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Frechet Differentiable Norm and Locally Uniformly Rotund Renormings

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Abstract

In this paper, we study briefly the role played by the locally uniformly rotund (LUR) norm and Frechet differentiability of a norm on the Banach space theory. Our old outstanding open Problem 3.8 mentioned below is the main object of this paper. We study nearly about it and find some additional assumptions on the space attached with this problem to obtain its positive or negative answer. We investigate different results related to these norms and their duals on different settings. In particular, we introduce reflexive spaces, Banach spaces with unconditional basis, weakly locally uniformly rotund (WLUR) norm, Almost locally uniformly rotund (ALUR) norm, strongly exposed point, sub-differentiability and ϵ-sub-differentiability, σ–slicely continuity, weakly compactly generated (WCG) Banach spaces with c^k –smooth norms, Symulian’s Theorem, and some technical lemmas.

Keywords: Locally uniformly rotund (LUR); Equivalent norms; Renorming ; Weakly locally uniformly rotund (WLUR); Almost locally uniformly rotund(ALUR) Unconditional basis; Day's norm; Symulian’s Theorem

MSC 2010 No.: 49J50, 46B20

1. Introduction

The notion of locally uniform rotund (LUR) norm was introduced by Lovaglia, A. R. (1955) and he generalized uniform convexity, localized this notion and then introduced LUR norm of a Banach space. He showed, as a straightforward consequence of a Theorem ‘‘ The norm of a Banach space is Frechet differentiable if the dual norm is LUR’’ which is mentioned in Asadi and Haghshenas (2012) and proved by Deville et al. (1983). The converse does not hold, even up to renormings. Asadi and Haghshenas (2012) showed that there exists a Banach space X with a Frechet differentiable norm, which does not admit any equivalent norm with a strictly convex dual norm. However, Smith and Troyanski (2008) mentioned that in the class of spaces with unconditional bases, we do have equivalence up to a renorming. Many efforts have been dedicated in the renorming theory to obtain sufficient conditions for a Banach space to admit an equivalent LUR norm. Molto et al. (2009) stated the first characterization of existence of LUR renormings. The spaces with this property are at the core of renorming theory in Banach spaces and consequently have been extensively studied, see for example, Deville et al. (1993). It is well known that the spaces with a LUR norm have the Kadec property. As is now evident, the notion of LUR is of fundamental importance for renorming theory and we refer to Deville et al. (1993) and the more recent Malto et al. (2009) for extensive results.

Differentiability of convex continuous functions has been studied extensively during the recent fifty years. Among all functions in this class, the norm function is of the great importance and deserves to be studied in more detail, since the geometric structure of the space is highly related to it. Asadi et al. (2009); Haghshenas, H. (2011) introduced Frechet differentiability of a norm. In this paper, we emphasize this norm and use Smulian’s Theorem to prove some geometric results in connection with LUR norm. Examples of spaces that admit Frechet differentiable norm are spaces with separable dual, reflexive spaces, C(K) space with $K^{(\omega_1)} = \phi$, $C_0(\Gamma)$ for arbitrary Γ, and C ([0, α]) for arbitrary ordinal α. In 1969, a proof published under the name Rainwater computed the local uniform convexity of Day’s norm. This norm has been the only known locally uniformly rotund norm on $c_0(\Gamma)$ and this renormability remains a standard tool for further renorming Theorems. For example, the basic technique for finding a strictly convex norm for a Banach space is to exhibit a continuous linear injection into a $c_0(\Gamma)$. Also, the local uniform rotundity of $c_0(\Gamma)$ is used in Trojonski’s conjecture settling proof for the existence of uniformly rotund norms for reflexive spaces. It was shown by Molto et al. (2009) that a Banach space X with a Frechet differentiable norm which has Gateaux differentiable dual norm, admits an equivalent LUR norm. It is proved in Troyanski, S. (1968) that ‘‘If a Banach space X has unconditional basis, then X admits an equivalent LUR norm’’.

Combination of the Frechet differentiability and the LUR property: In many situations one is interested in obtaining a new norm on a given space X which shares good smoothness and rotundity properties. A classical result in this direction is a method generally known as Asplund averaging: A Banach space X which admits a LUR norm $||\cdot||_1$ and a norm $||\cdot||_2$ the dual norm of which is LUR, admits also a norm $||\cdot||_3$ which is LUR and whose dual norm is LUR too. In particular, $||\cdot||_3$ is LUR and Frechet differentiable simultaneously. The Asplund averaging took a surprising twist with the recent deep result of Haydon, R. (2008), ‘‘If X admits a norm the dual of which is LUR, then it admits also an LUR norm’’.

