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On a double integral involving the I-function of two variables

Shantha K. Kumari[1] and Vasudevan T. M. Nambisan[2]
[1]Department of Mathematics
P.A. College of Engineering
Mangalore, INDIA
shanthapace@gmail.com
[2]Department of Mathematics
College of Engineering and Technology
Payyanur, INDIA
tmvasudevannambisan@yahoo.com

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Abstract

In this paper we establish an interesting double integral involving the I-function of two variables recently introduced in the literature. Since *I*-function of two variables is a very generalized function of two variables and it includes as special cases many of the known functions appearing in the literature, a number of integrals can be obtained by reducing the I-function of two variables to simpler special functions by suitably specializing the parameters. A few special cases of our result are also discussed.

Keywords: I-function; Mellin-Barnes Contour integral; H-function; Double integral

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1. Introduction

In 1997, Rathie (1997) introduced a generalization of the *H*-function of Fox (1961) in the literature namely the "*I*-function" which is useful in Mathematics, Physics and other branches of Applied Mathematics. Recently, the I-function introduced by Rathie (1997) and the generalized hypergeometric function of two variables introduced by Agarwal (1965) and

Sharma (1965) have found interesting and useful applications in wireless communications. [See Ansari et al. (2013, 2014) and Xia et al. (2012)].

Motivated by this, very recently, Shantha Kumari et al. (2014) introduced the *I*-function of two variables, which gives a natural generalization of *H*-function of two variables introduced by Mittal and Gupta (1972).

The *I*-function of two variables defined and studied by Kumari et al. (2014) is represented by means of the double Mellin-Barnes contour integral as follows:

$$\mathrm{I}\,[z_1, z_2] = \mathrm{I}^{0,n_1:m_2,n_2;\ m_3,n_3}_{p_1,q_1:p_2,q_2;\ p_3,q_3}\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\middle|\begin{matrix} (a_j;\alpha_j,A_j;\xi_j)_{1,p_1} : (c_j,C_j;U_j)_{1,p_2};\ (e_j,E_j;P_j)_{1,p_3} \\ (b_j;\beta_j,B_j;\eta_j)_{1,q_1} : (d_j,D_j;V_j)_{1,q_2};(f_j,F_j;Q_j)_{1,q_3} \end{matrix}\right]$$

$$= \frac{1}{(2\pi \mathrm{i})^2}\int_{L_s}\int_{L_t} \phi(s,t)\ \theta_1(s)\ \theta_2(t)\ z_1^s\ z_2^t ds\, dt\ , \tag{1.1}$$

where $\phi(s,t)$, $\theta_1(s)$ and $\theta_2(t)$ are given by

$$\phi(s,t) = \frac{\prod_{j=1}^{n_1}\Gamma^{\xi_j}(1-a_j+\alpha_j s+A_j t)}{\prod_{j=n_1+1}^{p_1}\Gamma^{\xi_j}(a_j-\alpha_j s-A_j t)}\frac{1}{\prod_{j=1}^{q_1}\Gamma^{\eta_j}(1-b_j+\beta_j s+B_j t)}\ , \tag{1.2}$$

$$\theta_1(s) = \frac{\prod_{j=1}^{n_2}\Gamma^{U_j}(1-c_j+C_j s)}{\prod_{j=n_2+1}^{p_2}\Gamma^{U_j}(c_j-C_j s)}\frac{\prod_{j=1}^{m_2}\Gamma^{V_j}(d_j-D_j s)}{\prod_{j=m_2+1}^{q_2}\Gamma^{V_j}(1-d_j+D_j s)}\ , \tag{1.3}$$

$$\theta_2(t) = \frac{\prod_{j=1}^{n_3}\Gamma^{P_j}(1-e_j+E_j t)}{\prod_{j=n_3+1}^{p_3}\Gamma^{P_j}(e_j-E_j t)}\frac{\prod_{j=1}^{m_3}\Gamma^{Q_j}(f_j-F_j t)}{\prod_{j=m_3+1}^{q_3}\Gamma^{Q_j}(1-f_j+F_j t)}\ . \tag{1.4}$$

