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Some New Sequence Spaces

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Abstract

In the present paper we introduce some new sequence spaces defined by a Musielak-Orlicz function on semi normed spaces. We make an effort to study some topological properties and inclusion relations between these spaces. The study of sequence spaces over n-normed spaces has also been initiated in this paper.

Keywords: Orlicz function; Musielak-Orlicz function; modulus function; paranorm space; sequence space; n-normed spaces

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1. Introduction and Preliminaries

Definition 1.1.

Let X be a linear metric space. A function $p : X \to R$ is called paranorm, if

(1) $p(x) \geq 0$ for all $x \in X$;

(2) $p(-x) = p(x)$ for all $x \in X$;

(3) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$; and

(4) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called *total paranorm* and the pair (X, p) is called a *total paranormed space*.

It is well known that the metric of any linear metric space is given by some total paranorm, see Wilansky (1984),Theorem 10.4.2, pp. 183.

Definition 1.2.

An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Lindenstrauss and Tzafriri (1971) used the idea of Orlicz function to define the following sequence space:

Let w be the space of all real or complex sequences $x = (x_k)$. Then,

$$l_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{for all } \rho > 0 \right\}, \tag{1}$$

which is called as an Orlicz sequence space. The space l_M is a Banach space with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}. \tag{2}$$

It is shown by Lindenstrauss and Tzafriri (1971) that every Orlicz sequence space l_M contains a subspace isomorphic to l_p $(p \ge 1)$. An Orlicz function M satisfies Δ_2 – condition if and only if for any constant $L > 1$ there exists a constant $K(L)$ such that $M(Lu) \le K(L)M(u)$, for all values of $u \ge 0$. An Orlicz function M can always be represented in the following integral form:

$$M(x) = \int_0^x \eta(t)dt, \tag{3}$$

where η is known as the kernel of M, is right differentiable for $t \ge 0$, $\eta(0) = 0$, $\eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

Definition 1.3.

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function, see Maligranda (1989), and Musielak (1983). A sequence $\mathcal{N} = (N_k)$ defined by

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\}, \quad k = 1,2,... \tag{4}$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows:

$$t_{\mathcal{M}} = \{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\}, \tag{5}$$

$$h_{\mathcal{M}} = \{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0\}, \tag{6}$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), \; x = (x_k) \in t_{\mathcal{M}}. \tag{7}$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$\|x\| = \inf\left\{k > 0 : I_{\mathcal{M}}\left(\frac{x}{k}\right) \leq 1\right\} \tag{8}$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf\left\{\frac{1}{k}\big(1 + I_{\mathcal{M}}(kx)\big) : k > 0\right\}. \tag{9}$$

The notion of difference sequence spaces was introduced by Kizmaz (1981), who studied the difference sequence $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Colak (1995) by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let m, n be non-negative integers, then for Z a given sequence space, we have

$$Z(\Delta_m^n) = \{x = (x_k) \in w : (\Delta_m^n x_k) \in Z\}, \tag{10}$$

for $Z = c, c_0$ and l_∞, where $\Delta_m^n x = (\Delta_m^n x_k) = (\Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+m})$ and $\Delta_m^0 x_k = x_k$, for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation:

$$\Delta_m^n x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+mv}. \tag{11}$$

Taking $m=1$, we get the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$ studied by Et and Colak (1995). Taking $m=n=1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kizmaz (1981). The difference sequence spaces were also discussed in Altinok et al. (2006), Isik (2004), and Tripathy et al. (2006). For more details about sequence spaces see [Altinok (2008), Et et al. (2006), Isik et al. (2013), Mursaleen (1983, 1983), Mursaleen et al. (1999), Raj et al. (2010, 2011), Raj and Sharma (2011, 2011), Savas (2004)] and references therein.

Let X be a complex linear space and (X,q) be a semi normed space with semi norm q. By $S(X)$ we denote the linear space of all sequences $x=(x_k)$ with $(x_k)\in X$ and the usual co-ordinate wise operations:

$$\alpha x = (\alpha x_k)$$

and

$$x + y = (x_k + y_k) \tag{12}$$

for each $\alpha \in \mathbb{C}$, where $\mathbb{C}$ denotes the set of all complex numbers. A study of sequence spaces $l_M(p,q,s)$ on semi normed spaces one can see in Bektas and Altin (2003).