Some Notations and Terminologies

In the sequel, $(X, \|\cdot\|)$ is the real Banach space with norm $\|\cdot\|$; $S(X)$ is the unit sphere in X; $(X^*, \|\cdot\|^*)$ is the dual space of X; $S(X^*)$ is the unit sphere of X^*; $B(X)$ is the unit ball of X; $B(X^*)$ is the unit ball of X^*. LUR, ALUR. WLUR, FD, WCG, and lsc are the short forms of locally uniformly rotund, weakly locally uniform rotund, Frechet differentiable, weakly compactly generated and lower- semi-continuity respectively**.** All undefined terms and notation are standard and can be found, for example, in Fabian et al. (2001) ; Molto et al. (2009).

2. Some Definitions

Definition 2.1. Deville et al. (1993).

The space X (or the norm $||.||$ on X) is said to be locally uniformly convex (in short LUR) if

$$\forall \ (x_n) \subseteq S(X),\ x \in S(X),\ \lim_{n\to\infty} || x_n || = || x || = ||(x_n + x)/2|| =1 \Rightarrow \lim_{n\to\infty} x_n = x.$$

Equivalently,

$$\lim_{n\to\infty} (2||x||^2 + 2|| x_n ||^2 - ||x + x_n ||^2) = 0 \Rightarrow \lim_{n\to\infty} || x - x_n || = 0, \text{ for all } (x_n) \subseteq X,\ x \in X.$$

Definition 2.2. Deville et al. (1993**);** Fabian et al. (2001).

A norm $||.||$ on X is said to be Frechet differentiable at $x \in S(X)$ if there exists $F \in S(X^*)$ such that for every $h \in S(X)$,

$$\lim_{h\to 0} \left| \frac{||x+h|| - ||x|| - F(x)h}{||h||} \right| = 0,$$

where $F(x)$ is called Frechet derivative of the norm function $\|\cdot\|$ at x and $F(x, h)$ is called the Frechet differential of $\|\cdot\|$ at x in the direction h. If the norm of X is Frechet differentiable at each $x \in S(X)$, then we say that X is Frechet differentiable.

Definition 2.3.

Let a linear space X be equipped with two norms $||\cdot||_1$ and $||\cdot||_2$. Then these two norms on a common space X are **equivalent** if there are two positive real numbers a and b such that

$$\forall x \in X,\quad a||x||_1 \le ||x||_2 \le b||x||_1.$$

Definition 2.4. Johnson et al**.** (2003).

Renorming of Banach space consists of replacing the given norm, which is usually provided by the very definition of the space, by another norm which may have better (or sometimes worse) geometric properties of Convexity or smoothness, or both.

3. Some Results

Theorem 3.1. Deville et al. (1993); Haghshenas,H. (2011); Fabian et al. (2001).

Suppose that $||.||$ is a norm on a Banach space X with dual norm $||.||^*$ and $x \in S(X)$. Then, the following are equivalent:

- (i) $||\cdot||$ is Frechet differentiable at x.
- (ii) $\forall (f_n), (g_n) \subseteq S(X^*)$ $f_n(x) \to 1, g_n(x) \to 1 \Rightarrow ||f_n - g_n||^* \to 0$.
- (iii) $\forall \{f_n\} \subseteq S(X^*)$ $f_n(x) \to 1 \Rightarrow \exists f \in S(X^*): f_n \to f$.

As a consequence of Smulian's Theorem we obtain the following result.

Theorem 3.2. Deville et al. (1993)**;** Asadi et al. (2009).

If X^* is LUR, then the norm function is Frechet differentiable on X.

Proof:

Let $x \in S(X), (f_n) \subseteq S(X^*)$. Then there is $f \in S(X^*)$ such that $f(x) = 1$, $\lim_{n\to\infty} f_n(x) = 1$.

