Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Vol. 6, Issue 1 (June 2011) pp. 232 – 239
(Previously, Vol. 6, Issue 11, pp. 1973 – 1980)

Applications and Applied Mathematics:
An International Journal
(AAM)

Positive Solutions for *nth* Order Differential Equations Under Some Conditions

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Received: January 13, 2011; Accepted: May 26, 2011

Abstract

This paper presents an existence and nonexistence of positive solutions for the nonlinear boundary value problems. We prove that the *nth* order nonlinear differential equation has at least one positive solution by using appropriate fixed point theorems.

Keywords: Fixed-point theorem, Banach space, nonlinear *nth* order differential equations, Positive solutions

AMS 2010 No.: 34 B15, 34 B18

1. Introduction

We are concerned, in this paper, with the existence of positive solutions for the following boundary value problem for *nth* order differential equations

$$u^{(n)}(t) = f\ (t, u(t)), \qquad 0 < t < 1\ , \tag{1.1}$$

$$u(0) = u'(0) = u'''(0) = \cdots = u^{(n-2)}(0) = u^{(n-1)}(1) = 0, \tag{1.2}$$

$$\alpha u'(1) + \beta u''(1) = 0\ ,\ \text{where}\quad \beta, \alpha \geq 0,\quad \alpha + \beta > 0 \tag{1.3}$$

Problems of the above type occur frequently in science, engineering, mathematical physics, economics and biology [Zhang (2006)].

The nonlinear *nth* order differential equations studied in this paper are an existence and nonexistence of positive solutions by using object of mathematical investigations [El-shahed (2009), El-Shahed and Hassan (2010), Guo and Lakshmikantham (1988), Sun and Wen (2006), Agarwal and O'Regan (1999) and Agarwal et al. (1999)]. However, there are few papers investigating the existence of positive solutions of nth impulsive differential equations by using the fixed point theorem of cone expansion and compression. The objective of the present paper is to fill this gap and the results presented are new and original. Also, several results obtained in Agarwal et al. (1999) are generalized.

2. Notation, Definition and Auxiliary Results

Theorem 2.1 [Agarwal et al. (2001), Agarwal and O'Regan (1999)]:

Assume that U is a relatively open subset of convex set K in Banach space E. Let

$N : \overline{U} \to K$ be a compact map with $0 \in U$. Then, either

(i) N has a fixed point in $\overline{U}$; or

(ii) There is $u \in U$ and $\lambda \in (0,1)$ such that $u = \lambda\, N\, u$.

At first, we find the solution $u(t)$**,** for the problem

$$u^{(n)}(t) = y(t), \qquad 0 < t < 1\ , \tag{2.1}$$

$$u(0) = u'(0) = u'''(0) = \cdots = u^{(n-2)}(0) = u^{(n-1)}(1) = 0, \tag{2.2}$$

$$\alpha u'(1) + \beta u''(1) = 0\ ,\ \text{where}\quad \beta, \alpha \geq 0,\quad \alpha + \beta > 0 \quad . \tag{2.3}$$

Applying Laplace transforms to equation (2.1), we have:

$$s^n \overline{u}(s) - s^{n-1}u(0) - s^{n-2}u'(0) - s^{n-3}u''(0) - \cdots - s\ u^{(n-2)}(0) - u^{(n-1)}(0) = \overline{y}(s)\,, \tag{2.4}$$

where $\bar{u}(s)$ and $\bar{y}(s)$ is the Laplace transform of $u(t)$ and y(t) respectively. So,

$$\bar{u}(s) = \frac{s^{n-1}u(0)}{s^n} + \frac{s^{n-2}u'(0)}{s^n} + \frac{s^{n-3}u''(0)}{s^n} + \cdots + \frac{su^{(n-2)}(0)}{s^n} + \frac{u^{(n-1)}(0)}{s^n} + \frac{\bar{y}(s)}{s^n}.$$

Therefore, by the inverse Laplace transform and using the boundary condition (2.2) and (2.3), we obtain the final form of $u(t)$ as:

$$u(t) = \frac{\alpha + (n-2)\beta}{[\alpha+\beta](n-2)!}\int_0^1 \frac{t^2}{2!} y(s)ds - \frac{\alpha}{[\alpha+\beta](n-2)!}\int_0^1 \frac{t^2(1-s)^{n-2}}{2!} y(s)ds -$$

$$\frac{\beta}{[\alpha+\beta](n-3)!}\int_0^1 \frac{t^2(1-s)^{n-3}}{2!} y(s)ds - \int_0^1 \frac{t^{n-1}}{(n-1)!} y(s)ds + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s)ds. \tag{2.5}$$

Definition 2.1.