The following inequality will be used throughout the paper. If $0 \le p_k \le \sup p_k = H$, $K = \max(1, 2^{H-1})$, then

$$|a_k + b_k|^{p_k} \le K\{|a_k|^{p_k} + |b_k|^{p_k}\}, \quad \text{for all } k \text{ and } a_k, b_k \in \mathbb{C}. \tag{13}$$

Also $|a|^{p_k} \le \max(1, |\alpha|^H)$, for all $\alpha \in \mathbb{C}$.

2. Sequence Spaces on Semi Normed Spaces Defined by a Musielak-Orlicz Function

Let $\lambda = (\lambda_k)$ be a scalar sequence and $x \in S(X)$ then we shall write $\lambda x = (\lambda_k x_k)$. Let U be the set of all sequences $u=(u_k)$ such that $u_k \ne 0$, $p=(p_k)$ be a sequence of strictly positive real numbers and $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. In the present section we define the sequence spaces:

$$l_{\mathcal{M}}(\Delta_m^n, u, p, q, s) =$$

$$\left\{ x \in S(X) : \sum_{k=1}^{\infty} k^{-s} \left[M_k \left(q\left(\frac{u_k \Delta_m^n x_k}{\rho} \right)\right)\right]^{p_k} < \infty, \text{ for some } \rho > 0 \text{ and } s \geq 0 \right\}. \tag{14}$$

If we take $p = (p_k) = 1$, for all $k \in \mathbb{N}$, then

$$l_{\mathcal{M}}(\Delta_m^n, u, q, s) =$$

$$\left\{ x \in S(X) : \sum_{k=1}^{\infty} k^{-s} \left[M_k \left(q\left(\frac{u_k \Delta_m^n x_k}{\rho} \right)\right)\right] < \infty, \text{ for some } \rho > 0 \text{ and } s \geq 0 \right\}. \tag{15}$$

If we take $s = 0$, we have

$$l_{\mathcal{M}}(\Delta_m^n, u, p, q) = \left\{ x \in S(X) : \sum_{k=1}^{\infty} \left[M_k \left(q\left(\frac{u_k \Delta_m^n x_k}{\rho} \right)\right)\right]^{p_k} < \infty, \text{ for some } \rho > 0 \right\}. \tag{16}$$

If we take $p = (p_k) = 1$, for all k and $s = 0$, we get

$$l_{\mathcal{M}}(\Delta_m^n, u, q) = \left\{ x \in S(X) : \sum_{k=1}^{\infty} \left[M_k \left(q\left(\frac{u_k \Delta_m^n x_k}{\rho} \right)\right)\right] < \infty, \text{ for some } \rho > 0 \right\}. \tag{17}$$

If we take $s = 0$, $q(x) = |x|$ and $X = \mathbb{C}$, we have

$$l_{\mathcal{M}}(\Delta_m^n, u, p) = \left\{ x \in S(X) : \sum_{k=1}^{\infty} \left[M_k \left(\frac{|u_k \Delta_m^n x_k|}{\rho} \right)\right]^{p_k} < \infty, \text{ for some } \rho > 0 \right\}. \tag{18}$$

The main purpose of this section is to study some topological properties and some inclusion relations between the sequence spaces which we have defined above.

Theorem 2.1.

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers. Then $l_{\mathcal{M}}(\Delta^n_m, u, p, q, s)$ is a linear space over the field of complex numbers $\mathbb{C}$.

Proof:

Let $x = (x_k)$, $y = (y_k) \in l_{\mathcal{M}}(\Delta^n_m, u, p, q, s)$ and $\alpha, \beta \in \mathbb{C}$. Then, there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k \Delta^n_m x_k}{\rho_1}\right)\right)\right]^{p_k} < \infty \tag{19}$$

and

$$\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k \Delta^n_m y_k}{\rho_2}\right)\right)\right]^{p_k} < \infty . \tag{20}$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_k)$ is nondecreasing, convex and q is a semi norm by using inequality (13), we have

$$\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k(\alpha\Delta^n_m x_k + \beta\Delta^n_m y_k)}{\rho_3}\right)\right)\right]^{p_k}$$

