Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 9, Issue 1 (June 2014), pp. 295-310

Reichenbach Fuzzy Set of Transitivity

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Received:September 22, 2013; Accepted: May 6, 2014

Abstract

Fuzzy implicators are the basic ingredients of many applications. So it becomes essential to study the various features of an implicator before implementing it in any practical application. This paper discusses the properties of transitivity of a fuzzy relation on a given universe and measure of fuzzy transitivity defined in terms of the Reichenbach fuzzy implicator which is an s-implicator.

Keywords: Fuzzy Transitivity; Reichenbach Fuzzy Implicator

MSC 2010 No.: Primary 03B52, 03E72, 8A72, 28E10; Secondary 46S40, 68T37, 93C42

1. Introduction and Preliminaries

A lot of work in fuzzy set theory and fuzzy logic has progressed since the inception of these fields in 1965. One of the basic notion of mathematics, a crisp equivalence relation (a reflexive, symmetric and transitive crisp relation) was fuzzified into the fuzzy counterpart by the founder of fuzzy sets himself in 1971 (Zadeh, 1971). Later, in subsequent decades it was observed that both the crisp as well as the $max-min$ fuzzy transitivity proposed by Zadeh are sources of many paradoxical situations [Klawonn, 1999), (Klawonn, 2003) and (DeCock and Kerre, 2003)]. This situation gave rise to many versions of the definition of transitivity [(Boixader et al, (2000), (Jacas, 1998), (Ovchinnikov and Riera, 19836), and (Valverde, 1985)]. Beg and Ashraf [(Beg and Ashraf, 2008) and (Beg and Ashraf, 2010)] reformulated the definition of fuzzy transitivity and defined as a fuzzy relation on a universe X. They discovered several novel properties of this new notion in terms of r-implicators. This paper is an effort to identify the different properties of the fuzzy sets of transitivity by using Reichenbach fuzzy implicator (S-implicator).

A fuzzy set A is a mapping from a universe X to $[0,1]$. For any $x \in X$, the value $A(x)$ denotes the degree of membership of x in A. Moreover, $F(X)$ is the set of all fuzzy subsets of X. For a crisp universe X, a fuzzy subset of $X \times X$ is called a fuzzy binary relation, which, and throughout this paper will be termed fuzzy relations. Given a crisp universe X and $A, B \in F(X)$, A is said to be a subset of B (in Zadeh's sense (Zadeh, 1965)) denoted by $A \subseteq B$, if and only if $A(x) \leq B(x)$ for all $x \in X$.

Definition 1.1. (Menger, 1951)

The triangular norm (t-norm) T and triangular conorm (t-conorm) S are increasing, associative, commutative mapping $[0,1]^2 \to [0,1]$ satisfying $T(1,x) = x$ and $S(x,0) = x$ for all $x \in [0,1]$.

The following t-norm and t-conorm are used in this paper for the sake of conjunction and disjunction:

Minimum operator $M: M(x,y) = \min(x,y)$,

Maximum operator $M^*: M^*(x,y) = \max(x,y)$.

Definition 1.2. (Fodor and Yager, 2000)

A negator N is an order-reversing $[0,1] \to [0,1]$ mapping such that $N(0) = 1$ and $N(1) = 0$. A strictly decreasing negator satisfying $n(n(x)) = x$ for all $x \in [0,1]$ is called a strong negator.

The negator defined as: $N(x) = 1 - x$ for all $x \in X$, is called standard negator and was defined by Zadeh himself.

Definition 1.3. (Valverde, 1985)

Given a t-norm T, a T −equivalence relation on a set X is a fuzzy relation E on X that satisfies:

(i) $E(x,x) = 1$ for all $x \in X$; (Reflexivity),

(ii) $E(x,y) = E(y,x)$ for all $x, y \in X$; (Symmetry),

(iii) $T\big(E(x,y), E(y,z)\big) \leq E(x,z)$ for all $x, y, z \in X$; (T −transitivity).

If $T = \min$, then E is called a similarity relation. A $\min$ transitive fuzzy relation satisfies:

$$\max_{y \in X}\Big(\min\big(E(x,y), E(y,z)\big)\Big) \leq E(x,z) \text{ for all } x, z \in X.$$

Commonly, it is called a $\max - \min$ transitive relation.