An operator is called completely continuous if it is continuous and maps bounded sets into precompacts.

Definition 2.2.

Let E be a real Banach space. A nonempty closed convex set $K \subset E$ is called cone of E if it satisfies the following conditions:

(*i*) $x \in K, \sigma \geq 0$ *implies* $\sigma x \in K$; and

(ii) $x \in K, -x \in K$ implies $x = 0$.

3. Main Result

Consider the family of problems:

$$u^{(n)}(t) = f(t, u(t)), \qquad 0 < \mathrm{t} < 1, \tag{3.1}$$

$$u(0) = u'(0) = u'''(0) = \cdots = u^{(n-2)}(0) = u^{(n-1)}(1) = 0, \tag{3.2}$$

$$\alpha u'(1) + \beta u''(1) = 0, \text{ where } \beta, \alpha \geq 0, \quad \alpha + \beta > 0. \tag{3.3}$$

Hence, (3.1), (3.2) and (3.3) are equivalent to the integral equation

$$\begin{aligned} u(t) = \frac{\alpha+(n-2)\beta}{[\alpha+\beta](n-2)!}\int_0^1 \frac{t^2}{2!} f(s,u(s))ds - \frac{\alpha}{[\alpha+\beta](n-2)!}\int_0^1 \frac{t^2(1-s)^{n-2}}{2!} f(s,u(s))ds \\ - \frac{\beta}{[\alpha+\beta](n-3)!}\int_0^1 \frac{t^2(1-s)^{n-3}}{2!} f(s,u(s))ds - \int_0^1 \frac{t^{n-1}}{(n-1)!} f(s,u(s))ds \\ + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s,u(s))ds. \end{aligned} \tag{3.4}$$

Defining $T : X \to X$ as**:**

$$\begin{aligned} Tu(t) = \frac{\alpha+(n-2)\beta}{[\alpha+\beta](n-2)!}\int_0^1 \frac{t^2}{2!} f(s,u(s))ds - \frac{\alpha}{[\alpha+\beta](n-2)!}\int_0^1 \frac{t^2(1-s)^{n-2}}{2!} f(s,u(s))ds \\ - \frac{\beta}{[\alpha+\beta](n-3)!}\int_0^1 \frac{t^2(1-s)^{n-3}}{2!} f(s,u(s))ds - \int_0^1 \frac{t^{n-1}}{(n-1)!} f(s,u(s))ds + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s,u(s))ds, \end{aligned} \tag{3.5}$$

where $X=C[0,1]$ is the Banach space endowed with the supper norm.

We have the following result for operator T.

Lemma 3.1:

Assume that $f : [0,1]\times R \to R$ is continuous function, then T is completely continuous operator.

Proof:

It is easy to see that T is continuous. For $u \in M = \{u \in X : \|u\| \le l, l > 0\}$, we obtain,

$$Tu(t) = \left| \begin{aligned} \frac{\alpha+(n-2)\beta}{[\alpha+\beta](n-2)!}\int_0^1 \frac{t^2}{2!} f(s,u(s))ds - \frac{\alpha}{[\alpha+\beta](n-2)!}\int_0^1 \frac{t^2(1-s)^{n-2}}{2!} f(s,u(s))ds \\ - \frac{\beta}{[\alpha+\beta](n-3)!}\int_0^1 \frac{t^2(1-s)^{n-3}}{2!} f(s,u(s))ds - \int_0^1 \frac{t^{n-1}}{(n-1)!} f(s,u(s))ds + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s,u(s))ds \end{aligned} \right|$$

$$\begin{aligned} \le \frac{\alpha+(n-2)\beta}{[\alpha+\beta](n-2)!}\int_0^1 \frac{t^2}{2!} |f(s,u(s))| ds + \frac{\alpha}{[\alpha+\beta](n-2)!}\int_0^1 \frac{t^2(1-s)^{n-2}}{2!} |f(s,u(s))| ds \\ + \frac{\beta}{[\alpha+\beta](n-3)!}\int_0^1 \frac{t^2(1-s)^{n-3}}{2!} |f(s,u(s))| ds + \int_0^1 \frac{t^{n-1}}{(n-1)!} |f(s,u(s))| ds + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} |f(s,u(s))| ds \end{aligned}$$

$$\le \frac{\alpha+(n-2)\beta}{[\alpha+\beta](n-2)!}\ \frac{L}{2!} + \frac{\alpha}{[\alpha+\beta](n-1)!}\ \frac{L}{2!} + \frac{\beta}{[\alpha+\beta](n-2)!}\ \frac{L}{2!} + \frac{L}{(n-1)!} + \frac{L}{n!},$$

where $L = \max_{0\le t\le 1,\|u\|\le 1}|f(t,u(t))|+1$, so $T(M)$ is bounded. Next we shall show the equi-continuity of $\overline{T(M)}$, $\forall u \in M$, $\forall \varepsilon > 0, t_1 < t_2 \in [0,1]$.