$$\leq \sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{\alpha u_k \Delta^n_m x_k}{\rho_3}\right) + q\left(\frac{\beta u_k \Delta^n_m y_k}{\rho_3}\right)\right)\right]^{p_k}$$

$$\leq \sum_{k=1}^{\infty} \frac{1}{2^{p_k}} k^{-s}\left[M_k\left(q\left(\frac{u_k \Delta^n_m x_k}{\rho_1}\right)\right) + M_k\left(q\left(\frac{u_k \Delta^n_m y_k}{\rho_2}\right)\right)\right]^{p_k}$$

$$\leq \sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k \Delta^n_m x_k}{\rho_1}\right)\right) + M_k\left(q\left(\frac{u_k \Delta^n_m y_k}{\rho_2}\right)\right)\right]^{p_k}$$

$$\leq K\sum_{k=1}^{\infty}k^{-s}\left[M_k\left(q\left(\frac{u_k\Delta_m^n x_k}{\rho_1}\right)\right)\right]^{p_k}+K\sum_{k=1}^{\infty}k^{-s}\left[M_k\left(q\left(\frac{u_k\Delta_m^n y_k}{\rho_2}\right)\right)\right]^{p_k}. \tag{21}$$

Thus, $\alpha x+\beta y\in l_{\mathcal{M}}(\Delta_m^n,u,p,q,s)$. Hence, $l_{\mathcal{M}}(\Delta_m^n,u,p,q,s)$ is a linear space.

Theorem 2.2.

Suppose $\mathcal{M}=(M_k)$ is a Musielak-Orlicz function and $p=(p_k)$ be a bounded sequence of positive real numbers. Then $\mathrm{l}_{\mathcal{M}}(\Delta_m^n,u,p,q,s)$ is a paranormed space with the paranorm defined by

$$g(x)=\inf\left\{\rho^{\frac{p_n}{M}}:\left(\sum_{k=1}^{\infty}k^{-s}\left[M_k\left(q\left(\frac{u_k\Delta_m^n x_k}{\rho}\right)\right)\right]^{p_k}\right)^{\frac{1}{M}}\leq 1, s\geq 0, n=1,2,3,\ldots\right\}, \tag{22}$$

where

$M=\max(1,\sup_k p_k)$.

Proof:

(i) Clearly, $g(x)\geq 0$, for $x=(x_k)\in l_{\mathcal{M}}(\Delta_m^n,u,p,q,s)$. Since $M_k(0)=0$, we get $g(0)=0$.

(ii) $g(-x)=g(x)$.

(iii) Let $x=(x_k)$, $y=(y_k)\in l_{\mathcal{M}}(\Delta_m^n,u,p,q,s)$. Then, there exist ρ_1, $\rho_2>0$ such that

$$\sum_{k=1}^{\infty}k^{-s}\left[M_k\left(q\left(\frac{u_k\Delta_m^n x_k}{\rho_1}\right)\right)\right]^{p_k}\leq 1 \tag{23}$$

and

$$\sum_{k=1}^{\infty}k^{-s}\left[M_k\left(q\left(\frac{u_k\Delta_m^n x_k}{\rho_2}\right)\right)\right]^{p_k}\leq 1. \tag{24}$$

If $\rho=\rho_1+\rho_2$, then by Minkowski's inequality we have

$$\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k\Delta_m^n(x_k+y_k)}{\rho}\right)\right)\right]^{p_k} = \sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k\Delta_m^n x_k + u_k\Delta_m^n y_k}{\rho_1+\rho_2}\right)\right)\right]^{p_k}$$

$$\leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right)\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k\Delta_m^n x_k}{\rho_1}\right)\right)\right]^{p_k} + \left(\frac{\rho_2}{\rho_1+\rho_2}\right)\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k\Delta_m^n x_k}{\rho_2}\right)\right)\right]^{p_k}$$

$$\leq 1, \tag{25}$$

and thus,

$$g(x+y) = \inf\left\{(\rho)^{\frac{p_n}{M}} : \left(\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k\Delta_m^n x_k + u_k\Delta_m^n y_k}{\rho}\right)\right)\right]^{p_k}\right)^{\frac{1}{M}} \leq 1, s \geq 0, \rho > 0\right\}$$