Definition 1.4. (Smets and Magrez, 1987)

A fuzzy implicator I is a binary operation on $[0,1]$ with order reversing first partial mappings and order preserving second partial mappings satisfying:

$$I(0,1) = I(0,0) = I(1,1) = 1 \text{ and } I(1,0) = 0.$$

Reichenbach fuzzy implicator is used in this paper to study the various properties of fuzzy set transitivity and it is defined as:

$$I(x,y) = 1 - x + xy \text{ for all } x, y \in X.$$

Definition 1.5. (Bandler and Kohout, 1980)

Given two fuzzy relations R and S on X, the direct product or sup$-T$ product of R and S is defined as:

$$R \circ^T S(x,z) = \sup_{y \in X} T\big(R(x,y), S(y,z)\big).$$

Definition 1.6. (Garmendia, 2005)

Let (X, ρ) be a measurable space. A function $m: \rho \to [0, \infty[$ is a fuzzy measure if it satisfies the following properties:

$$m_1: m(\emptyset) = 0, \text{ and } m(X) = 1,$$

$$m_2: A \subseteq B \text{ implies that } m(A) \leq m(B).$$

The concept of measure considers that $\rho \subseteq \{0,1\}^X$, but this consideration can be extended to a set $\mathfrak{I}$ of fuzzy subsets of X, i.e., $\mathfrak{I} \subseteq F(X)$, satisfying the properties of the measurable space $(F(X), \mathfrak{I})$.

For any $A \in F(X)$, the following two fuzzy measures will be used for the construction of the results on any measure of fuzzy transitivity:

1. $m_1(A) = plinth(A) = \inf_{x \in X} A(x)$,
2. $m_2(A) = \frac{|A|}{|X|} = \frac{\sum_{i=1}^{n} A(xi)}{n}$ (in case of finite universes with n elements).

Definition 1.7. (Beg and Ashraf, 2008)

The fuzzy inclusion $Incl$ is a mapping $Incl: F(X) \times F(X) \to F(X)$, which assigns to every $A, B \in F(X)$ a fuzzy set $Incl(A,B) \in F(X)$ defined for all $x \in X$ as:

$$Incl(A,B)(x) = I\big(A(x), B(x)\big),$$

where I is any implicator. The composition of fuzzy measure to this fuzzy set gives the measure of fuzzy inclusion $mIncl(A,B)$, i.e.,

$$mIncl(A,B) = m\big(Incl(A,B)\big) = m\Big(I\big(A(x),B(x)\big)\Big).$$

If $m = \inf$, then $mIncl = Inc$, i.e., for all $A,B \in F(X)$,

$$Inc(A,B) = \inf_{x\in X} I\big(A(x),B(x)\big),$$

where I is any implicator. This definition of Inc was given by Bandler and Kohout (Bandler and Kohout, 1980).

2. Reichenbach Fuzzy set of Transitivity

Definition 2.1 (Beg and Ashraf, 2008)

Let R be a fuzzy relation defined on the universe X. The fuzzy set of transitivity $tr^{I,T}(R)$ is a fuzzy relation on X defined as:

$$tr^{I,T}(R)(x,z) = \inf_{y\in X} I\Big(T\big(R(x,y),R(y,z)\big),R(x,z)\Big). \tag{1}$$

The transitivity function assigns a degree of transitivity to the relation at each point of $X \times X$. Thus a degree of transitivity of a relation may be different at different points of the domain $X \times X$. A relation R is called a fuzzy transitive relation if the set of fuzzy transitivity is non-empty, i.e., $tr^{I,T}(R) \neq \emptyset$ otherwise, it is known as a nontransitive fuzzy relation. A transitive fuzzy relation is called a strong (otherwise weak) fuzzy transitive if $tr^{I,T}(R) \geq 0.5$. A fuzzy relation R is called a strong (weak) fuzzy equivalence relation if R is reflexive, symmetric and a strong (weak) fuzzy transitive relation.