Let

$$\eta < \left\{ 2!\frac{\varepsilon[\alpha+\beta](n-2)!}{5L[\alpha+(n-2)\beta]}, 2!\frac{\varepsilon[\alpha+\beta](n-1)!}{5L\alpha}, 2!\frac{\varepsilon[\alpha+\beta](n-2)!}{5L\beta} \right\},$$

$$\sigma < \left\{ \frac{\varepsilon(n-1)!}{5L} \right\},$$

$$\gamma < \left\{ \varepsilon\frac{n!}{5L} \right\}, \quad t_2^2 - t_1^2 < \eta, \; t_2^{n-1} - t_1^{n-1} < \sigma, \; t_2^n + t_1^n < \gamma .$$

Then, we have

$$|Tu(t_2)-Tu(t_1)| = \left| \begin{aligned} &\frac{\alpha+(n-2)\beta}{[\alpha+\beta](n-2)!}\int_0^1 \frac{t_2{}^2-t_1^2}{2!} f(s,u(s))ds - \frac{\alpha}{[\alpha+\beta](n-2)!}\int_0^1 \frac{(t_2{}^2-t_1^2)(1-s)^{n-2}}{2!} f(s,u(s))ds \\ &- \frac{\beta}{[\alpha+\beta](n-3)!}\int_0^1 (\frac{t_2{}^2-t_1^2)(1-s)^{n-3}}{2!} f(s,u(s))ds - \int_0^1 \frac{t_2{}^{n-1}-t_1^{n-1}}{(n-1)!} f(s,u(s))ds \\ &+ \int_0^{t_2} \frac{(t_2-s)^{n-1}}{(n-1)!} f(s,u(s))ds - \int_0^{t_1} \frac{(t_1-s)^{n-1}}{(n-1)!} f(s,u(s))ds \end{aligned} \right|$$

$$\begin{aligned} &\le \frac{\alpha+(n-2)\beta}{[\alpha+\beta](n-2)!}\frac{L(t_2^2-t_1^2)}{2!} + \frac{\alpha}{[\alpha+\beta](n-1)!}\frac{L(t_2^2-t_1^2)}{2!} \\ &\quad + \frac{\beta}{[\alpha+\beta](n-2)!}\frac{L(t_2^2-t_1^2)}{2!} + \frac{L(t_2^{n-1}-t_1^{n-1})}{(n-1)!} + \frac{Lt_1^n}{n!} + \frac{Lt_1^n}{n!} \\ &\le \frac{\varepsilon}{5}+\frac{\varepsilon}{5}+\frac{\varepsilon}{5}+\frac{\varepsilon}{5}+\frac{\varepsilon}{5}. \end{aligned}$$

Thus, $\overline{T(M)}$ is equicontinuous. The Arzela-Ascoli theorem implies that the operator T is completely continuous.

Theorem 3.1:

Assume that $f : [0,1]\times R \to R$ is continuous function, and there exist constants

$$0<c_1 < (\frac{2!(\alpha+\beta)(n-1)!}{n\alpha+(n-1)^2\beta}, \quad \frac{n!}{n+1}), \qquad c_2 > 0,$$

such that $\left| f(t,u(t)) \right| \le c_1 |u| + c_2$, for all $t \in [0,1]$. Then, the boundary value problem (3.1)-(3.3) has a solution.

Proof:

Following [Yang (2005), Zhang (2006) and Odda (2010)], we will apply the nonlinear alternative theorem to prove that T has one fixed point. Let $\Omega = \{u \in X : \|u\| < R\}$, be open subset of X , where

$$R > \left(4\left\{ \frac{n\alpha + (n-1)^2\beta}{2!(n-1)!(\alpha+\beta)}|u|c_1, \frac{(n+1)c_1}{n!}|u|, \frac{n\alpha + (n-1)^2\beta}{2!(n-1)!(\alpha+\beta)}|u|c_2, \frac{(n+1)c_2}{n!}|u| \right\} \right).$$

