quad \text{(A4)}$$

in which the sum is extended over all the positive roots of equation (A3),

$$C_{n,m} = \pi \int_{\xi_i}^{\xi_o} (\cosh 2\xi - \Theta_{2n,m}) B_{2n}^2(\xi, q_{2n,m}) d\xi , \quad \text{(A5)}$$

and

$$\Theta_{2n,m} = \frac{1}{\pi} \int_0^{2\pi} \cos 2\eta \, ce_{2n}^2(\eta, q_{2n,m}) d\eta . \quad \text{(A6)}$$

(II) Debnath (1963) introduced the Jacobi transform with $(1-z)^{\alpha}(1+z)^{\beta}$ as weight function of the Kernel as

$$f^{(\alpha,\beta)}(z) = \int_{-1}^{1} (1-z)^{\alpha} (1+z)^{\beta} P_{\gamma}^{(\alpha,\beta)}(z) F(z) dz , \quad \text{(A7)}$$

and its anti-transforming formula as

$$F(z) = \sum_{\gamma=0}^{\infty} f^{(\alpha,\beta)}(\gamma) P_{\gamma}^{(\alpha,\beta)}(z) / \delta_{\gamma} , \quad \text{(A8)}$$

in which $P_{\gamma}^{(\alpha,\beta)}(z)$ is the Jacobi polynomial of degree γ and orders $\alpha(>-1)$ and $\beta(>-1)$ and

$$\delta_{\gamma} = \frac{2^{\alpha+\beta+1} \Gamma(\gamma+\alpha+1) \Gamma(\gamma+\beta+1)}{\gamma! (2\gamma+\alpha+\beta+1) \Gamma(\gamma+\alpha+\beta+1)} , \quad \text{(A9)}$$

satisfying the differential equation

$$(1-z^2) y_{,zz} + \{(\beta-\alpha) - (\alpha+\beta+2)z\} y_{,z} + \gamma(\gamma+\alpha+\beta+1) y = 0 . \quad \text{(A10)}$$

Moreover the integral transform has the following orthogonal property for $P_{\gamma}^{(\alpha,\beta)}(z)$ as

$$\int_{-1}^{1}(1-z)^{\alpha}(1+z)^{\beta}P_{\gamma}^{(\alpha,\beta)}(z)P_{\not\gamma}^{(\alpha,\beta)}(z)\,dz \begin{cases} =0, & \gamma \neq \not\gamma, \\ =\delta_{\gamma}, & \gamma = \not\gamma. \end{cases} \quad \text{(A11)}$$

(III) **The extended transformation and its essential property**

Following Fujiwara (1966) and Bhonsle (1976) we explicitly show in this paper that the study of integral transforms involving classical orthogonal polynomials as kernel can be unified. Analogous to the finite Mathieu transform defined by Gupta (1964) in above section 3, we introduce a repeated integral transform with

$$(\cosh 2\xi - \cos 2\eta)\, B_{2n}(\xi, q_{2n,m})\, ce_{2n}(\eta, q_{2n,m})(1-z)^{\alpha}(1+z)^{\beta} P_{\gamma}^{(\alpha,\beta)}(z), \quad \text{(A12)}$$

as a kernel defined over the range $\xi_i \le \xi \le \xi_o$, $0 \le \eta \le 2\pi$ and $-1 \le z \le 1$.

Now if $\overline{M}(\alpha,\beta,\gamma,\xi_i,\xi_o,q_{2n,m})$ be the kernel of $M(\xi,\eta,z)$ then, Mathieu-Jacobi transformation can be proposed as (13) and its inversion theorem can be defined as (14). The basic operational properties holds - if (i) the function $M(\xi,\eta,z)$ is in the domain $\xi_i \le \xi \le \xi_o$, $0 \le \eta \le 2\pi$ and $-1 \le z \le 1$. (ii) $M(\xi,\eta,z)$ satisfies Dirichlet conditions for $\xi_i \le \xi \le \xi_o$, $0 \le \eta \le 2\pi$. (iii) $M(\xi,\eta,z), \partial M/\partial z$ are bounded in the above said domain. (iv) $M,_{zz}$ be bonded and integrable in each of the sub-interval of $\xi_i \le \xi \le \xi_o$, $0 \le \eta \le 2\pi$ and $-1 \le z \le 1$, (v) $\overline{M}(\alpha,\beta,\gamma,\xi_i,\xi_o,q_{2n,m})$ the Mathieu-Jacobi transform of $M(\xi,\eta,z)$ exists, (vi) $M(\xi,\eta,z)=0$ at $\xi=\xi_o$, (vii) $M,_{\xi}=0$ at $\xi=\xi_o$, and

(viii) $\lim_{z\to\pm1}(1-z)^{\alpha+1}(1+z)\beta^{\alpha+1}M(\xi,\eta,z) = \lim_{z\to\pm1}(1-z)^{\alpha+1}(1+z)\beta^{\alpha+1}M,_{z}=0$,

Then, from equation (A11), we obtain

$$(1-z^2)\,\overline{M},_{zz}+\{(\beta-\alpha)-(\alpha+\beta+2)z\}\overline{M},_{z} = -\gamma(\gamma+\alpha+\beta+1)\overline{M}.$$