In this paper, the Definition 2.1 is revisited by restricting it to the use of Reichenbach fuzzy implicator and min t-norm, so the superscripts I,T will not be used. Hence

$$\begin{aligned} tr(R)(x,z) &= \inf_{y\in X} I\Big(M\big(R(x,y),R(y,z)\big),R(x,z)\Big) \\ &= \inf_{y\in X}\Big[I\Big(\min\big(R(x,y),R(y,z)\big),R(x,z)\Big)\Big] \\ &= \inf_{y\in X}\big[1-\min\big(R(x,y),R(y,z)\big)+\min\big(R(x,y),R(y,z)\big)R(x,z)\big] \\ &= \inf_{y\in X}\begin{cases}1-R(x,y)+R(x,y)R(x,z); \text{ if } R(x,y)\leq R(y,z),\\ 1-R(y,z)+R(y,z)R(x,z); \text{ if } R(x,y)\geq R(y,z).\end{cases}\end{aligned}$$

Theorem 2.2.

A fuzzy relation R is strong fuzzy transitive if either $R(x,z) \leq 0.5$ or $R(x,z) \geq 0.5$ for all $x,z \in X$.

Proof:

Let us suppose that $R(x,z) \leq 0.5$ for all $x,z \in X$, then for all $x,y,z \in X$,

$$\min\big(R(x,y),R(y,z)\big) \leq R(x,z) \leq 0.5$$

$$1-\min\big(R(x,y),R(y,z)\big) \geq 0.5 \text{ and } \big[\min\big(R(x,y),R(y,z)\big)\big]\, R(x,z) \leq 0.5$$

$$1-\min\big[\big(R(x,y),R(y,z)\big)\big] + \big[\min\big(R(x,y),R(y,z)\big)R(x,z)\big] \geq 0.5$$

$$1-\min\big(R(x,y),R(y,z)\big) + \min\big(R(x,y),R(y,z)\big)R(x,z) \geq 0.5$$

Thus, for all $x,z \in X$,

$$\inf_{y\in X}\big[1-\min\big(R(x,y),R(y,z)\big) + \min\big(R(x,y),R(y,z)\big)\, R(x,z)\big] \geq 0.5$$

Hence, for all $x,z \in X$,

$$tr(R)(x,z) = \inf_{y\in X}\big[1-\min\big(R(x,y),R(y,z)\big) + \min\big(R(x,y),R(y,z)\big)R(x,z)\big] \geq 0.5$$

Similarly, suppose $R(x,z) \geq 0.5$ for all $x,z \in X$, then there exists $y \in X$, such that

$$R(x,z) \geq \min\big(R(x,y),R(y,z)\big) > 0$$

$$\min\big(R(x,y),R(y,z)\big) > 0 \text{ and } 1-R(x,z) \leq 0.5$$

$$\min\big(R(x,y),R(y,z)\big)[1-R(x,z)] \leq 0.5$$

$$1-\min\big(R(x,y),R(y,z)\big)[1-R(x,z)] \geq 0.5$$

$$1-\min\big(R(x,y),R(y,z)\big) + \min\big(R(x,y),R(y,z)\big)R(x,z) \geq 0.5$$

Thus there exists a $y \in X$, such that for all $x,z \in X$

$$\inf_{y\in X}\big[1-\min\big(R(x,y),R(y,z)\big) + \min\big(R(x,y),R(y,z)\big)\, R(x,z)\big] \geq 0.5$$

Hence for all $x,z \in X$,

$$tr(R)(x,z) = \inf_{y\in X}\big[1-\min\big(R(x,y),R(y,z)\big) + \min\big(R(x,y),R(y,z)\big)\, R(x,z)\big] \geq 0.5$$

Theorem 2.3.

Let R be a fuzzy relation on a universe X. Then $tr(R)(x,z)=1$ if and only if $R(x,z)=1$ or $\min\big(R(x,y),R(y,z)\big)=0$ for all $y\in X$.

