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Oscillation Results for Even Order Trinomial Functional Differential Equations with Damping

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Abstract

In this paper, we investigate the oscillatory behavior of solutions to a certain class of nonlinear functional differential equations of the even order with damping. By using the integral averaging technique and Riccati type transformations, we prove four new theorems on the subject. Several examples are also considered to illustrate the main results.

Keywords: Oscillation; even order; delay differential equations

AMS MSC 2010: 34C10, 34K11

1. Introduction

In this paper, we will study the oscillatory behavior of solutions of the even order trinomial functional differential equation with damping term

$$\left(a(t)\left|x^{(n-1)}(t)\right|^{p-2}x^{(n-1)}(t)\right)'+r(t)\left|x^{(n-1)}(t)\right|^{p-2}x^{(n-1)}(t)+q(t)\left|x(g(t))\right|^{p-2}x(g(t))=0, \quad (1.1)$$

where $t \geq t_0 > 0$, $n > 2$ is an even integer, and $p > 1$ is a real number. We assume throughout this paper, and without further mention, that the following conditions hold:

(C1) $g : [t_0, \infty) \to (0, \infty)$ is a real valued continuous function such that $g(t) \leq t$, $\lim_{t\to\infty} g(t) = \infty$;

(C2) $a : [t_0, \infty) \to (0, \infty)$ is a real valued continuously differentiable function, and $r : [t_0, \infty) \to [0, \infty)$ is a real valued continuous function such that $a'(t) + r(t) \geq 0$;

(C3) $q : [t_0, \infty) \to (0, \infty)$ is a real valued continuous function.

By a solution of equation (1.1) we mean a nontrivial function

$$x : [T_x, \infty) \to R,\ T_x \geq t_0,$$

such that

$$x \in C^{n-1}([T_x, \infty), R),\ a(t)\left|x^{(n-1)}(t)\right|^{p-2} x^{(n-1)}(t) \in C^1([T_x, \infty), R),$$

and $x(t)$ satisfies (1.1) for all sufficiently large t. Here, we only consider those solutions $x(t)$ of equation (1.1) satisfying $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$, and we assume that (1.1) possesses such solutions. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_x, \infty)$, and is called nonoscillatory, otherwise. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

The problem of determining the oscillation of solutions of higher order functional differential equations with deviating arguments has been a very active area of research in the last few decades. Much of the literature on the subject has been concerned with equations of types

$$x^{(n)}(t) + q(t)x(g(t)) = 0,$$

$$\left(\left|x^{(n-1)}(t)\right|^{\alpha-1} x^{(n-1)}(t)\right)' + q(t)\left|x(g(t))\right|^{\alpha-1} x(g(t)) = 0,$$

$$\left(\left|x^{(n-1)}(t)\right|^{\alpha-1} x^{(n-1)}(t)\right)' + F(t, x(g(t))) = 0,$$

and

$$\left(a(t)\left|x^{(n-1)}(t)\right|^{\alpha-1}x^{(n-1)}(t)\right)' + F\left(t, x\left(g(t)\right)\right) = 0.$$

A number of oscillation criteria for higher order delay differential equations without a damping term of these types can be found in the research papers of Agarwal et al. (2001), Zhang and Yan (2006), Zhang et al. (2010), Grace and Lalli (1984), Xu and Xia (2004), Xu and Lv (2011), Zhang et al. (2013) and the references cited therein. On the other hand very little is known for higher order functional differential equations with damping term, and so it becomes useful to prove new oscillation results in the case with a damping term. In the presence of damping, Liu et al. (2011) considered equation (1.1) and obtained the following oscillation criterion.

Theorem 1.1. [Liu et al. (2011), Theorem 1].