It follows that

$$2 \geq || f_n + f || \geq (f_n + f)(x) \to 2 \text{ as } n \to \infty$$

$$\Rightarrow || f_n + f || \to 2.$$

So,

$$\lim_{n\to\infty} (2||f_n||^2 + 2||f||^2 - ||f_n + f||^2) = 0.$$

Since X^* is locally uniformly convex, $\lim_{n\to\infty} || f_n - f || = 0$, then by Smulian's Theorem $||\cdot||$ is Frechet differentiable at $x \in S(X)$. □

Theorem 3.3: Fabian et al. (2001); Haydon, R. (2008); Orihuela, J. (2007).

If X^* has a dual LUR norm then X admits an equivalent LUR norm.

Theorem 3.4. Wulbert, D. (1986)**.**

The norm $||\cdot||$ on $C_0(\Gamma)$ is Frechet differentiable.

Proof:

Theorem 3.3 showed that if the dual space X^* has a dual LUR norm, then X admits an equivalent LUR norm. But the dual norm $||\cdot||$ on $C_0(\Gamma)^*$ *is LUR*. So by Theorem 3.2 the Theorem 3.4 is proved.

Example 3.5. Lin, P.K. (2004).

The ℓ_1 space has an equivalent LUR norm.

Since $C_0(\Gamma)^* = \ell_1$, Example 3.5 implies Theorem 3.4 and thus, Theorem 3.3 is true.

Theorem 3.7. Wulbert, D. (1986, p. 69).

The norm $||\cdot||$ on $C_0(\Gamma)$ is LUR.

Proof:

After defining Day's norm on $C_0(\Gamma)$, it is proved in (Deville et al. (1993), pp.69-71).

We are now face with the following problem which is raised in Asadi and Haghshenas (2012); Orihuela, J. (2007).

Problem 3.8.

Assume X admits an equivalent Frechet differentiable norm. Does X admit an equivalent LUR norm ?

This problem has been solved with additional conditions on X.

Theorem 3.9. Fabian et al. (2001, p. 272).

Let the norm $||\cdot||$ on a Banach space X be Frechet differentiable. If the dual norm $||\cdot||^*$ on X^* is Frechet differentiable, then $||\cdot||$ and $||\cdot||^*$ are both LUR.

Proof:

Let $(x_n) \subseteq S(X)$ and $x \in S(X)$. Assume that $||(x_n + x)/2|| \to 1$. We have to show that $||x_n - x|| \to 0$. Let f_n be the Frechet derivative of the norm $||\cdot||$ at $(x_n + x)/2$ and f be the Frechet derivative of the norm $||\cdot||$ at x, i. e., $f_n = ||(x_n + x)/2||'$ and $f = ||x||'$ respectively. Since f_n and f are Frechet differentiable, then they are linear and bounded. Frechet derivative of a function f at $x \in X$ is

$$||x||'h = \lim_{h \to 0} \frac{||x + h|| - ||x||}{||h||},$$

so here,

$$f(x) = ||x||'x = \lim_{x\to 0} \frac{||x+x|| - ||x||}{||x||} = 1.$$

Thus,

$$|f(x)| = |\lim_{x\to 0} \frac{||x+x|| - ||x||}{||x||}|$$
$$\leq \lim_{x\to 0} ||x + x - x|| / ||x||$$
$$= 1$$

Therefore,

$$|f(x)| \leq 1 \text{ and } ||f|| \leq 1.$$

But,

$$||f|| = sup\{|f(x)|: \text{for all } f \in X^*\}$$

$$\Rightarrow ||f|| \geq |f(x)| \geq f(x) = 1.$$

It follows that

$$1 \geq ||f|| \geq 1 \Rightarrow ||f|| = 1.$$

So,

$$f \in S(X^*), f(x) = 1.$$

Similarly, we get

$$f_n \in S(X^*), f_n(x_n + x) = ||x_n + x||.$$

We claim that

$$\lim_n f_n(x_n) = 1, \lim_{n\to\infty} f_n(x) = 1.$$

Suppose that $\lim_n inf(f_n(x)) < 1$. Then for some $0 < \alpha < 1$ and some sub sequence f_{n_k} of the sequence (f_n) such that