Proof:

For any $x,z\in X$,

$$tr(R)(x,z)=1$$

$$\Leftrightarrow \qquad \inf_{y\in X}\big[1-\min\big(R(x,y),R(y,z)\big)+\min\big(R(x,y),R(y,z)\big)\,R(x,z)\big]=1$$

For all $y\in X$

$$\Leftrightarrow \qquad 1-\min\big(R(x,y),R(y,z)\big)+\min\big(R(x,y),R(y,z)\big)R(x,z)=1$$

$$\Leftrightarrow \qquad \min\big(R(x,y),(y,z)\big)-\min\big(R(x,y),R(y,z)\big)R(x,z)=0$$

$$\Leftrightarrow \qquad \min\big(R(x,y),(y,z)\big)[1-R(x,z)]=0$$

$$\Leftrightarrow \qquad \text{either } \min\big(R(x,y),(y,z)\big)=0 \text{ or } R(x,z)=1.$$

Theorem 2.4.

Let R be a fuzzy relation on a universe X. The fuzzy set of inclusion (defined in terms of I) of $R\circ R$ into R is equal to the fuzzy set of transitivity of R, i.e.,

$$Incl(R\circ R,R)=tr(R)$$

Proof: Let $x,z\in X$, then using Definition 1.7, we get

$$Incl(R\circ R,R)(x,z)=I\left(\sup_{y\in X}\min\big(R(x,y),R(y,z)\big),R(x,z)\right)$$

$$=\inf_{y\in X} I\left(\min\big(R(x,y),R(y,z)\big),R(x,z)\right)$$

$$=tr(R)(x,z).$$

Proposition 2.5.

Let R be a fuzzy relation. Then following inclusion holds:

$$R\subseteq tr(R),\ \text{ i.e.,}\qquad R(x,z)\le tr(R)(x,z) \text{ for all } x,z\in X.$$

Proof:

Let $x, z \in X$,

$$tr(R)(x,z) = \inf_{y \in X} I\left(M\big(R(x,y), R(y,z)\big), R(x,z)\right)$$

$$= \inf_{y \in X}\left[1 - \min\big(R(x,y), R(y,z)\big) + \min\big(R(x,y), R(y,z)\big)\, R(x,z)\right]$$

Since for all $x, y, z \in X$,

$$R(x,z), R(x,y), R(y,z) \in [0,1]$$

so

$$R(x,y)\; R(x,z) \leq R(x,y)$$

$$\min\left[\big(R(x,y), R(y,z)\big)\right] R(x,z) \leq \min\left[\big(R(x,y), R(y,z)\big)\right]$$

$$1 - \min\left[\big(R(x,y), R(y,z)\big)\right] R(x,z) \leq 1 - \min\left[\big(R(x,y), R(y,z)\big)\right]$$

$$R(x,z) - \min\big(R(x,y), R(y,z)\big) R(x,z) \leq 1 - \min\big(R(x,y), R(y,z)\big)$$

$$R(x,z) \leq 1 - \min\big(R(x,y), R(y,z)\big) + \min\big(R(x,y), R(y,z)\big) R(x,z).$$

This implies that for each $x, z \in X$,

$$R(x,z) \leq \inf_{y \in X}\left[1 - \min\big(R(x,y), R(y,z)\big) + \min\big(R(x,y), R(y,z)\big)\, R(x,z)\right].$$

Hence for every $x, z \in X$,

$$R(x,z) \leq tr(R)(x,z).$$

Corollary 2.6.

The fuzzy transitivity relation $tr(R)$ of a reflexive, symmetric and fuzzy transitive relation R on a universe X is itself a reflexive, symmetric and fuzzy transitive relation on the same universe X. Proof: For reflexivity; let $x, z \in X$,

$$tr(R)(x,x) = \inf_{y \in X} I\left(M\big(R(x,y), R(y,x)\big), R(x,x)\right)$$

$$= \inf_{y \in X} I\left(\min\big(R(x,y), R(x,y)\big), R(x,x)\right), \text{ by symmetry of } R$$

$$= \inf_{y \in X} I\big(R(x,y), R(x,x)\big)$$

$$= \inf_{y \in X} [1 - R(x,y) + R(x,y)R(x,x)]$$

$$= \inf_{y \in X} [1 - R(x,y) + R(x,y)], \text{ by refelxivity of } R$$

$$= \inf_{y \in X} [1]$$

$$= 1.$$

For Symmetry; let $x, z \in X$,

$$tr(R)(z,x) = \inf_{y \in X} I\left(M\big(R(z,y), R(y,x)\big), R(z,x)\right)$$

$$= \inf_{y \in X} I\left(\min\big(R(y,z), R(x,y)\big), R(x,z)\right), \text{ by symmetry of } R$$