Assume (C1) – (C3), $a'(t) \geq 0,$ and let $n \geq 2$ be even, $g \in C^1[t_0, \infty),$ $g'(t) > 0$ for $t \geq t_0,$ and

$$\int_{t_0}^{\infty}\left[\frac{1}{a(s)}\exp\left(-\int_{t_0}^{s}\frac{r(\tau)}{a(\tau)}d\tau\right)\right]^{\frac{1}{p-1}}ds = \infty. \tag{1.2}$$

Suppose that there exists a continuous function

$H : D \equiv \{(t,s) : t \geq s \geq t_0\} \to R$ such that $H(t,t) = 0,$ $t \geq t_0;$ $H(t,s) > 0,$ $t > s \geq t_0,$ and H has a nonpositive continuous partial derivative with respect to second variable in $D_0 \equiv \{(t,s) : t > s \geq t_0\}.$ Assume further that there exist functions $h \in C(D_0, R),$ $K, \delta \in C^1([t_0, \infty), (0, \infty))$ such that

$$-\frac{\partial}{\partial s}\left(H(t,s)K(s)\right) - H(t,s)K(s)\left(\frac{\delta'(s)}{\delta(s)} - \frac{r(s)}{a(s)}\right) = h(t,s), \ \forall (t,s) \in D_0. \tag{1.3}$$

If for some constant $\theta \in (0,1)$ and for all constants $M > 0,$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[\delta(s)q(s)H(t,s)K(s) - \left(\frac{|h(t,s)|}{p}\right)^{p}\frac{\delta(s)a(s)}{\left(H(t,s)G(s)K(s)\right)^{p-1}}\right]ds = \infty,$$

where $G(s) := \theta M g^{n-2}(s) g'(s),$ then equation (1.1) is oscillatory.

Very recently, using the integral averaging technique and a comparison method, Zhang et al. (2014) also established some new criteria for oscillation and asymptotic behavior of equation (1.1). The results obtained in Zhang et al. (2014) not only improve some known results in the literature, but also suggest a different method to investigate (1.1) in the case where n is odd. By employing double Riccati transformation, Zhang et al. (2014) presented the following oscillation criterion for (1.1) in the case where $n \geq 4$ is even.

Theorem 1.2. [Zhang et al. (2014), Theorem 2.4].

Assume (C1) – (C3), (1.2), $n \geq 4$ is even, and let D, D_0, H be as in Theorem 1.1. Assume further that there exist functions $h \in C(D_0, R)$, $K, \delta \in C^1([t_0,\infty),(0,\infty))$ such that (1.3) holds and for some constant $\lambda_0 \in (0,1)$,

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)K(s)\delta(s)q(s)\left(\frac{g^{n-1}(s)}{s^{n-1}}\right)^{p-1} - \frac{\delta(s)a(s)\left(\frac{|h(t,s)|}{p}\right)^{p}}{\left(H(t,s)K(s)\frac{\lambda_0 s^{n-2}}{(n-2)!}\right)^{p-1}} \right] ds = \infty. \tag{1.4}$$

Suppose also there exist a continuous function $H_* : D \to R$ such that

$$H_*(t,t) = 0,\ t \geq t_0; \quad H_*(t,s) > 0,\ t > s \geq t_0, \tag{1.5}$$

and H_* has a nonpositive continuous partial derivative with respect to second variable in D_0. If there exist functions $h_* \in C(D_0, R)$, $K_*, \rho \in C^1([t_0,\infty),(0,\infty))$ such that

$$-\frac{\partial}{\partial s}\left(H_*(t,s)K_*(s)\right) - H_*(t,s)K_*(s)\frac{\rho'(s)}{\rho(s)} = h_*(t,s),\ \forall (t,s) \in D_0 \tag{1.6}$$

and

$$\limsup_{t\to\infty} \frac{1}{H_*(t,t_0)} \int_{t_0}^{t} \left[H_*(t,s)K_*(s)\rho(s)Q(s) - \frac{\rho(s)|h_*(t,s)|^2}{4H_*(t,s)K_*(s)} \right] ds = \infty, \tag{1.7}$$

where

$$Q(t) := \frac{\int_t^\infty (\eta - t)^{n-4} \left[\frac{\int_\eta^\infty q(s) \left(\frac{g(s)}{s} \right)^{p-1} ds}{a(\eta)} \right]^{1/(p-1)} d\eta}{(n-4)!},$$

then equation (1.1) is oscillatory.