$$\lim_n inf f_{n_k} < \alpha \text{ for all } k.$$

Then,

$$f_{n_k}(x + x_n) = f_{n_k}(x) + f_{n_k}(x_{n_k}) < 1 + \alpha < 2,$$

a contradiction with

$$f_{n_k}(x + x_{n_k}) = ||x + x_{n_k}|| \to 2.$$

Similarly, a contradiction is **o**btained with

$$f_n(x_n) \to 1.$$

Therefore,

$$\lim_n inf\ f_n(x_n) = 1,$$

since for all *n*,

$$|f_n(x)| \leq ||\, f_n \,||\ ||\, x\,|| = 1,\ \lim_n f_n(x) = 1.$$

As ||·|| is Frechet differentiable, so by Symulian Theorem,

$$||f_n - f\,||^* \to 0,\quad |\,(f_n - f\,)(x_n)| \leq ||\, f_n - f\,||\ ||\, x_n\,|| \to 0.$$

$$\Rightarrow f(x_n) = f_n(x_n) + (f - f_n)(x_n) \to 1.$$

Since $||\cdot||^*$ is Frechet differentiable so by Symulian Theorem $||x_n - x|| \to 0$. Thus, the norm $||\cdot||$ is LUR.

***Proof of second part*:**

As the dual norm $||\cdot||^*$ is Frechet differentiable, X is reflexive and the norm function is convex and lower-semi continuous (lsc), so we have $||\cdot||^{**} = ||\cdot||$ for all $x \in X$. By proof of first part of this Theorem, $||\cdot||^*$ is LUC. But the dual norm $||\cdot||^*$ is LUR implies $||\cdot||$ is LUR by Theorem 3.3. So $||\cdot||^*$ is also LUC. □

Theorem 3.10. Molto et al. (2009)

Let the norm $||\cdot||$ on a Banach space X be Frechet differentiable. If it has Gateaux differentiable dual norm, then X admits an equivalent LUR norm.

Theorem 3. 11. Borwein and Vanderwerff *(* 2010, compare to remark 5.1.3, Ex.5.1.17, p, 225)

Suppose X is a reflexive Banach space. Then a norm ||.|| on X is Frechet differentiable and LUR if and only if its dual norm is Frechet differentiable and LUR.

Proof:

Suppose the dual norm to ||.|| is Frechet differentiable and locally uniformly convex. Then we have to show ||.|| on X is LUR and Frechet differentiable. Suppose $x_n,\ x \in S(X)$ are such that $||\, x_n + x\,|| \to 2$. Choose $f \in S(X^*)$ so that $f(x) = 1$ and $f_n \in S(X^*)$ so that

$$f_n(x_n + x) = ||x_n + x||.$$

Then,

$$f(x_n) \to 1, \quad (\because f_n(x_n) \le |f_n(x_n)| \le ||f_n||\, ||x_n|| \to 1)$$

Similarly,

$$f_n(x) \le |f_n(x)| \le ||f_n||\, ||x|| \to 1.$$

As the dual norm is Frechet differentiable at f and $f(x) = 1$, Smulian's Theorem implies $x_n \to x$. So by definition, the norm $\|\cdot\|$ is LUR. Again, as dual norm is LUR .That is,

$$f, f_n \in S(X^*), ||f_n + f|| \to 2 \Rightarrow || f_n - f || \to 0.$$

So for

$$\begin{aligned} & f_n \in S(X^*), f(x_n) \to 1 \\ \Rightarrow\ & 2 \ge ||f_n + f|| \ge (f_n + f)(x) \ge 2. \\ \Rightarrow\ & || f_n + f || \to 2. \end{aligned}$$

Then,

$$|| f_n - f || \to 0. \ (\because \text{by assumption})$$

Therefore, by definition the norm $||.||$ is Frechet differentiable by Smulian's Theorem. ☐

4. Unconditional Basis

The existence of a Schauder basis in a Banach space does not give very much information on the structure of the space. If one wants to study in more detail the structure of a Banach space by using bases one is led to consider bases with various special properties. Undoubtedly, the most useful and widely studied special class of bases is that of **unconditional bases.** A basis (x_n) of a Banach space X is said to be unconditional if for every $x \in X$, its expansion in terms of the basis $\sum_{n=1}^{\infty} a_n x_n$ converges unconditionally. It is defined in Lindenstrauss and Tzafriri (1977). For example natural basis in l_p, $1 \le p < \infty$ and C_0 is unconditional basis. In the spaces $C\,[0, 1]$ and $L_1[0,1]$ there is no unconditional basis . Recall that a basis of X is unconditional if and only if it remains a basis under any rearrangement of the sequence basis in X. With the aid of this type of basis to a Banach space X, we get a positive answer to the Problem 3.8 which is stated as follows.