$$= \inf_{y \in X} I\left(\min\big(R(x,y), R(y,z)\big), R(x,z)\right), \text{ by symmetry of } min$$

$$= tr(R)(x,z).$$

As it is observed in Proposition 2.5, $tr(R)(x,z) \geq R(x,z)$, for all $x, z \in X$. Hence,

$$tr\big(tr(R)\big)(x,z) \geq tr(R)(x,z) \geq R(x,z).$$

Since $R(x,z)$ is a fuzzy transitive relation and it is observed that fuzzy set of transitivity remains non-empty, so for each $x, z \in X$,

$$tr\big(tr(R)\big)(x,z) \geq tr(R)(x,z) \geq R(x,z) > 0.$$

Corollary 2.7.

If $tr(R)(x,z) = 0$ for some $x, z \in X$, then $tr\big(tr(R)(x,z)\big) = 0$ and $tr\left(tr\big(tr(R)(x,z)\big)\right) = 0$.

Proof:

Suppose that $tr(R)(x,z) = 0$. This implies that, there exists some $y \in X$, such that for all $x, z \in X$,

$$tr(R)(x,z) = \inf_{y \in X} \big[1 - \min\big(R(x,y), R(y,z)\big) + \min\big(R(x,y), R(y,z)\big)R(x,z)\big] = 0.$$

This implies that

$$1 - \min\big(R(x,y), R(y,z)\big) + \min\big(R(x,y), R(y,z)\big)R(x,z) = 0$$

$$\min\big(R(x,y), R(y,z)\big)[1 - R(x,z)] = 1$$

$$\min\big(R(x,y), R(y,z)\big) = 1 \text{ and } 1 - R(x,z) = 1$$

$$R(x,y) = 1 \text{ and } R(y,z) = 1 \text{ and } R(x,z) = 0.$$

It further implies that

$$tr\big(R(x,y)\big) = 1 \text{ and } tr\big(R(y,z)\big) = 1 \text{ and } tr\big(R(x,z)\big) = 0.$$

So the triplet (1,1,0) i.e., $tr(R)(x,y) = 1 = tr(R)(y,z)$ and $tr(R)(x,z) = 0$, remains fixed every time the transitivity operator is applied. So $tr(R)(x,z) = 0$ implies $tr\big(tr(R)(x,z)\big) = 0$, $tr\left(tr\big(tr(R)(x,z)\big)\right) = 0$ and so on.

Theorem 2.8.

Let R be fuzzy relation on a universe X. If $R(x,z) \geq c$ for all $x, z \in X$, then $tr(R)(x,z) \geq c$, where $c \in [0,1]$.

Proof:

Suppose that $R(x,z) \geq c$ for any $x, z \in X$ and for some $c \in [0,1]$. This implies that, there exists some $y \in X$, such that $R(x,y), R(y,z)$ and $R(x,z) \in [0,1]$, so, $R(x,y)R(x,z) \leq R(x,y)$ and $R(y,z)R(x,z) \leq R(y,z)$. As it has already been proved in Proposition 2.5 that $R(x,z) \leq tr(R)(x,z)$, and $R(x,z) \geq c$, so $tr(R)(x,z) \geq R(x,z) \geq c$.

The converse of this does not hold.

Corollary 2.9.

Let R be a fuzzy relation on a universe X. If $R(x,z) > 0$ for all $x, z \in X$, then $tr(R)(x,z) > 0$ for all $x, z \in X$.

Proof: Suppose that $c = 0$. By Theorem 2.8, if $R(x,z) \geq 0$, then $tr(R)(x,z) \geq 0$.

Corollary 2.10.

Let R be a fuzzy relation on a universe X. If $supp(R) = X$, then $supp\big(tr(R)\big) = X$, for all $x, z \in X$.

Proof:

Suppose that $supp(R) = X$. This implies that, there exists $x, z \in X$, such that $R(x,z) > 0$. By Corollary 2.9, $tr(R)(x,z) > 0$ for all $x, z \in X$. That is $supp(R) = X$.