$$\leq \inf\left\{(\rho_1)^{\frac{p_n}{M}} : \left(\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k\Delta_m^n x_k}{\rho_1}\right)\right)\right]^{p_k}\right)^{\frac{1}{M}} \leq 1, s \geq 0, \rho_1 > 0\right\}$$

$$\leq \inf\left\{(\rho_2)^{\frac{p_n}{M}} : \left(\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k\Delta_m^n x_k}{\rho_2}\right)\right)\right]^{p_k}\right)^{\frac{1}{M}} \leq 1, s \geq 0, \rho_2 > 0\right\}$$

$$\leq g(x) + g(y). \tag{26}$$

(iv) Finally, we prove that scalar multiplication is continuous. Let λ be any complex number,

$$g(\lambda x) = \inf\left\{\rho^{\frac{p_n}{M}} : \left(\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{\lambda u_k\Delta_m^n x_k}{\rho}\right)\right)\right]^{p_k}\right)^{\frac{1}{M}} \leq 1, s \geq 0, n = 1,2,3,\ldots\right\},$$

$$= \inf\left\{ |\lambda| r^{\frac{p_n}{M}} : \left(\sum_{k=1}^{\infty} k^{-s} \left[M_k \left(q\left(\frac{u_k \Delta_m^n x_k}{\rho} \right) \right) \right]^{p_k} \right)^{\frac{1}{M}} \leq 1, s \geq 0, n = 1,2,3,\ldots \right\}, \tag{27}$$

where $r = \dfrac{\rho}{|\lambda|}$. Hence, $l_{\mathcal{M}}(\Delta_m^n, u, p, q, s)$ is a paranormed space.

Theorem 2.3.

(i) Let $0 < p_k < t_k < \infty$, for each $k \in \mathbb{N}$. Then, $l_{\mathcal{M}}(\Delta_m^n, u, p, q) \subseteq l_{\mathcal{M}}(\Delta_m^n, u, t, q)$,

(ii) Let $0 < p_k < t_k < \infty$, for each $k \in \mathbb{N}$. Then, $l_{\mathcal{M}}(\Delta_m^n, u, p) \subseteq l_{\mathcal{M}}(\Delta_m^n, u, t)$,

(iii) $l_{\mathcal{M}}(\Delta_m^n, u, q) \subseteq l_{\mathcal{M}}(\Delta_m^n, u, q, s)$,

(iv) $l_{\mathcal{M}}(\Delta_m^n, u, p, q) \subseteq l_{\mathcal{M}}(\Delta_m^n, u, p, q, s)$.

Proof:

(i) Let $x = (x_k) \in l_{\mathcal{M}}(\Delta_m^n, u, p, q, s)$. Then, there exists some $\rho > 0$ such that

$$\sum_{k=1}^{\infty} k^{-s} \left[M_k \left(q\left(\frac{u_k \Delta_m^n x_k}{\rho} \right) \right) \right]^{p_k} < \infty, \tag{28}$$

which implies that $M_k\left(q\left(\dfrac{u_k \Delta_m^n x_k}{\rho} \right) \right) \leq 1$ for sufficiently large value of k, say $k \geq m_0$ for some fixed $m_0 \in \mathbb{N}$. Since $\mathcal{M} = (M_k)$ is nondecreasing, we get

$$\sum_{k=1}^{\infty} k^{-s} \left[M_k \left(q\left(\frac{u_k \Delta_m^n x_k}{\rho} \right) \right) \right]^{t_k} < \infty. \tag{29}$$

Thus, we have

$$\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k\Delta_m^n x_k}{\rho}\right)\right)\right]^{t_k} \leq \sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k\Delta_m^n x_k}{\rho}\right)\right)\right]^{p_k} < \infty . \tag{30}$$

This shows that $x=(x_k) \in l_{\mathcal{M}}(\Delta_m^n, u, p, q, s)$. Hence, $l_{\mathcal{M}}(\Delta_m^n, u, p, q) \subseteq l_{\mathcal{M}}(\Delta_m^n, u, t, q)$.

Similarly, we can prove (ii), (iii) and (iv) in view of (i).

Corollary 2.4 .

(i) If $0 < p_k \leq 1$, for all $k \in \mathbb{N}$, then $\mathrm{l}_{\mathcal{M}}(\Delta_m^n, u, p, q) \subseteq \mathrm{l}_{\mathcal{M}}(\Delta_m^n, u, q)$.