Lemma 4.1. Deville et al. (1993, pp. 68, 69); Diestel (1975, p. 95)

Let $(s_k)_{(k \in \mathbb{N})}$, $(t_k)_{(k \in \mathbb{N})}$ be two non-increasing sequences of non-negative numbers such that $s_k = t_k = 0$ for all large $k \in \mathbb{N}$. Let $\pi: \mathbb{N} \to \mathbb{N}$ be an injective surjection (permutation of natural numbers).

Then,

$$\sum_{k=1}^{\infty} s_k(t_k - t_{\pi(k)}) \geq 0$$

and for all $K \in \mathbb{N}$ either

$$\pi \{1, \cdots, K\} = \{1, \cdots, K\}$$

or ,

$$(s_K - s_{K+1})(t_K - t_{K+1}) \leq \sum_{k=1}^{\infty} s_k(t_k - t_{\pi(k)}).$$

Lemma 4.2. Deville et al. (1993).

Let Γ be an infinite set, let $u \in l_\infty(\Gamma)$, $\epsilon > 0$, and assume that the set

$$\{\gamma \in \Gamma : |\, u(\Gamma)\,| > \epsilon\} < \infty .$$

Let $u_n \in l_\infty(\Gamma)$, $n \in \mathbb{N}$, be such that

$$2||x||_D^2 + 2||x_n||_D^2 - ||x + u_n||_D^2 \to 0 \text{ as } n \to \infty.$$

Then,

$$\limsup_{n\to\infty} ||\, u - u_n \,||_\infty \leq 3\epsilon.$$

Using Lemma 4.1, Lemma 4.2 the above Theorem is proved in elaborate form due to Rainwater and found in Diestel (1975, pp. 94-100). So it is not necessary to give its proof. If $u \in c_0(\Gamma)$ in the Lemma 4.2, Day's norm $||\cdot||_D$ is LUR at u. Thus, we get a well known result that this norm on $c_0(\Gamma)$ is LUR.

Theorem 4.3. Smith and Troyanski (2008).

Let X have an unconditional basis. Then X admits an equivalent norm with LUR dual norm whenever X admits an equivalent Frechet smooth norm. (This result has been known since the 1960's).

***Proof*:**

Let X have an unconditional basis and X admits an equivalent Frechet smooth norm. We have to show that X admits an equivalent norm with LUR dual norm. Let $||\cdot||$ be the original norm on X and $(e_\gamma)_{\gamma \in \Gamma}$ an unconditional basis with conjugate system $(f_\gamma)_{\gamma \in \Gamma}$. By a renorming, we may assume that $(e_\gamma)_{\gamma \in \Gamma}$ is 1-unconditional with respect to $||\cdot||$.

By lemma 4.2 Day's norm $||\cdot||_{Day}$ on $\ell_\infty(\Gamma)$ can be seen in Deville et al. (1993, Def., II.7.2, 22) and the fact that $||\cdot||_{Day}$ norm is LUR when restricted to $c_0(\Gamma)$, it is proved in Deville et al. (1993, Theorem II.7.3) ; Haydon, R. (2008).

Define T: $X \to c_0(\Gamma)$ by

$$(T\,x)(\gamma) = f(x) \text{ for } x \in X \text{ and } \gamma \in \Gamma,$$

and set

$$|||\, x \,|||^2 = ||\, T\,x\,||^2_{Day} + \sum 2^{-n}\, ||\, x\,||^2_n\,,$$

where

$$||x||^2{}_n = sup\,\{\, ||\,\Sigma_{\gamma\, \in \Gamma - A}\; f_\gamma\,(\,x\,)\,e_\gamma\,|| + 2\,\Sigma_{\gamma\, \in A}\,|\,f_\gamma\,(x)|\!:\, A \in \Gamma \text{ and card } A \leq n\}.$$

Then, |||·||| is LUR. It is proof is compared in Deville et al. *(1993*, pp. 69-71).

