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# Laminar Boundary Layer Flow of Sisko Fluid

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## Abstract:

The problem of steady two dimensional laminar boundary layer flow of non-Newtonian fluid is analyzed in the present paper. Sisko fluid model, one of the various fluid models of non-Newtonian fluid, is considered for stress-strain relationship. Similarity and numerical solutions obtained for the defined flow problem.

**Keywords:** Group theory, boundary layer, non-Newtonian, Sisko fluid, laminar flow, Stress, Strain, Similarity solutions

**MSC2010:** 65L99, 76A05, 76D05, 76D10, 76W05

## Nomenclature:

$u$, v - Velocity components in $X$ and $Y$ directions, respectively
$U$ - Main stream velocities in $X$ direction
$\overline{\tau}$ - Stress component
$\overline{\Delta}$ - Strain rate component
$a,b$ - Sisko fluid parameters
$n$- Flow behaviour indices
$\eta$ - Similarity variable
$f$, g - Similarity functions
$A, \alpha_1, ..., \alpha_5$ - Real constants

## 1. Introduction:

The complex rehology of biological fluids has motivated investigations involving different non-Newtonian fluids. In recent years, non-Newtonian fluids have become more and more important industrially. Academic curiosity and practical applications have generated considerable interest in finding the solutions of differential equations governing the motion of non-Newtonian fluids. The property of these fluids is that the stress tensor is related to the rate of deformation tensor by some non-linear relationship. These fluids flow problems present some interesting challenges to researchers in engineering, applied mathematics and computer science. Many materials such as drilling mud, clay coating and other suspensions, certain oils and greases, polymer melts, blood, paints and certain oils, elastomers and many emulsions and some other thin and thick oils have been treated as non-Newtonian fluids.

Because of the difficulty to suggest a single model, which exhibits all properties of non-Newtonian fluids, they cannot be described as Newtonian fluids, and there has been much confusion in the constitutive classification of non-Newtonian fluids. Non-Newtonian fluids are usually classified as: (i) fluids for which shear stress depends only on the rate of shear; (ii) fluids for which relation between shear stress and rate of strain depends on time; (iii) the viscoinelastic fluids which possess both elastic and viscous properties. Thus, for any non-Newtonian fluid the mathematical structure of the shearing stress and the rate of shear is always important. But derivation of such mathematical formulation is indeed a difficult task.

In the past few decades, several researchers have analyzed the problems of boundary layer flow of non-Newtonian fluids past different geometries (Bird et al. (1960), Hansen et al. (1968), Kapur et al. (1982), Lee et al. (1966), Manisha et al. (2005, 2008, 2009, 2010), Timol et al. (1986, 2004), Wells (1964)). Many fluids in the real world are non-Newtonian by nature. The study of such fluids is very important due to their vast applications in the field of engineering sciences and industries. Several fluid models of non-Newtonian fluid have been investigated. A brief instruction and classification on various fluid models of non-Newtonian fluid is discussed in detail by Manisha et al. (2010, 2013). Similarity analysis of three-dimensional boundary layer equations of a class of non-Newtonian fluids in which the stress is an arbitrary function of rates of strain is made have been discussed by Pakdemirli (1994). Kapur et al. (1963) have developed the theory for similar solutions of the boundary layer equations for non-Newtonian fluids. They have also discussed briefly important particular cases like the boundary layer flow along a wedge, along a flat plate, in a convergent channel and two dimensional stagnation point flow. Bognar (2011) has derived the similarity solutions of the Prandtl boundary layer equations describing a non-Newtonian power law fluid past an impermeable flat plate, driven by a power law velocity profile. Further, analytical solutions are obtained for the steady laminar boundary layer of Power-Law non-Newtonian flow with non-linear viscosity over a flat moving plate by Nagler (2014). Also, lots of work have been carried out by many other scientist for Power-Law fluid model such as (Bizzell et al. (1962), Djukic (1973, 1974), Na et al.(1967), Patel et al. (2011,2012,2014). The self-modeling flow regime in a laminar boundary layer of non-Newtonian fluid is studied in the general case, without the restriction to a certain region of positive values of '$n$' was discussed by Zhizhin (1987). Many investigators have worked on Powell-Eyring fluid model such as Patel et al. (2009), Sirohi et al. (1984)).