(ii) If $p_k \leq 1$, for all $k \in \mathbb{N}$, then $\mathrm{l}_{\mathcal{M}}(\Delta_m^n, u, q) \subseteq \mathrm{l}_{\mathcal{M}}(\Delta_m^n, u, p, q)$.

Proof:

(i) If we take $t_k = 1$ for all $k \in \mathbb{N}$, in Theorem 2.3 (i), we get

$$l_{\mathcal{M}}(\Delta_m^n, u, p, q) \subseteq l_{\mathcal{M}}(\Delta_m^n, u, q). \tag{31}$$

(ii) If we take $p_k = 1$ for all $k \in \mathbb{N}$, in Theorem 2.3 (i), we get

$$l_{\mathcal{M}}(\Delta_m^n, u, q) \subseteq l_{\mathcal{M}}(\Delta_m^n, u, p, q). \tag{32}$$

Theorem 2.5.

The sequence space $\mathrm{l}_{\mathcal{M}}(\Delta_m^n, u, p, q, s)$ is solid.

Proof:

Let $x=(x_k) \in l_{\mathcal{M}}(\Delta_m^n, u, p, q, s)$ i.e.

$$\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k\Delta_m^n x_k}{\rho}\right)\right)\right]^{p_k} < \infty . \tag{33}$$

Let (α_k) be a sequence of scalars such that $|\alpha_k| \leq 1$, for all $k \in \mathbb{N}$. Thus, we have

$$\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{\alpha_k u_k \Delta_m^n x_k}{\rho}\right)\right)\right]^{p_k} \le \sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k \Delta_m^n x_k}{\rho}\right)\right)\right]^{p_k} < \infty. \tag{34}$$

This shows that $(\alpha_k x_k) \in l_{\mathcal{M}}(\Delta_m^n, u, p, q, s)$, for all sequences of scalars (α_k) with $|\alpha_k| \le 1$, for all $k \in \mathbb{N}$ whenever $(x_k) \in l_{\mathcal{M}}(\Delta_m^n, u, p, q, s)$. Hence, the space $l_{\mathcal{M}}(\Delta_m^n, u, p, q, s)$ is a solid sequence space.

Theorem 2.6.

The sequence space $\mathrm{l}_{\mathcal{M}}(\Delta_m^n, u, p, q, s)$ is monotone.

Proof:

The proof is trivial so we omit it.

Corollary 2.7.

(i) Let $|u_k| \le 1$ for all $k \in \mathbb{N}$. Then $\mathrm{l}_{\mathcal{M}}(\Delta_m^n, p, q, s) \subseteq \mathrm{l}_{\mathcal{M}}(\Delta_m^n, u, p, q, s)$.

(ii) Let $|u_k| \ge 1$ for all $k \in \mathbb{N}$. Then $\mathrm{l}_{\mathcal{M}}(\Delta_m^n, u, p, q, s) \subseteq \mathrm{l}_{\mathcal{M}}(\Delta_m^n, p, q, s)$.

Proof:

It is obvious.

3. Sequence Spaces Defined by a Musielak-Orlicz Function Over n-Normed Spaces

In this section we define some sequence spaces defined by a Musielak-Orlicz function over n-normed spaces. We also study some topological properties on these spaces. The concept of 2-normed spaces was initially developed by Gähler (1965) in the mid of 1960's, while that of n-normed spaces can be found in Misiak (1989). Since, many others have studied this concept and obtained various results, see Gunawan (2001, 2001) and Gunawan and Mashadi (2001).

Let $n \in \mathbb{N}$ and X be a real vector space of dimension d, where $n \ge d$. A real valued function $\|\cdot, \ldots, \cdot\|$ on X^n satisfying the following four conditions:

(1) $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,

(2) $\|x_1, x_2, \ldots, x_n\|$ is invariant under permutation,

(3) $\|\alpha x_1, x_2, ..., x_n\| = |\alpha| \|x_1, x_2, ..., x_n\|$ for any $\alpha \in \mathbb{R}$, and

(4) $\|x + x', x_2, ..., x_n\| \leq \|x, x_2, ..., x_n\| + \|x', x_2, ..., x_n\|$ is called an n-norm on X, and the pair $(X, \|.,...,.\|)$ is called an n-normed space.