Indeed, Troyanski, S. (1968), proved that ‘‘If *X* has an unconditional basis then *X* admits an equivalent LUR norm’’. Let *X* have a 1-unconditional basis $(e_\gamma)_{\gamma \in \Gamma}$ with conjugate system $(f_\gamma)_{\gamma \in \Gamma}$ If *X* admits an equivalent Frechet smooth norm then it cannot contain an isomorphic copy of ℓ_1, for ℓ_1 admits no such norm Deville et al. (1993, Corollary II.3.3). Hence $(e_\gamma)_{\gamma \in \Gamma}$ is shrinking Lindenstrauss and Tzafriri (1977, Theorem 1.c .9), and it follows that the conjugate system $(f_\gamma)_{\gamma \in \Gamma}$ is an unconditional basis of X*. Now it is a straightforward matter to verify that the LUR norm |||·||| defined above, but on *X**, |||·||| is w*-lower semi continuous and thus, the dual of an equivalent norm on *X* Fabian et al. (2001, Theorem 8.8). □

5. Weakly Locally Uniform Rotund (WLUR) Norm

Here in Problem 3.8 WLUR condition is added because on one hand, it gives the positive answer to the former problem. On the other hand it is used to construct LUR norm |||· |||| on the space whose dual is a Vasak space Deville et al. (1993, chap. VII, Theorem 2.7 p.276), in particular, |||·||| is Frechet differentiable .We say the space *X* (or the norm ||.|| on *X*) is said to be weakly locally uniformly convex (in short WLUR) if, for all $(x_n) \subseteq S(X),\, x \in S(X)$,

$$\lim_{n\to\infty} ||\, x_n\,|| = ||\, x\,|| = ||(x_n + x)\,/\,2|| = 1$$
$$\Rightarrow \lim_{n\to\infty} x_n = x \text{ in weak topology of } X\,.$$

Theorem 5.1. Malto et al. (1999).

A weakly locally uniform rotund (WLUR) norm on Banach space has an equivalent LUR norm.

Theorem 5. 2. Malto et al.(1999).

A Banach space *X* with a WLUR and Frechet differentiable norm must be LUR renormable.

This theorem is proved in Deville et al. (1993, Chap. VII, Proposition 2.6, pp. 276-278).

6. Almost Locally Uniform Rotund (ALUR) Norm

Definition 6. 1. Aizpurlt and Garcia-Pacheco (2005)

We say that a norm ||·|| on a Banach space X is almost locally uniformly rotund (ALUR) if for every pair of sequences

$$(x_n)_{n\in\mathbb{N}} \text{ in } S(X),\ (f_m)_{m\in\mathbb{N}} \text{ in } S(X^*) \text{ such that}$$

$$\lim_m(\lim_n(f_m((x_n+x)/2))) = 1 \Rightarrow (x_n)_{n\in\mathbb{N}} \text{ converges to } x.$$

Remarks 6.2.

We have implication LUR⇒ ALUR⇒ R (rotund)

Remarks 6 .3.

If $x\in S(X)$ is a strongly exposed point and a smooth point of B(X), then it is an almost locally uniformly rotund point of $B(X)$.

Theorem 6.4. Aizpurlt and Garcia-Pacheco (2005).

Let X be a Banach space and let $x\in S(X)$. If x is a strongly exposed point of $B(X)$ and a strongly smooth point of $B(X)$, then it is a locally uniformly rotund point of $B(X)$.

Theorem 6.5. Aizpurlt and Garcia-Pacheco (2005)

Let X be a Banach space. Then X is locally uniformly rotund if it is almost locally uniformly rotund and its norm is Frechet differentiable in $S(X)$.

Proof:

Assume that a Banach space X is ALUR and its norm is Frechet differentiable. We have to show that X is LUR. Let $x\in S(X)$.Then x is strongly exposed point of $B(X)$ because "x is an almost locally uniformly rotund point of B(X) if and only if it is a strongly exposed point of B(X) for each $f\in S(X^*)$ such that $f(x) = 1$". Further as the norm of X is Frechet differentiable at x it is strongly smooth point of B(X). Thus, by Theorem 6.4 norm of X is LUR. □

The converse of Theorem 3.2 is not true in general, since there are spaces with Frechet differentiable norm which admits no dual LUR norm. However, the following theorem shows the converse is true.

Theorem 6.7.

If $x\in X$ and $||x||=1$ all points of $S(X)$ is strongly exposed point of $B(X)$ and norm $||\cdot||$ is Frechet differentiable at $x\in S(X)$ so that $f\in S(X^*)$ where $f(x) = ||x||$. Then the dual norm $||\cdot||^*$ is LUR.