Moreover,

(i) $z_1 \neq 0,\ z_2 \neq 0$;
(ii) $\mathrm{i} = \sqrt{-1}$;
(iii) an empty product is interpreted as unity;
(iv) the parameters $n_j, p_j, q_j (j = 1,2,3)$, $m_j (j = 2,3)$ are nonnegative integers such that $0 \le n_j \le p_j (j = 1,2,3)$, $q_1 \ge 0$, $0 \le m_j \le q_j (j = 2,3)$ (not all zero simultaneously);
(v) $\alpha_j, A_j (j = 1,\dots,p_1)$, $\beta_j, B_j\ (j = 1,\dots,q_1)$, $C_j (j = 1,\dots,p_2)$, $D_j (j = 1,\dots,q_2)$, $E_j (j = 1,\dots,p_3)$, $F_j (j = 1,\dots,q_3)$ are assumed to be positive quantities for standardization purpose.
(vi) $a_j (j = 1,\dots,p_1)$, $b_j (j = 1,\dots,q_1)$, $c_j (j = 1,\dots,p_2)$, $d_j (j = 1,\dots,q_2)$, $e_j (j = 1,\dots,p_3)$ and $f_j (j = 1,\dots,q_3)$ are complex numbers;
(vii) The exponents $\xi_j (j = 1,\dots,p)$, $\eta_j (j = 1,\dots,q)$, $U_j\ (j = 1,\dots,p_2)$, $V_j (j = 1,\dots,q_2)$, $P_j (j = 1,\dots,p_3)$, $Q_j (j = 1,\dots,q_3)$ of various gamma functions involved in (1.2), (1.3) and (1.4) may take non-integer values.

(viii) L_s and L_t are suitable contours of Mellin - Barnes type. Moreover, the contour L_s is in the complex s-plane and runs from $\sigma_1 - i\infty$ to $\sigma_1 + i\infty$, (σ_1 real) so that all the singularities of $\Gamma^{V_j}(d_j - D_j s)(j = 1, \dots, m_2)$ lie to the right of L_s and all the singularities of $\Gamma^{U_j}(1 - c_j + C_j s)(j = 1, \dots, n_2)$, $\Gamma^{\xi_j}(1 - a_j + \alpha_j s + A_j t)(j = 1, \dots, n_1)$ lie to the left of L_s; The other contour L_t follows similar conditions in the complex t-plane.

The function defined by (1.1) is an analytic function of z_1 and z_2 if

$$R = \sum_{j=1}^{p_1} \xi_j \alpha_j + \sum_{j=1}^{p_2} U_j C_j - \sum_{j=1}^{q_1} \eta_j \beta_j - \sum_{j=1}^{q_2} V_j D_j < 0, \tag{1.5}$$

$$S = \sum_{j=1}^{p_1} \xi_j A_j + \sum_{j=1}^{p_3} P_j E_j - \sum_{j=1}^{q_1} \eta_j B_j - \sum_{j=1}^{q_3} Q_j F_j < 0. \tag{1.6}$$

Further, the integral (1.1) is convergent if

$$\Delta_1 = \Big[\sum_{j=1}^{n_1} \xi_j \alpha_j - \sum_{j=n_1+1}^{p_1} \xi_j \alpha_j - \sum_{j=1}^{q_1} \eta_j \beta_j + \sum_{j=1}^{n_2} U_j C_j - \sum_{j=n_2+1}^{p_2} U_j C_j + \sum_{j=1}^{m_2} V_j D_j - \sum_{j=m_2+1}^{q_2} V_j D_j\Big] > 0, \tag{1.7}$$

$$\Delta_2 = \Big[\sum_{j=1}^{n_1} \xi_j A_j - \sum_{j=n_1+1}^{p_1} \xi_j A_j - \sum_{j=1}^{q_1} \eta_j B_j + \sum_{j=1}^{n_3} P_j E_j - \sum_{j=n_3+1}^{p_3} P_j E_j + \sum_{j=1}^{m_3} Q_j F_j - \sum_{j=m_3+1}^{Q_3} Q_j F_j\Big] > 0, \tag{1.8}$$

$$|\arg(z_1)| < \frac{1}{2}\Delta_1 \pi, \quad |\arg(z_2)| < \frac{1}{2}\Delta_2 \pi. \tag{1.9}$$