Example 2.11.

If the relations R and S are defined on $X = \{1,2,3\}$ by:

$$R = \begin{bmatrix} 1 & 0.9 & 0.3 \\ 0.9 & 1 & 0.5 \\ 0.3 & 0.5 & 1 \end{bmatrix} \text{ and } \qquad S = \begin{bmatrix} 1 & 0.25 & 0.85 \\ 0.25 & 1 & 0.65 \\ 0.85 & 0.65 & 1 \end{bmatrix},$$

then the fuzzy sets of transitivity are:

$$tr(R) = \begin{bmatrix} 1 & 0.91 & 0.65 \\ 0.91 & 1 & 0.75 \\ 0.65 & 0.75 & 1 \end{bmatrix}, \quad tr(S) = \begin{bmatrix} 1 & 0.5125 & 0.8725 \\ 0.5125 & 1 & 0.7725 \\ 0.8725 & 0.7725 & 1 \end{bmatrix}.$$

It is evident from the above example that the converse of the Theorem 2.8 may not be true. Reason, for $c = 0.65 \in tr(R)(x,z)$ there exists a value $0.3 \in R(x,z)$ such that $R(x,z) < c$, that is, for $c \in [0,1]$, $tr(R)(x,z) \geq c$ $R(x,z) \geq c$ for all $x, z \in X$.

Definition 2.12, (Beg and Ashraf, 2008)

The measure of fuzzy transitivity is a mapping $Tr: F(X \times X) \to [0,1]$ defined as:

$$Tr(R) = m\big(tr(R)\big),$$

where $F(X \times X)$ denotes the set of all fuzzy relations on X and m is a fuzzy measure. A fuzzy relation R is called $\varepsilon -$transitive if $Tr(R) = \varepsilon$. A reflexive, symmetric and $\varepsilon -$transitive fuzzy

relation is called an $\varepsilon -$equivalence fuzzy relation.

Example 2.13.

If the measures m_1 and m_2 applied in the Example 2.11, then we get:

$$Tr_1(R) = m_1\big(tr(R)\big) = 0.65 \text{ and } Tr_2(R) = m_2\big(tr(R)\big) = 0.8467,$$

and

$$Tr_1(S) = m_1\big(tr(S)\big) = 0.5125 \text{ and } Tr_2(S) = m_2\big(tr(S)\big) = 0.812778.$$

Theorem 2.14.

Let R be an $\varepsilon -$equivalence relation on X, then $tr(R)$ is a fuzzy equivalence relation of degree greater than ε.

Proof:

Let R be an $\varepsilon -$equivalence fuzzy relation. Then R is reflexive, symmetric and $\varepsilon -$transitive i.e., $Tr(R) = m\big(tr(R)\big) = \varepsilon$. Now $tr(R)(x, x) = 1$ and $tr(R)(z, x) = tr(R)(x, z)$, i.e., reflexive and symmetric has already been proved in Corollary 2.6.

Now it follows from Proposition 2.5 that: $R \subseteq tr(R) \subseteq tr\big(tr(R)\big)$. By applying the Sugeno's fuzzy measure and using its monotonic nature we get:

$$m(R) \leq m\big(tr(R)\big) \leq m\Big(tr\big(tr(R)\big)\Big)$$

$$Tr\big(tr(R)\big) = m\Big(tr\big(tr(R)\big)\Big) \geq m\big(tr(R)\big) = \varepsilon$$

$$Tr\big(tr(R)\big) > \varepsilon.$$

Hence, the fuzzy set of transitivity of an ε- equivalence relation R is a fuzzy relation with measure of transitivity greater than ε.

Theorem 2.15.

Let R be a fuzzy relation on X and $\circ$ stands for $sup - T$ product. If Reichenbach implicator is used for both then:

$$mIncl(R \circ R, R) = Tr(R).$$

Proof:

From the Theorem 2.4, we have

$$Incl(R \circ R, R) = tr(R).$$

By applying measure, we have

$$mIncl(R \circ R, R) = m\big(Incl(R \circ R, R)\big) = m\big(tr(R)\big) = Tr(R).$$

In case measure m_1 is used, we obtain: $Inc(R \circ R, R) = Tr(R)$, where Inc is defined in Definition 1.7.