For example, we may take $X = \mathbb{R}^n$ being equipped with the n-norm $\|x_1, x_2, ..., x_n\|_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, ..., x_n$, which may be given explicitly by the formula

$$\|x_1, x_2, ..., x_n\|_E = |\det(x_{i,j})|, \tag{35}$$

where $x_i = (x_{i1}, x_{i2}, ..., x_{in}) \in \mathbb{R}^n$ for each $i = 1,2,...,n$, and E denote the Euclidean norm. Let $(X, \|.,...,.\|)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, ..., a_n\}$ be linearly independent set in X. Then, the following function $\|.,...,.\|_\infty$ on X^{n-1} defined by

$$\|x_1, x_2, ..., x_{n-1}\|_\infty = \max\{\|x_1, x_2, ..., x_{n-1}, a_i\| : i = 1,2,...,n\} \tag{36}$$

defines an $(n-1)$ norm on X with respect to $\{a_1, a_2, ..., a_n\}$.

A sequence (x_k) in an n-normed space $(X, \|.,...,.\|)$ is said to converge to some $L \in X$ in the n-norm if $\lim_{k\to\infty} \|x_k - L, z_1, ..., z_{n-1}\| = 0,$ for every $z_1, ..., z_{n-1} \in X$.

A sequence (x_k) in an n-normed space $(X, \|.,...,.\|)$ is said to be Cauchy with respect to the n-norm if $\lim_{k,p\to\infty} \|x_k - x_p, z_1, ..., z_{n-1}\| = 0,$ for every $z_1, ..., z_{n-1} \in X$.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.
In this section we define the sequence space:

$$l_{\mathcal{M}}(\Delta_m^n, u, p, \|.,...,.\|)$$

$$= \left\{ x \in S(X) : \sum_{k=1}^{\infty} \left[M_k \left(\left\| \frac{u_k \Delta_m^n x_k}{\rho}, z_1, ..., z_{n-1} \right\| \right) \right]^{p_k} < \infty, \text{for some } \rho > 0 \right\}. \tag{37}$$

Theorem 3.1.

Let $\mathcal{M} = (M_k)$ be a Musielak Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers. Then $l_{\mathcal{M}}\left(\Delta^n_m, u, p, \|.,\ldots,.\|\right)$ is a linear space over the field of complex numbers $\mathbb{C}$.

Proof:

Let $x = (x_k), y = (y_k) \in l_{\mathcal{M}}\left(\Delta^n_m, u, p, \|.,\ldots,.\|\right)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\sum_{k=1}^{\infty}\left[M_k\left(\left\|\frac{u_k\Delta^n_m x_k}{\rho_1}, z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_k} < \infty, \tag{38}$$

and

$$\sum_{k=1}^{\infty}\left[M_k\left(\left\|\frac{u_k\Delta^n_m y_k}{\rho_2}, z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_k} < \infty. \tag{39}$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_k)$ is nondecreasing convex function, by using inequality (13) we have

$$\sum_{k=1}^{\infty}\left[M_k\left(\left\|\frac{u_k\Delta^n_m(\alpha x_k + \beta y_k)}{\rho_3}, z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_k}$$

$$= \sum_{k=1}^{\infty}\left[M_k\left(\left\|\frac{\alpha u_k\Delta^n_m x_k}{\rho_3}, z_1,\ldots,z_{n-1} + \frac{\beta u_k\Delta^n_m y_k}{\rho_3}, z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_k}$$

$$\le K\sum_{k=1}^{\infty}\frac{1}{2^{p_k}}\left[M_k\left(\left\|\frac{u_k\Delta^n_m x_k}{\rho_1}, z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_k} + K\sum_{k=1}^{\infty}\frac{1}{2^{p_k}}\left[M_k\left(\left\|\frac{u_k\Delta^n_m y_k}{\rho_2}, z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_k}$$

$$\le K\sum_{k=1}^{\infty}\left[M_k\left(\left\|\frac{u_k\Delta^n_m x_k}{\rho_1}, z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_k} + K\sum_{k=1}^{\infty}\left[M_k\left(\left\|\frac{u_k\Delta^n_m y_k}{\rho_2}, z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_k} < \infty. \tag{40}$$

Thus, $\alpha x+\beta y\in l_{\mathcal{M}}\left(\Delta_m^n,u,p,\|.,\ldots,.\|\right)$. Therefore, $l_{\mathcal{M}}\left(\Delta_m^n,u,p,\|.,\ldots,.\|\right)$ is a linear space.