In this paper and for the sake of brevity, we shall use the following contracted notation for the I-function defined in (1.1) :

$$\mathrm{I}[z_1, z_2] = \mathrm{I}\,{}^{0, n_1 : m_2, n_2;\ m_3, n_3}_{p_1, q_1 : p_2, q_2;\ p_3, q_3}\left[{}^{z_1}_{z_2}\middle|{}^{\mathcal{A}:\mathcal{C};\ \mathcal{E}}_{\mathcal{B}:\mathcal{D};\ \mathcal{F}}\right]. \tag{1.10}$$

Further, if $V_j = 1 (j = 1, \dots, m_2)$, $Q_j = 1 (j = 1, \dots, m_3)$ in (1.1), then the function will be denoted by

$$\overline{\mathrm{I}}[z_1, z_2] = \mathrm{I}\,{}^{0, n_1 : m_2, n_2;\ m_3, n_3}_{p_1, q_1 : p_2, q_2;\ p_3, q_3}\left[{}^{z_1}_{z_2}\middle|{}^{\mathcal{A}:\mathcal{C};\ \mathcal{E}}_{\mathcal{B}:\acute{\mathcal{D}};\ \acute{\mathcal{F}}}\right], \tag{1.11}$$

where

- $\mathcal{A}$ stands for $(a_j; \alpha_j, A_j; \xi_j)_{1,p_1} \equiv (a_1; \alpha_1, A_1; \xi_1), \dots, (a_{p_1}; \alpha_{p_1}, A_{p_1}; \xi_{p_1})$;
- $\mathcal{B}$ stands for $(b_j; \beta_j, B_j; \eta_j)_{1,q_1} \equiv (b_1; \beta_1, B_1; \eta_1), \dots, (b_{q_1}; \beta_{q_1}, B_{q_1}; \eta_{q_1})$;
- $\mathcal{C}$ stands for $(c_j, C_j; U_j)_{1,p_2} \equiv (c_1, C_1; U_1), \dots, (c_{p_2}, C_{p_2}; U_{p_2})$;
- $\mathcal{D}$ stands for $(d_j, D_j; V_j)_{1,q_2} \equiv (d_1, D_1; V_1), \dots, (d_{q_2}, D_{q_2}; V_{q_2})$;

- $\mathcal{E}$ stands for $\left(e_j, E_j; P_j\right)_{1,p_3} \equiv (e_1, E_1; P_1), \dots, \left(e_{p_3}, E_{p_3}; P_{p_3}\right)$;
- $\mathcal{F}$ stands for $\left(f_j, F_j; Q_j\right)_{1,q_3} \equiv (f_1, F_1; Q_1), \dots, \left(f_{q_3}, F_{q_3}; Q_{q_3}\right)$;
- $\acute{\mathcal{D}}$ stands for $\left(d_j, D_j; 1\right)_{1,m_2}$, $\left(d_j, D_j; V_j\right)_{m_2+1,\, q_2}$;
- $\acute{\mathcal{F}}$ stands for $\left(f_j, F_j; 1\right)_{1,m_3}$, $\left(f_j, F_j; Q_j\right)_{m_3+1,\, q_3}$.

A more detailed account of *I*-function, its behaviour and various special cases in one and two variables can be found in the paper by Shantha Kumari et al. (2014).

For a detailed study of some double Mellin-Barnes type integrals known as general H-functions of two variables and their applications in convolution theory, we refer a book by Hai and Yakubovich (1992) to the readers.

2. Results Required

We recall an interesting double integral recorded in Edwards (1922, p. 145):

$$\int_0^1 \int_0^1 \frac{y^{\alpha}\,(1-x)^{\alpha-1}\,(1-y)^{\beta-1}}{(1-xy)^{\alpha+\beta-1}}\, dx\, dy = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

provided $\Re(\alpha) > 0$ and $\Re(\beta) > 0$.

3. Main Result

In this section, the following general double integral will be established.