The motivation for this paper comes from the ideas of Liu et al. (2011), Zhang et al. (2014) and the cited papers in the references. By using Riccati type transformations and the integral averaging technique, we establish some new sufficient conditions which guarantee the oscillation of solutions of equation (1.1). The obtained results improve upon those reported in Liu et al. (2011) in the sense that our conditions do not require the signs of $a'(t)$ and $g'(t)$, and any constant $M > 0$. On the other hand, Zhang et al. (2014) said that the sign of derivative x'' is not known, and so Theorem 1.2 involved double assumptions, as for instance, (1.4) and (1.7) (see Zhang et al. (2014, Remark 2.13). Here, we are also able to determine the sign of the derivative x''. It is therefore hoped that the present paper will contribute to the studies for oscillatory behavior of solutions of functional differential equations with damping. We want to emphasize that the results in this work are different from those of Liu et al. (2011) and Zhang et al. (2014); that is, they are neither a special case nor a generalized form of the results in Liu et al. (2011) and Zhang et al. (2014). They are interesting in their own right. Finally, some examples are given to illustrate our results.

2. Main Results

Before presenting our main results, we begin with the following lemmas that are essential in the proofs of our theorems.

Lemma 2.1. [Philos (1981)].

Let $f \in C^n([t_0, \infty), R^+)$. If $f^{(n)}$ is eventually of one sign for all large t, then there exist a $t_x \geq t_0$ and an integer l, $0 \leq l \leq n$ with $n + l$ even for $f^{(n)} \geq 0$, or $n + l$ odd for $f^{(n)} \leq 0$ such that

$$l > 0, \qquad \text{yields} \quad f^{(k)}(t) > 0 \quad \text{for} \quad t \geq t_x, \quad k = 0, 1, 2, ..., l-1 \tag{2.1}$$

and

$$l \leq n-1, \qquad \text{yields} \quad (-1)^{1+k} f^{(k)}(t) > 0 \quad \text{for} \quad t \geq t_x, \quad k = l, l+1, ..., n-1. \tag{2.2}$$

Lemma 2.2. [Kiguradze and Chanturiya (1993), Graef and Saker (2013)].

If a function $f(t)$ satisfies $f^{(i)}(t)>0,\ i=0,1,2,...,m,$ and $f^{(m+1)}(t)\le 0,$ then

$$\frac{f(t)}{t^m/m!}\ge\frac{f'(t)}{t^{m-1}/(m-1)!}.$$

Lemma 2.3. [Hardy et al. (1988)].

If X and Y are nonnegative and $\lambda>1,$ then

$$\lambda XY^{\lambda-1}-X^{\lambda}\le(\lambda-1)Y^{\lambda},$$

where equality holds if and only if $X=Y.$

Theorem 2.1.

Assume that conditions (C1) – (C3), and (1.2) hold, $n\ge 4$ is even, and

$$\int_{t_0}^{\infty}\left[\int_{v}^{\infty}\left(\frac{1}{a(u)}\int_{u}^{\infty}q(s)\,ds\right)^{1/(p-1)}du\right]dv=\infty. \tag{2.3}$$

Let D_0,D,H be as in Theorem 1.1. Suppose further that there exist functions $h\in C(D_0,R),$ $K,\delta\in C^1([t_0,\infty),(0,\infty))$ such that (1.3) is satisfied and for some constant $k\in(0,1),$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)K(s)\delta(s)q(s)\left(\frac{g^{n-1}(s)}{s^{n-1}}\right)^{p-1}-\frac{\delta(s)a(s)\left(\frac{|h(t,s)|}{p}\right)^{p}}{\left[H(t,s)K(s)\frac{k^{n-2}s^{n-2}}{(n-2)!}\right]^{p-1}}\right]ds=\infty. \tag{2.4}$$

Then equation (1.1) is oscillatory.