We suppose that there is a point $u \in \partial\Omega$ and $\lambda \in (0,1)$ such that $u = Tu$. For $u \in \partial\Omega$, we have:

$$Tu(t) = \left| \begin{array}{l} \frac{\alpha+(n-2)\beta}{[\alpha+\beta](n-2)!}\int_0^1 \frac{t^2}{2!} f(s,u(s))ds - \frac{\alpha}{[\alpha+\beta](n-2)!}\int_0^1 \frac{t^2(1-s)^{n-2}}{2!} f(s,u(s))ds \\ \quad - \frac{\beta}{[\alpha+\beta](n-3)!}\int_0^1 \frac{t^2(1-s)^{n-3}}{2!} f(s,u(s))ds - \int_0^1 \frac{t^{n-1}}{(n-1)!} f(s,u(s))ds + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s,u(s))ds \end{array} \right|$$

$$\le \frac{\alpha+(n-2)\beta}{[\alpha+\beta](n-2)!}\int_0^1 \frac{t^2}{2!}\left| f(s,u(s)) \right| ds + \frac{\alpha}{[\alpha+\beta](n-2)!}\int_0^1 \frac{t^2(1-s)^{n-2}}{2!}\left| f(s,u(s)) \right| ds$$
$$+ \frac{\beta}{[\alpha+\beta](n-3)!}\int_0^1 \frac{t^2(1-s)^{n-3}}{2!}\left| f(s,u(s)) \right| ds + \int_0^1 \frac{t^{n-1}}{(n-1)!}\left| f(s,u(s)) \right| ds + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\left| f(s,u(s)) \right| ds$$
$$\le \frac{\alpha+(n-2)\beta}{[\alpha+\beta](n-2)!}\int_0^1 \frac{t^2}{2!}(c_1|u(s)|+c_2)ds + \frac{\alpha}{[\alpha+\beta](n-2)!}\int_0^1 \frac{t^2(1-s)^{n-2}}{2!}(c_1|u(s)|+c_2)ds$$
$$+ \frac{\beta}{[\alpha+\beta](n-3)!}\int_0^1 \frac{t^2(1-s)^{n-3}}{2!}(c_1|u(s)|+c_2)ds + \int_0^1 \frac{t^{n-1}}{(n-1)!}(c_1|u(s)|+c_2)ds + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}(c_1|u(s)|+c_2)ds$$

$$\le \frac{\alpha+(n-2)\beta}{2![\alpha+\beta](n-2)!}(c_1|u(s)|+c_2) + \frac{\alpha}{2![\alpha+\beta](n-1)!}(c_1|u(s)|+c_2)$$
$$+ \frac{\beta}{2![\alpha+\beta](n-2)!}(c_1|u(s)|+c_2) + \frac{1}{(n-1)!}(c_1|u(s)|+c_2) + \frac{1}{(n)!}(c_1|u(s)|+c_2)$$

$$\le \frac{\alpha+(n-2)\beta}{2![\alpha+\beta](n-2)!}(c_1|u(s)|) + \frac{\alpha}{2![\alpha+\beta](n-1)!}(c_1|u(s)|)$$
$$+ \frac{\beta}{2![\alpha+\beta](n-2)!}(c_1|u(s)|) + \frac{1}{(n-1)!}(c_1|u(s)|) + \frac{1}{(n)!}(c_1|u(s)|)$$

$$+ \frac{\alpha+(n-2)\beta}{2![\alpha+\beta](n-2)!}(c_2) + \frac{\alpha}{2![\alpha+\beta](n-1)!}(c_2) + \frac{\beta}{2![\alpha+\beta](n-2)!}(c_2) + \frac{1}{(n-1)!}(c_2) + \frac{1}{(n)!}(c_2)$$

$$< \frac{R}{4} + \frac{R}{4} + \frac{R}{4} + \frac{R}{4} = \text{R},$$

which implies that $\|T\| \neq R = \|u\|$, that is a contradiction. Then, the nonlinear alternative theorem implies that T has a fixed point $u \in \overline{\Omega}$, that is, problem (3.1)-(3.3), has a solution $u \in \overline{\Omega}$.

Finally, we give an example to illustrate the results obtained in this paper.

Example.

From the equation (3.1) –(3.3) we solve the boundary value problem

$$u^{(5)}(t) = \frac{u+1}{u^2+7} \ . \qquad (3.6)$$

Theorem 3.1 with $\alpha = 1$ and $\beta = 1$, we find $0 < c_1 < \min(\frac{2!(2)4!}{5+(4)^2}, \frac{5!}{6})$. So, we conclude that the problem (3.6) has a solution.

Acknowledgements

The author is grateful to the reviewers for their useful comments.

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