Theorem 3.2.

If $\mathcal{M}=(M_k)$ is a Musielak-Orlicz function and $p=(p_k)$ be a bounded sequence of positive real numbers, then $l_{\mathcal{M}}\left(\Delta_m^n,u,p,\|.,\ldots,.\|\right)$ is a paranormed space with the paranorm defined by

$$g(x)=\inf\left\{\rho^{\frac{p_n}{M}}:\left(\sum_{k=1}^{\infty}\left[M_k\left(\left\|\frac{u_k\Delta_m^n x_k}{\rho},z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{M}}\leq 1, n=1,2,3,\ldots\right\}, \tag{41}$$

where $M=\max(1,\sup_k p_k)$.

Proof:

(i) Clearly, $g(x)\geq 0$ for $x=(x_k)\in l_{\mathcal{M}}\left(\Delta_m^n,u,p,\|.,\ldots,.\|\right)$. Since $M_k(0)=0$, we get $g(0)=0$.
(ii) $g(-x)=g(x)$.

(iii) Let $x=(x_k), y=(y_k)\in l_{\mathcal{M}}\left(\Delta_m^n,u,p,\|.,\ldots,.\|\right)$. Then there exist $\rho_1,\rho_2>0$ such that

$$\sum_{k=1}^{\infty}\left[M_k\left(\left\|\frac{u_k\Delta_m^n x_k}{\rho_1},z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\leq 1 \tag{42}$$

and

$$\sum_{k=1}^{\infty}\left[M_k\left(\left\|\frac{u_k\Delta_m^n x_k}{\rho_2},z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\leq 1. \tag{43}$$

If $\rho=\rho_1+\rho_2$, then by Minkowski's inequality, we have

$$\sum_{k=1}^{\infty}\left[M_k\left(\left\|\frac{u_k\Delta_m^n x_k}{\rho},z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_k}=\sum_{k=1}^{\infty}\left[M_k\left(\left\|\frac{u_k\Delta_m^n x_k}{\rho_1+\rho_2},z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_k}$$

$$\leq\frac{\rho_1}{\rho_1+\rho_2}\sum_{k=1}^{\infty}\left[M_k\left(\left\|\frac{u_k\Delta_m^n x_k}{\rho_1},z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_k}+\frac{\rho_2}{\rho_1+\rho_2}\sum_{k=1}^{\infty}\left[M_k\left(\left\|\frac{u_k\Delta_m^n y_k}{\rho_2},z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_k}$$

$$\leq 1 \tag{44}$$

and thus,

$$g(x+y)=\inf\left\{(\rho_1+\rho_2)^{\frac{p_n}{M}}:\left(\sum_{k=1}^{\infty}\left[M_k\left(\left\|\frac{u_k\Delta_m^n x_k+u_k\Delta_m^n y_k}{\rho_1+\rho_2},z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{M}}\right\},$$

$$\leq \inf\left\{(\rho_1)^{\frac{p_n}{M}}:\left(\sum_{k=1}^{\infty}\left[M_k\left(\left\|\frac{u_k\Delta_m^n x_k}{\rho_1},z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{M}}\right\}$$

$$+\inf\left\{(\rho_2)^{\frac{p_n}{M}}:\left(\sum_{k=1}^{\infty}\left[M_k\left(\left\|\frac{u_k\Delta_m^n y_k}{\rho_2},z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{M}}\right\}$$

$$\leq g(x)+g(y). \tag{45}$$

Finally, we prove that scalar multiplication is continuous. Let λ be any complex number,

$$g(\lambda x)=\inf\left\{\rho^{\frac{p_n}{M}}:\left(\sum_{k=1}^{\infty}\left[M_k\left(\left\|\frac{\lambda u_k\Delta_m^n x_k}{\rho},z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{M}}\right\}$$

$$=\inf\left\{(|\lambda|r)^{\frac{p_n}{M}}:\left(\sum_{k=1}^{\infty}\left[M_k\left(\left\|\frac{u_k\Delta_m^n x_k}{\rho},z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{M}}\right\}. \tag{46}$$

where $r=\frac{\rho}{|\lambda|}$. Hence $l_{\mathcal{M}}\left(\Delta_m^n,u,p,\|.,\ldots,.\|\right)$ is a paranormed space.