Example 2.16.

Let a relation R be defined on $X = \{1,2,3,4\}$ as:

$$R = \begin{bmatrix} 1 & 0.6 & 1 & 0.9 \\ 0.6 & 1 & 0.9 & 1 \\ 1 & 0.9 & 1 & 0.6 \\ 0.9 & 1 & 0.6 & 1 \end{bmatrix}, \text{ then } tr(R) = \begin{bmatrix} 1 & 0.64 & 1 & 0.91 \\ 0.64 & 1 & 0.91 & 1 \\ 1 & 0.91 & 1 & 0.64 \\ 0.91 & 1 & 0.64 & 1 \end{bmatrix}$$

$$tr\big(tr(R)\big) = \begin{bmatrix} 1 & 0.6724 & 1 & 0.9181 \\ 0.6724 & 1 & 0.9181 & 1 \\ 1 & 0.9181 & 1 & 0.6724 \\ 0.9181 & 1 & 0.6724 & 1 \end{bmatrix}$$

$$tr\Big(tr\big(tr(R)\big)\Big) = \begin{bmatrix} 1 & 0.69923 & 1 & 0.924808 \\ 0.69923 & 1 & 0.924808 & 1 \\ 1 & 0.924808 & 1 & 0.69923 \\ 0.924808 & 1 & 0.69923 & 1 \end{bmatrix}.$$

Theorem 2.17.

The class of fuzzy transitive relations is closed under fuzzy intersection.

Proof:

Let R and S be the two fuzzy transitive relations. By hypothesis for all $x, y, z \in X$, we have

$$I\Big(\min\big(R(x,y), R(y,z)\big), R(x,z)\Big) > 0$$

and

$$I\Big(\min\big(S(x,y), S(y,z)\big), S(x,z)\Big) > 0.$$

Contrarily suppose that there exist $x, y, z \in X$ such that

$$I\Big(M\big(M(R,S)(x,y), M(R,S)(y,z)\big), M(R,S)(x,z)\Big) = 0$$

$$1 - M\big(M(R,S)(x,y), M(R,S)(y,z)\big)[1 - M(R,S)(x,z)] = 0$$

$$M\big(M(R,S)(x,y), M(R,S)(y,z)\big)[1 - M(R,S)(x,z)] = 1$$

$$M\big(M(R,S)(x,y), M(R,S)(y,z)\big) = 1 \text{ and } 1 - M(R,S)(x,z) = 1$$

$$M\big(M(R,S)(x,y), M(R,S)(y,z)\big) = 1 \text{ and } M(R,S)(x,z) = 0.$$

It further implies that

$$M(R,S)(x,y) = 1 \text{ and } M(R,S)(y,z) = 1 \text{ and } M(R,S)(x,z) = 0$$

$$R(x,y) = 1 \text{ and } R(y,z) = 1 \text{ and } R(x,z) = 0$$

and

$$S(x,y) = 1 \text{ and } S(y,z) = 1 \text{ and } S(x,z) = 0$$

So,

$$I\left(\min\big(R(x,y), R(y,z)\big), R(x,z)\right) = 0$$

and

$$I\left(\min\big(S(x,y), S(y,z)\big), S(x,z)\right) = 0.$$

i.e., $tr(R)(x,z) = 0$ and $tr(S)(x,z) = 0$, a contradiction to the hypothesis. Thus $R \cap S$ is fuzzy transitive.

Theorem 2.18.

Let (X,d) be an ultra-metric space. Define a fuzzy relation R on X as:

$$R(x,y) = \frac{1}{1 + d(x,y)}$$

Then R is a strong fuzzy equivalence relation on X.