Very little information is available in the literature about the fluid model proposed by Sisko (1958). Na and Hansen (1967) have examined the laminar flow of Sisko fluid between two circular parallel disks. Bhrami et al. (1996) analyzed the isothermal and axial laminar flow of Power-law fluid and Sisko fluid in an annuli. The thin film flow problem of Sisko fluid and Oldroyd fluid on a moving vertical belt was discussed by Nemati et al. (2009). They have applied Homotopy analysis method (HAM) to solve the equations of flow problem. The problem of Sisko fluid passing through an axisymmetric uniform tube was solved analytically using the perturbation method and HAM by Nadeem et al. (2010). Moallemi et al. (2011) have applied the Homotopy perturbation method to solve the flow of a Sisko fluid in pipes. Sari et al. (2012) have applied Lie group analysis to obtain similarity solutions of the boundary layer flow of Sisko fluids.

The numerical solution for the time-dependent free convective flow of Sisko fluid past flat plate moving through a binary mixture has been obtained by Olanrewaju et al. (2013). Siddiqui et al. (2013) have examined the drainage of Sisko fluid film down a vertical belt. The approximate solution of the governing equations was obtained using Perturbation method and Adomian decomposition method in their paper. Asghar et al. (2014) have presented the equations for the peristaltic flow of MHD Sisko fluid in a channel. They have considered the effect of strong and weak magnetic fields.

Mathematically, Sisko Model can be written as in Sisko (1958)

$$\bar{\tau} = -\left\{ a + b \left| \sqrt{\frac{1}{2}\left(\bar{\Delta} : \bar{\Delta}\right)} \right|^{(n-1)} \right\} \bar{\Delta},$$

where $\bar{\tau}$ and $\bar{\Delta}$ are the stress tensor and the rate of deformation tensor, respectively; *a*, *b* and *n* are defined differently for different fluids. In the present paper, the problem of steady two dimensional laminar boundary layer flow of non-Newtonian fluid is analyzed. Sisko fluid model, one of the various fluid models of non-Newtonian fluid, is considered for stress-strain relationship. The cases of Newtonian fluid and Power-law fluid are also discussed. Similarity and numerical solutions are obtained for the defined flow problem. All the cases are presented graphically.

## 2. Governing Equations

The two dimensional laminar boundary layer flows past a semi-infinite flat plate is considered. The geometry of present flow problem is shown in Figure 1. The governing equations of continuity and momentum of laminar boundary layer flow of Sisko fluid past a semi-infinite flat plate are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \frac{\partial}{\partial y}\left\{ \left[ a + b\left(\frac{\partial u}{\partial y}\right)^{n-1} \right] \frac{\partial u}{\partial y} \right\}, \tag{2}$$

with boundary conditions:

$$y = 0 : u = 0,\ v = 0, \tag{3}$$

$$y = \infty : u = \underline{U}(x), \tag{4}$$

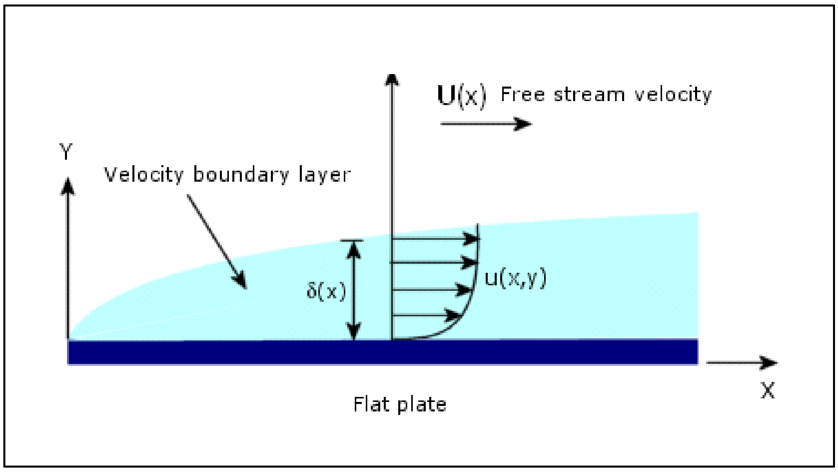


**Figure 1.** Flow Geometry

Taking one parameter scaling Group Transformation,

$$x = A^{\alpha_1}\bar{x},\ y = A^{\alpha_2}\bar{y},\ u = A^{\alpha_3}\bar{u},\ v = A^{\alpha_4}\bar{v},\ U = A^{\alpha_5}\bar{U}, \tag{5}$$

introducing (5) in Equations (1) - (2) and using simple chain rule we get:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = A^{\alpha_3-\alpha_1}\frac{\partial \bar{u}}{\partial \bar{x}} + A^{\alpha_4-\alpha_2}\frac{\partial \bar{v}}{\partial \bar{y}},$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - U\frac{dU}{dx} - a\frac{\partial^2 u}{\partial y^2} - nb\left(\frac{\partial u}{\partial y}\right)^{n-1}\frac{\partial^2 u}{\partial y^2}$$

$$= A^{2\alpha_3-\alpha_1}\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + A^{\alpha_4+\alpha_3-\alpha_2}\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} - A^{2\alpha_5-\alpha_1}\bar{U}\frac{d\bar{U}}{d\bar{x}} - A^{\alpha_3-2\alpha_2}a\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$

$$-nbA^{(n-1)\alpha_3+\alpha_3-(n-1)\alpha_2-2\alpha_2}\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^{n-1}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}.$$

The above set of equations remain invariant provided:

$$\alpha_3 - \alpha_1 = \alpha_4 - \alpha_2,$$
$$2\alpha_3 - \alpha_1 = \alpha_4 + \alpha_3 - \alpha_2 = 2\alpha_5 - \alpha_1 = \alpha_3 - 2\alpha_2 = n\alpha_3 - (n+1)\alpha_2.$$

Solving the above relations for $\alpha$ 's we get:

$$\alpha_1 = 3\alpha_2 = 3\alpha_3 = -3\alpha_4 = 3\alpha_5,$$

put $\alpha = \dfrac{\alpha_2}{\alpha_1} = \dfrac{1}{3}$

$$\alpha = \alpha_3 = -\alpha_4 = \alpha_5.$$

Following Seshadri et al. (1985) one can derive the absolute invariants, so called similarity independent variable $\eta$ and similarity dependent variables $f(\eta)$ $g(\eta)$ and $h(\eta)$ as follows:

$$\eta = \frac{y}{x^{\alpha}} = \frac{y}{x^{\frac{1}{3}}} = x^{-\frac{1}{3}} y, \tag{6}$$

$$\mathrm{f}'(\eta) = \frac{u}{x^{\frac{1}{3}}}, \tag{7}$$

$$g(\eta) = \frac{v}{x^{-\frac{1}{3}}}, \tag{8}$$

$$h(\eta) = \frac{U}{x^{\frac{1}{3}}} = c_1. \tag{9}$$

Using Equations (6)-(9), Equations (1) and (2) are transformed into the following:

$$f'(\eta) - \eta f''(\eta) + 3g'(\eta) = 0, \tag{10}$$

$$f'^2(\eta) - \eta f'(\eta) f''(\eta) + 3g(\eta) f''(\eta) = c_1^2 + 3af'''(\eta) + 3nb\left(f''(\eta)\right)^{n-1} f'''(\eta). \tag{11}$$