$$\int_0^1 \int_0^1 \left\{ \frac{y^{\alpha}\,(1-x)^{\alpha-1}\,(1-y)^{\beta-1}}{(1-xy)^{\alpha+\beta-1}} \times \bar{\mathrm{I}}\left[\frac{z_1\, y^{\lambda_1}\,(1-x)^{\lambda_1-1}\,(1-y)^{\mu_1}}{(1-xy)^{\lambda_1+\mu_1}},\ \frac{z_2\, y^{\lambda_2}\,(1-x)^{\lambda_2-1}\,(1-y)^{\mu_2}}{(1-xy)^{\lambda_2+\mu_2}}\right]\right\} dx$$
$$= \mathrm{I}^{\,0,\ n_1+2:\ m_2, n_2;\ m_3, n_3}_{\,p_1+2, q_1+1:\ p_2, q_2;\ p_3, q_3}\left[\begin{matrix}\frac{z_1}{2^{2\lambda_1}} \\ \frac{z_2}{2^{2\lambda_2}}\end{matrix}\middle|\begin{matrix}(1-\alpha, \lambda_1, \lambda_2\,; 1), (1-\beta;\ \mu_1,\ \mu_2; 1), \mathcal{A}: \mathcal{C};\ \mathcal{E} \\ \mathcal{B},\ (1-\alpha-\beta;\ \lambda_1+\mu_1,\ \lambda_2+\ \mu_2;\ 1): \acute{\mathcal{D}};\ \acute{\mathcal{F}}\end{matrix}\right], \tag{3.1}$$

provided

(i) $\lambda_1 \geq 0,\ \lambda_2 \geq 0$ (both λ_1 and λ_2 are not simultaneously zero);
(ii) $\mu_1 \geq 0,\ \mu_2 \geq 0$ (both μ_1 and μ_2 are not simultaneously zero);
(iii) The conditions given in (1.7), (1.8) and (1.9) are satisfied with $V_j = 1 (j = 1, \dots, m_2),\ Q_j = 1 (j = 1, \dots, m_3)$;
(iv) $\Re\left[\alpha + \lambda_1\left(\frac{d_i}{D_i}\right) + \lambda_2\left(\frac{f_j}{F_j}\right)\right] > 0;\ i = 1, \dots, m_2,\ j = 1, \dots, m_3$;
(v) $\Re\left[\beta + \mu_1\left(\frac{d_i}{D_i}\right) + \mu_2\left(\frac{f_j}{F_j}\right)\right] > 0,\ i = 1, \dots, m_2;\ j = 1, \dots, m_3$.

Proof:

In order to prove the double integral (3.1), we proceed as follows. For this, denoting the left-hand side of (3.1) by S, we first express the $\bar{\mathrm{I}}$ - function of two variables in the integrand of (3.1) by its Mellin-Barnes contour integral given by (1.1). Now changing the order of integration which is permissible under the conditions stated with (3.1), we obtain the expression

$$S = \frac{1}{(2\pi i)^2} \int_{L_s} \int_{L_t} \Big[\phi(s,t)\, \theta_1(s)\, \theta_2(t)\, z_1^s\, z_2^t$$

$$\times \int_0^1 \int_0^1 \left\{ \frac{y^{\alpha+\lambda_1 s+\lambda_2 t-1}\,(1-x)^{\alpha+\lambda_1 s+\lambda_2 t-1}\,(1-y)^{\beta+\mu_1 s+\mu_2 t-1}}{(1-xy)^{(\alpha+\beta)+(\lambda_1+\mu_1)s+(\lambda_2+\mu_2)t-1}} \right\} dxdy \Big] \; dsdt \qquad (3.2)$$

Evaluating the inner double integral with the help of result (2.1) we obtain

$$S = \frac{1}{(2\pi i)^2} \int_{L_s} \int_{L_t} \phi(s,t)\, \theta_1(s)\, \theta_2(t)\, z_1^s\, z_2^t\, \frac{\Gamma(\alpha+\lambda_1 s+\lambda_2 t)\, \Gamma(\beta+\mu_1 s+\mu_2 t)}{\Gamma(\alpha+\beta+(\lambda_1+\mu_1)s+(\lambda_2+\mu_2)t)}\, ds\, dt \qquad (3.3)$$

Now interpreting the Mellin-Barnes contour integral in (3.3) with the help of the definition of the I-function of two variables (1.1), we arrive at the desired result.

4. Special Cases

On account of very general nature of the I-function of two variables, it includes as special cases many of the known functions appearing in the literature and hence the integral derived in this paper will serve as the key formula from which a large number of known and unknown results can be obtained by reducing the I-function of two variables into simpler special functions by suitably specializing the parameters. However, here we shall mention some of these results.

Special Case 4.1.