Proof:

Assume that all point $x \in S(X)$ is strongly exposed point of B(X) and norm $\|\cdot\|$ of X is Frechet differentiable at $x \in S(X)$. Let us to show dual norm $\|\cdot\|^*$ is *LUR*. Let f_n in $B(X^*)$ such that $\| f_n + f \| \to 2$.We have to show $\| f_n - f \| \to 0$. Choose x_n in $B(X)$ such that

$$(f_n + f)(x_n) \to 2.$$

It follows that

$$2 \geq \| f_n + f \| \geq (f_n + f)(x_n) \to 2$$
$$\Rightarrow \|f_n + f\| \to 2$$

Moreover,

$$f(x_n) \to 1 = f(x).$$

So,

$$\| x_n - x \| \to 0,$$

since $x \in S(X)$ is strongly exposed by f in $B(X)$. Now combining this with the fact that

$$f_n(x_n) \to 1 \Rightarrow f_n(x) \to 1.$$

But $\|\cdot\|$ is Frechet differentiable so by Symulian's Theorem we have,

$$\| f_n - f \|^* \to 0.$$

Thus, the dual norm $\|\cdot\|^*$ is LUR. □

Thus, Theorem 6.7 together with Theorem 3.3, gives the positive answer to the Problem 3.8.

7. Sub-differentiability and ϵ -sub differentiability

Definition 7.1. Borwein et al. (2010).

Let X be a Banach space. The Fenchel conjugate of the function $f: X \to [-\infty, +\infty]$ is the function $f^*: X^* \to [-\infty, +\infty]$ defined by

$$f^*(x^*) := \sup_{x \in X} \{x^*(x) - f(x)\}.$$

The function f^* is convex. Its conjugate is called bi-conjugate and it is denoted by f^{**} and it is a function on X^{**}. If it is finite then we say that f is co-finite.

Definition 7. 2.

Let f be a continuous convex function defined on a Banach space X. Then sub differential of the function f at $x \in X$ is the set

$$\partial f(x) = \{x^* \in X^*: x^*(y - x) \leq f(y) - f(x) \text{ for all } x \in X\}.$$

We write

$$\partial f(X) = \cup_{x \in X} \partial f(x).$$

The sub-differentiable of the mapping $\partial \|\cdot\|$ of some norm: $X \to 2^{B(X^*)}$ is defined by

$$\partial \|x\| = \{x^* \in X^* : \langle x^*, x \rangle = \|x\|, \langle x^*, z\rangle \leq \|z\| \text{ for all } z \in X\}.$$

We remark that the norm $\|\cdot\|$ is Gateaux differentiable at x if $\partial \|x\| = \{x^*\}$. In this case we say that x^* *is* Gateaux derivative of the norm $\|\cdot\|$ at x and this equivalent to

$$\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t} = \langle x^*, y \rangle \quad \text{for all } y \in X.$$

Definition 7. 3.

Let $f : X \to (-\infty, +\infty)$ be lower semi continuous function, $\epsilon > 0$, $x_0 \in \mathcal{D}(f)$. Then f is said to be ϵ - sub differential at x_0 if

$$\partial_\epsilon f(x_0) = \{\phi \in X^* : \phi(x) - \phi(x_0) \leq f(x) - f(x_0) + \epsilon \text{ for all } x \in X\}.$$

Remark 7.4.

(i) $x_0 \notin dom\, f \Rightarrow \partial_\epsilon f(x_0) = \phi$

(ii) Let f be lower semi continuous and $\epsilon > 0$ be given. Then, $\partial_\epsilon f(x_0) \neq \phi$.

7.1. Some Results

Symulian Lemma Deville et al. (1993, p. 347).

Let f be a continuous convex function on a Banach space X. Then the following are equivalent:

(i) f is Frechet differential at x_0,

(ii) $\|\phi_n - \phi\| \to 0$ if $\phi \in \partial f(x_0)$ $\phi_n \in \partial_{\epsilon_n} f(x_0)$, $\epsilon_n \downarrow 0$.

Theorem 7.1.1. Borwein et al. (2010, p .171).