Now, putting the value of $g = \frac{1}{3}\left[\eta f'(\eta) - 2f(\eta)\right]$ in (11), from (10) we obtain

$$f'^2(\eta) - 2f(\eta) f''(\eta) - 3af'''(\eta) - 3nb\left(f''(\eta)\right)^{n-1} f'''(\eta) - c_1^2 = 0, \tag{12}$$

with the boundary conditions (3) and (4):

$$f(0) = 0,\ f'(0) = 0,\ f'(\infty) = 1. \tag{13}$$

**Case I**:

If we take $a$ = 0, $b$ = 1 and $n$ = $n$ in the Sisko fluid model then we obtained the Power-law fluid model. For this case the similarity solution (12) will be reduced in the following equation with the same boundary conditions given in Equation (13):

$$f'^2(\eta) - 2f(\eta) f''(\eta) - 3n\left(f''(\eta)\right)^{n-1} f'''(\eta) - c_1^2 = 0. \tag{14}$$

**Case II**:

If we take $a$ = 1, $b$ = 0 and $n$ = 1 in the Sisko fluid model then we obtained the stress strain relationship of Newtonian fluid. For this case the similarity solution (12) will be reduced in the following equation with the same boundary conditions given in Equation (13):

$$f'^2(\eta) - 2f(\eta)f''(\eta) - 3f'''(\eta) - c_1^2 = 0. \quad (15)$$

## 3. Results and discussions

The similarity solutions obtained for the defined flow problem are solved numerically and presented graphically in this paper. Figures 2-4 are given for the velocity profile *f'* versus eta of laminar boundary layer flow of Sisko fluid past a semi-infinite plate. In Figure 2 the fluid parameters are kept constant and different values of fluid index are considered. Figure 3 and Figure 4 represent the velocity vs. eta for fixed values of fluid index and different values of fluid parameters $b$ and $a$, respectively. It shows that the velocity profile decreases with decreasing fluid parameters $b$ and $a$, respectively.  These types of results are also discussed by Siddiqui et al. (2013) in their paper entitled, *Analytic solution for the drainage of Sisko fluid film down a vertical belt*. Figure 5 represents the velocity versus eta for case I, i.e., Power-law model. Figure 6 represents the velocity versus eta for Newtonian fluid (case II). The velocity increases as eta increases.

## 4. Conclusion:

The methods for obtaining similarity transformations are devided into two categories: (i) Direct methods, and (ii) group thoretic methods. The direct methods such as, seperation of variables do not invoke group invariance. On the other hand, group theoretic methods are more elegant mathematically. The main concept of invariance under a group of transformation is always invoked. In the present paper, the governing non-linear partial differential equations of the laminar boundary layer flow of non-newtonian fluid are tranformed into non-linear ordinary differential equations using group thoretic method. The Sisko fluid model of non-Newtonian fluids is considered for the stress-strain relationship. Then the obtained ODE is solved by ODE solver. From Figure 2 and Figure 5, we conclude that the velocity profile increases more rapidly for Sisko fluid than it is in Power-law fluid.

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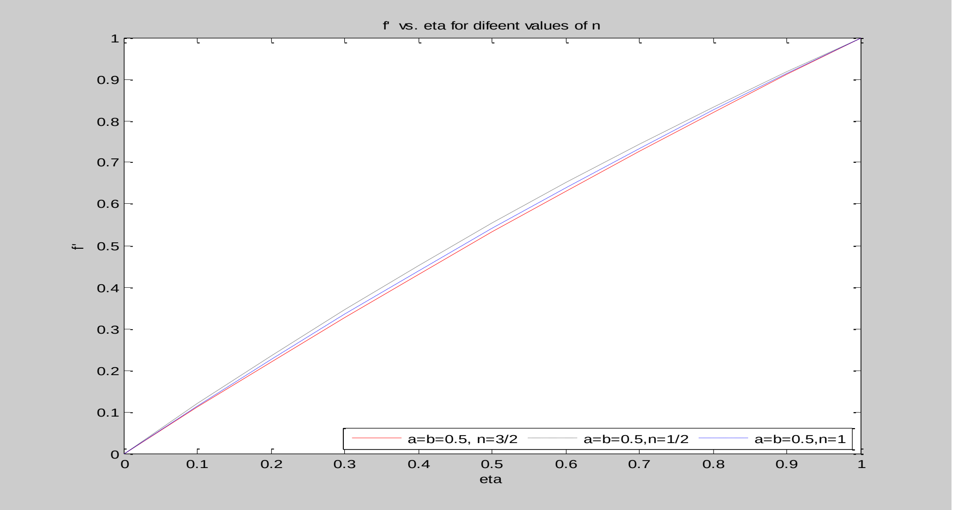