If $\mu_1 = \mu_2 = 0$, then (3.1) takes the following form:

$$\int_0^1 \int_0^1 \left\{ \frac{y^{\alpha}\,(1-x)^{\alpha-1}\,(1-y)^{\beta-1}}{(1-xy)^{\alpha+\beta-1}} \right.$$

$$\left. \times\, \bar{\mathrm{I}} \left[\frac{z_1\, y^{\lambda_1}\,(1-x)^{\lambda_1}}{(1-xy)^{\lambda_1}}, \; \frac{z_2\, y^{\lambda_2}\,(1-x)^{\lambda_2}}{(1-xy)^{\lambda_2}} \right] \right\} dx\, dy$$

$$= \Gamma(\beta)\, \mathrm{I}^{0,\, n_1+1:\, m_2, n_2;\, m_3, n_3}_{p_1+1, q_1+1:\, p_2, q_2;\, p_3, q_3} \left[\begin{matrix} z_1 \\ z_2 \end{matrix} \middle| \begin{matrix} (1-\alpha, \lambda_1, \lambda_2\,; 1),\; \mathcal{A} : \mathcal{C};\, \mathcal{E} \\ \mathcal{B}, (1-\alpha-\beta;\, \lambda_1,\, \lambda_2;\, 1) : \acute{\mathcal{D}};\, \acute{\mathcal{F}} \end{matrix} \right], \qquad (4.1)$$

provided that the conditions easily obtainable from (3.1) with $\mu_1 = \mu_2 = 0$ are satisfied.

Special Case 4.2.

If $\lambda_1 = \lambda_2 = 0$, then (3.1) takes the following form:

$$\int_0^1 \int_0^1 \left\{ \frac{y^{\alpha}\,(1-x)^{\alpha-1}\,(1-y)^{\beta-1}}{(1-xy)^{\alpha+\beta-1}} \right.$$

$$\left. \times\ \bar{\mathrm{I}} \left[\frac{z_1\ (1-y)^{\mu_1}}{(1-xy)^{\mu_1}}\ ,\ \ \frac{z_2\,(1-y)^{\mu_2}}{(1-xy)^{\mu_2}} \right] \right\} dx\, dy$$

$$= \Gamma(\alpha)\ \mathrm{I}\ \begin{matrix} 0,\ n_1+1:\ m_2, n_2;\ m_3, n_3 \\ p_1+1, q_1+1: p_2, q_2;\ p_3, q_3 \end{matrix} \left[\begin{matrix} z_1 \\ z_2 \end{matrix} \middle| \begin{matrix} (1-\beta,\ \mu_1,\ \mu_2\,;1),\ \ \mathcal{A}\ :\ \mathcal{C};\ \mathcal{E} \\ \mathcal{B},\ (1-\alpha-\beta;\mu_1,\ \ \mu_2;\ 1)\ :\ \acute{\mathcal{D}};\ \acute{\mathcal{F}} \end{matrix} \right], \quad (4.2)$$

provided that the conditions easily obtainable from (3.1) with $\lambda_1 = \lambda_2 = 0$ are satisfied.

Special Case 4.3.

When all exponents $\xi_j (j = 1, \dots, p_1)$, $\eta_j (j = 1, \dots, q_1)$, $U_j (j = 1, \dots, p_2)$, $V_j (j = 1, \dots, q_2)$, $P_j (j = 1, \dots, p_3)$, $Q_j (j = 1, \dots, q_3)$ are equal to unity, the I-function of two variables reduces to the *H*-function of two variables defined by Mittal and Gupta (1972) and therefore we obtain the corresponding double integrals involving H- function of two variables recorded in (Srivastava et al. (1982)).

Special Case 4.4

If we take $p_1 = q_1 = n_1 = n_3 = p_3 = f_1 = 0$, $m_3 = q_3 = 1$, $E_j = F_j = P_j = Q_j = 1$ and let $z_2 \to 0$, and further specializing the parameters of (4.1) and (4.2), these results reduce to double integrals involving I-function of one variable introduced by Rathie, (1997).

5. Conclusion

In this research paper we have evaluated a double integral involving the I-function of two variables recently introduced by Shantha Kumari et al. (2014). The integral established in this paper is of very general nature as it contains I-functions of two variables, which is a very general function of two variables studied so far. Thus, the integral established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving special functions of one and two variables can be obtained. Since the generalized function of two variables introduced by Agarwal (1965) and Sharma(1965) have found interesting applications in wireless communication (see Xia et al. (2012)) the double integral evaluated in this paper may be potentially useful.

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