Proof:

For all $x, y, z \in X,$ we have

(i) Reflexivity: $R(x,x) = \frac{1}{1+d(x,x)} = \frac{1}{1+0} = 1,$

(ii) Symmetry: $R(x,y) = \frac{1}{1+d(x,y)} = \frac{1}{1+d(y,x)} = R(y,x),$

(iii) Strong Fuzzy transitivity:

Suppose on contrary that there exist $x, y, z \in X,$ such that

$$tr(x,y,z) = I\left(M\big(R(x,y), R(y,z)\big), R(x,z)\right) < 0.5.$$

This implies that

$$I\left(M\left(\frac{1}{1+d(x,y)},\frac{1}{1+d(y,z)}\right),\frac{1}{1+d(x,z)}\right)<0$$

$$\Leftrightarrow \quad I\left(\min\left(\frac{1}{1+d(x,y)},\frac{1}{1+d(y,z)}\right),\frac{1}{1+d(x,z)}\right)<0.5$$

$$\Leftrightarrow \quad 1-\min\left(\frac{1}{1+d(x,y)},\frac{1}{1+d(y,z)}\right)+\min\left(\frac{1}{1+d(x,y)},\frac{1}{1+d(y,z)}\right)\left(\frac{1}{1+d(x,z)}\right)<0.5$$

$$\Leftrightarrow \quad \min\left(\frac{1}{1+d(x,y)},\frac{1}{1+d(y,z)}\right)-\min\left(\frac{1}{1+d(x,y)},\frac{1}{1+d(y,z)}\right)\left(\frac{1}{1+d(x,z)}\right)>0.5$$

$$\Leftrightarrow \quad \min\left(\frac{1}{1+d(x,y)},\frac{1}{1+d(y,z)}\right)\left(1-\frac{1}{1+d(x,z)}\right)>0.5$$

$$\Leftrightarrow \quad \min\left(\frac{1}{1+d(x,y)},\frac{1}{1+d(y,z)}\right)>0.5 \text{ and } 1-\frac{1}{1+d(x,z)}>0.5$$

$$\Leftrightarrow \quad \frac{1}{1+d(x,y)}>0.5 \text{ and } \frac{1}{1+d(y,z)}>0.5 \text{ and } \frac{1}{1+d(x,z)}<0.5$$

$$\Leftrightarrow \quad d(x,y)<1, \text{ and } d(y,z)<1,$$

and

$$d(x,z)>1. \tag{2}$$

$$\Leftrightarrow \quad \max\big(d(x,y),d(y,z)\big)<1.$$

Since d is an ultra metric it follows that, $d(x,z)\leq\max\big(d(x,y),d(y,z)\big)<1$. But eq. (2) also implies that $d(x,z)>1$. Hence these inequalities can not hold simultaneously, a contradiction.

3. Applications

Control theory, modeling, diagnosis or decision making are the key areas in which the outcomes can be improved by using fuzzy implicators. They can be successfully applied in many fields like Artificial Intelligence (AI), psychology, medicine, economics, sociology, music, etc. Reichenbach implications also play significant role in the approximate reasoning procedure in a fuzzy rule-based system as it is a many valued fuzzy implicator (Leski, 2003), (Li et al, 2002). Answer set programming (ASP) can be successfully implemented in planing the problems, configuring and verifying the software but fails to model these problems with continuous domain. Then fuzzy answer set programming (FASP) helps to resolve this difficulty. FASP combines the ASP and Fuzzy logic to cope with the problem of continuity. Reichenbach can be effectively used in FASP to model such problems (Janssens, 2011). It can be a good choice to design a fuzzy rulebase for the fuzzy controllers applied to problems of musical decisions like card inversions (Bonardi and Truck, 2010). It has very interesting applications in (AI). Reichenbach implication can be used in the fuzzy inference system as a implication based neuro fuzzy architectures Rutkowska and Hayashi, 2003). Reichenbach fuzzy implicator, being an equivalence relation can also be used for document retrieval in web data mining and data clustering analysis (Raut and Bamnote, 2010).

4. Conclusion

This paper presents the opportunity to discuss the concepts and properties of Fuzzy set transitivity and the measure of fuzzy transitivity on the universe X. As we have seen that the transitivity relation has a greater degree of transitivity as compared to the relation R itself, this gives us a way to find the transitive closure of the fuzzy relation with the help of transitivity operator. For an equivalence fuzzy relation R, fuzzy transitive relation $tr(R)$ is also an equivalence relation and is a strong fuzzy equivalence if the universe of discourse X is an ultra-metric space.

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