Theorem 3.3.

The sequence space $l_{\mathcal{M}}\left(\Delta_m^n,u,p,\|.,\ldots,.\|\right)$ is solid.

Proof:

Let $x = (x_k) \in l_{\mathcal{M}} \left(\Delta_m^n, u, p, \|.,...,.\|\right)$. Then,

$$\sum_{k=1}^{\infty}\left[M_k\left(\left\|\frac{u_k \Delta_m^n x_k}{\rho}, z_1, ..., z_{n-1}\right\|\right)\right]^{p_k} < \infty. \tag{47}$$

Let (α_k) be a sequence of scalars such that $|\alpha_k| \leq 1$, for all $k \in \mathbb{N}$. Thus, we have

$$\sum_{k=1}^{\infty}\left[M_k\left(\left\|\frac{\alpha_k u_k \Delta_m^n x_k}{\rho}, z_1, ..., z_{n-1}\right\|\right)\right]^{p_k} \leq \sum_{k=1}^{\infty}\left[M_k\left(\left\|\frac{u_k \Delta_m^n x_k}{\rho}, z_1, ..., z_{n-1}\right\|\right)\right]^{p_k} < \infty. \tag{48}$$

This shows that $(\alpha_k x_k) \in l_{\mathcal{M}} \left(\Delta_m^n, u, p, \|.,...,.\|\right)$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$. Hence, the space $l_{\mathcal{M}} \left(\Delta_m^n, u, p, \|.,...,.\|\right)$ is a solid sequence space.

Theorem 3.4.

The sequence space $l_{\mathcal{M}} \left(\Delta_m^n, u, p, \|.,...,.\|\right)$ is monotone.

***Proof*:**

The proof is trivial so we omit it.

4. Applications of Sequence Spaces

(a) Applications of Sequence Spaces in Matrix Theory

The theory of sequence spaces and their matrix maps has made remarkable advances in enveloping duality theory via unified techniques effecting matrix transformation from one sequence space into another. Thus, we have several important applications of the theory of sequence spaces. Apart from this, the theory of sequence spaces is powerful tool for obtaining positive results concerning Schauder basis and their associated type. Mathematicians, like Cesàro, Borel, Nörlund, Reisz and others have studied the general theory of matrix transformations motivated by special and classical results in summability theory. On the other hand, in most cases the general linear operators on one sequence space into another is actually given by an infinite matrix. The well-known German Mathematician O. Toeplitz, first observed in 1911, that the technique of linear space theory can be used to characterize matrix transformations. Later, the Banach-Steinhaus theorem and related results became useful tools in dealing with such problems.

(b) Applications of Sequence Spaces in Biomathematics:

The relationship between sequences and binding properties of an aptamer for immunoglobulin (IgE) can be investigated using custom DNA microarrays. Single, double and some triple mutations of the aptamer sequences can be created to evaluate the important of specific base composition on aptamer binding. The functional sequence space can be represented as a rugged landscape with sharp peaks defined by highly constrained base compositions for more details see Katillius et al. (2007). One more interesting application is an Analysis of Peptides from known Proteins Clusterization in Sequence Space. A combinatorial sequence space (CSS) can be introduce to represent sequence as a set of overlapping $K-$tupples of some fixed length which correspond to a point in CSS. In Strelets et al. (1994) analyse clusterization of protein sequences in the CSS and to test various hypothesis about possible evolutionary basis of this clusterization. Possible applications of sequence spaces were also discussed in Strelets et al. (1994).

5. Conclusion

In this paper we have constructed some new sequence spaces defined by a Musielak-Orlicz function over semi normed spaces. We have also made an attempt to introduce some sequence spaces over n-normed spaces. We have studied some topological properties and interesting inclusion relations between these sequence spaces. The solutions obtained are potentially significant and important for the explanation of some practical physical problems. The method may also be applied to other sequence spaces.

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