Let $f : X \to (-\infty, +\infty)$, $x \in X$, $x^* \in X^*$ and $\epsilon > 0$. Then,

$$x^* \in \partial f(x), \; f(x) + f^*(x^*) \leq x^*(x) + \epsilon \;.$$

Theorem 7.1.2. Borwein et al (2010, Theorem 4.4.5).

Let $f : X \to (-\infty, +\infty)$, $x_0 \in \mathcal{D}(f)$. Then

(a) If $\phi \in \partial f(x_0)$ then $x_0 \in \partial f^*(\phi)$.The converse is true if f is convex and lsc at x_0.
(b) If $\phi \in \partial_\epsilon f(x_0)$ then a point $x_0 \in \partial f^*(\phi)$. The converse is true if f is convex and lsc at x_0.

Theorem 7. 1.3. Borwein et al. (2010, pp. 171-225).

Suppose X is a reflexive Banach space and f: $X \to (-\infty,+\infty)$ is lsc proper function . Let f is continuous co -finite and f and f^* are both Frechet differentiable. Then f and f^* are both locally uniform convex (LUR).

8. Some typical Results

It is shown in Malto et al. (2009); Orihuela, J. (2007) that for a Banach space X with a Frechet differentiable norm which has Gateaux differentiable dual norm the duality map δ provides a σ -slicely continuous and co-σ -continuous map between the unit spheres of X and the dual space X^*, and therefore X admits an equivalent locally uniformly rotund norm. It seems to be possible that the requirement on the dual norm could be relaxed by looking for σ -continuous maps from the dual space X^* into X such that the composition with the duality map would provide enough σ -slicely continuous maps from X into X to approximate the identity map, and to finally get the LUR renormability of the space X itself. So we propose to study the following.

Question 8.1. Orihuela, J. (2007).

Let X be a Banach space with a Frechet differentiable norm. Is it possible to construct a sequence of σ -continuous maps for the norm topologies $\phi_p : X^* \to X$ such that the sequence $\{\phi_p o\, \delta : p \in \mathbb{N}\}$ will provide a way to approximate the identity map on the unit sphere of X ? For instance, in such a way that:

$$0 \in \overline{\{(Id - \phi_p)^n(x) : n, p \in \mathbb{N}\}}^{\sigma(X, X^*)} \text{ for all } x \in S(X).$$

A place to begin to look for the sequence $(\phi_p\, o\, \delta)$ could be modifications of the Toruncyck homeomorphism between the dual X^* and X for every Asplund space X since the density characters of X and X^* coincide. If so, the Banach space will be LUR renormable and we will have a positive answer to the above open Problem 3.8 Molto et al. (2009) .

Remark 8.2. Godefroy et al. (1993); Fabian et al. (1983).

Recall that X need not be WCG if X^* is WCG. Further, even if X is WCG and admits a C^k - smooth norm (Note that here, C^k - smoothness is in the Frechet sense) we **cannot ensure** that such a norm also can be made LUR.

Theorem 8.3. Yost, D. (1981)

Let X be a reflexive Banach space. Then a space X can be renormed so that X^* has Frechet differentiable norm, but X is not locally uniformly convex.

Remark 8.4. Wulbert, D. (1986)

An equivalent norm $|||\cdot|||$ on $C_0(\Gamma)$ is LUR and Frechet smooth.

Theorem 8.5. Zizler, V. (1984)

Suppose that X is a Banach space which admits a real valued continuously Frechet differentiable function with bounded non-empty support. Then a space X^* admits an equivalent locally uniformly rotund norm.

9. Conclusion

In this paper, we introduce and study the concept of LUR norm and Frechet derivative of a norm on a Banach space X. The open Problem 3.8 is seen side by side to get its solution and more observation is done in different settings. Thus, we summarize exactly the above open problem. In my research work no examples and no results have found the question has right or wrong answer. Though on observing Theorems 3.9, 3.10, 3.11, Theorem 4.3 together with Theorem 3.3, Theorems 5.1, 5.2, 6.5, 7. 1. 4, Theorem 6.7 together with Theorem 3.3, Question 8.1, and some typical results in Theorems 8.3, 8.5 and Remark 8.4 above, the Problem 3.8 has the positive answer. Further, Remark 8.2 gives negative answer. But the general case still remains open. These types of additional conditions to the above problem opens gate to the researchers to make fertile field in the garden of the renorming theory.

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