**Figure 2.** Velocity profile for a=b=0.5 and different values of fluid index *n*

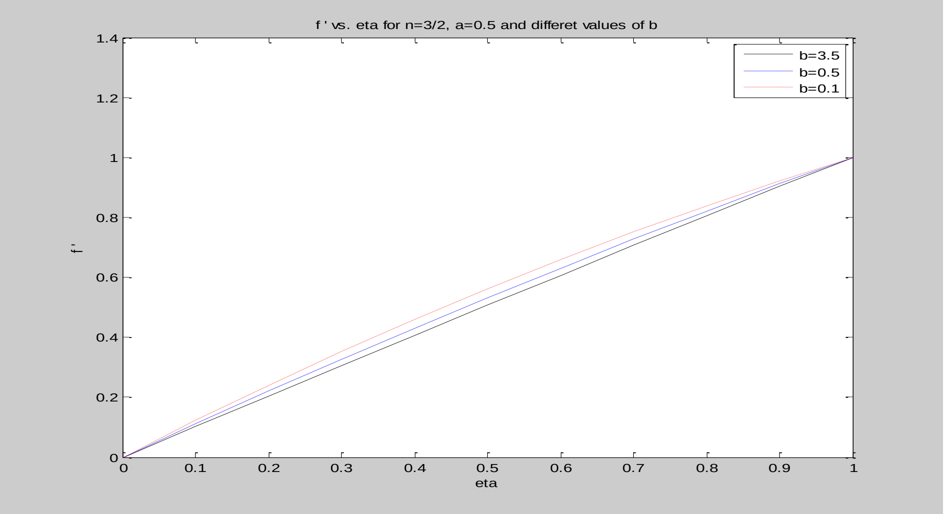


**Figure 3.** Velocity profile for *a*=0.5, *n*=3/2 and different values of fluid parameter ‘*b*’

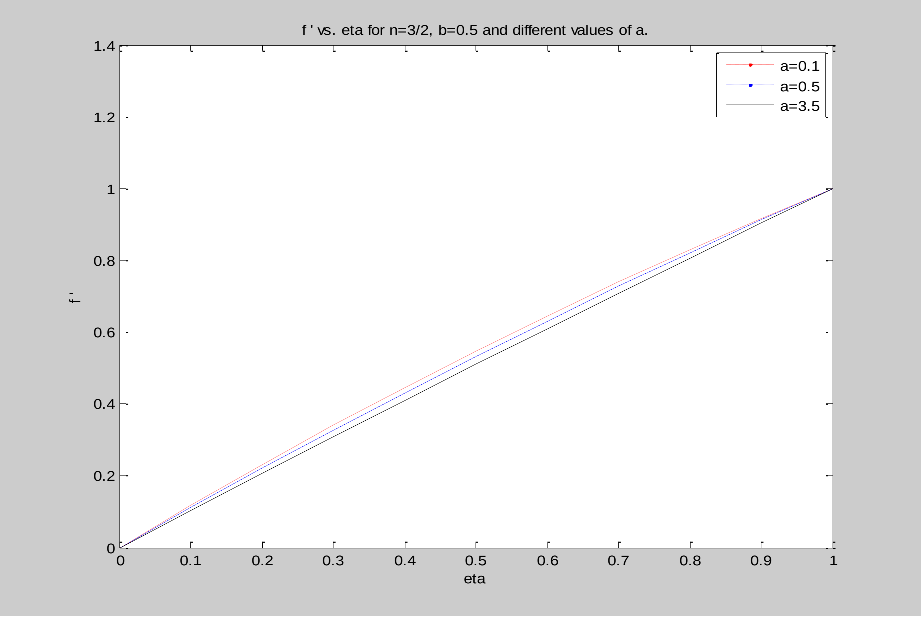


**Figure 4.** Velocity profile for *b*=0.5, *n*=3/2 and different values of fluid parameter ‘*a*’

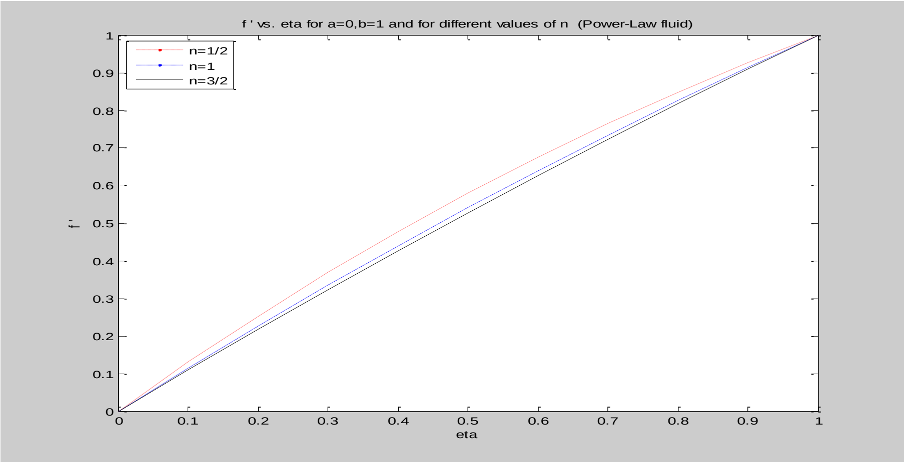


**Figure 5.** Velocity profile for $a$=0, $b$=1 and different values of fluid index '$n$' (Power-Law Fluid)

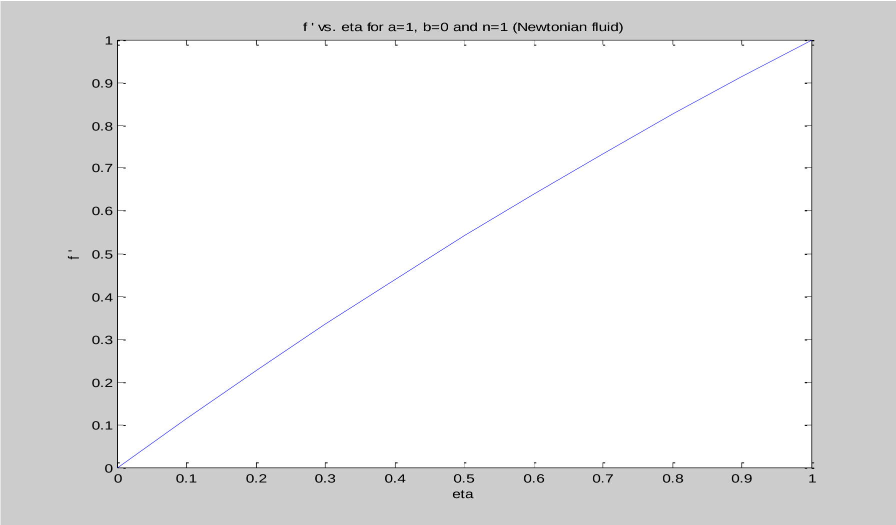


**Figure 6.** Velocity profile for Newtonian fluid, i.e., $a$=1, $b$=0 and $n$=1

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