Proof:

Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(t)$ is an eventually positive solution of equation (1.1). Then there exist a $t_1 \ge t_0$ such that $x(t) > 0$, and $x(g(t)) > 0$ for all $t \ge t_1$. In view of (1.1), we have, for $t \ge t_1$,

$$\left(a(t)\left|x^{(n-1)}(t)\right|^{p-2} x^{(n-1)}(t)\right)' + r(t)\left|x^{(n-1)}(t)\right|^{p-2} x^{(n-1)}(t) = -q(t)\left(x(g(t))\right)^{p-1} < 0,$$

which implies

$$\left(\exp\left(\int_{t_0}^{t} \frac{r(\tau)}{a(\tau)} d\tau\right) a(t)\left|x^{(n-1)}(t)\right|^{p-2} x^{(n-1)}(t)\right)' < 0 \qquad \text{for } t \ge t_1. \tag{2.5}$$

Thus, $\exp\left(\int_{t_0}^{t} \frac{r(\tau)}{a(\tau)} d\tau\right) a(t)\left|x^{(n-1)}(t)\right|^{p-2} x^{(n-1)}(t)$ is decreasing, and so $x^{(n-1)}(t)$ is eventually of one sign. We claim that

$$x^{(n-1)}(t) > 0, \qquad \text{for } t \ge t_1. \tag{2.6}$$

If this is not the case, then there exists $t_2 \in [t_1, \infty)$ such that $x^{(n-1)}(t) < 0$ for $t \ge t_2$. In view of (2.5), we have, for $t \ge t_2$,

$$\begin{aligned}\exp\left(\int_{t_0}^{t} \frac{r(\tau)}{a(\tau)} d\tau\right) a(t)\left|x^{(n-1)}(t)\right|^{p-2} x^{(n-1)}(t) &\le \exp\left(\int_{t_0}^{t_2} \frac{r(\tau)}{a(\tau)} d\tau\right) a(t_2)\left|x^{(n-1)}(t_2)\right|^{p-2} x^{(n-1)}(t_2) \\ &= -M^{p-1} \exp\left(\int_{t_0}^{t_2} \frac{r(\tau)}{a(\tau)} d\tau\right),\end{aligned}$$

where $M = \left(a(t_2)\right)^{1/(p-1)} \left|x^{(n-1)}(t_2)\right| > 0$. Hence,

$$x^{(n-1)}(t) \le -M\left(\frac{1}{a(t)} e\text{xp}\left(-\int_{t_2}^{t} \frac{r(\tau)}{a(\tau)} d\tau\right)\right)^{1/(p-1)}, \qquad \text{for } t \ge t_2.$$

Integration yields

$$x^{(n-2)}(t) \le x^{(n-2)}(t_2) - M\int_{t_2}^{t}\left(\frac{1}{a(s)}e\text{xp}\left(-\int_{t_2}^{s}\frac{r(\tau)}{a(\tau)}d\tau\right)\right)^{1/(p-1)} ds.$$

In view of (1.2), it follows that $\lim_{t\to\infty} x^{(n-2)}(t) = -\infty$. Similarly, we find

$$\lim_{t\to\infty} x^{(n-3)}(t) = \lim_{t\to\infty} x^{(n-4)}(t) = \ldots = \lim_{t\to\infty} x'(t) = \lim_{t\to\infty} x(t) = -\infty,$$

which contradicts the fact that $x(t) > 0$ for all $t \ge t_1$. So, (2.6) is satisfied. Using (2.6) in (2.5), we obtain

$$\exp\left(\int_{t_0}^{t}\frac{r(\tau)}{a(\tau)}d\tau\right)\left[a'(t)+r(t)\right]\left(x^{(n-1)}(t)\right)^{p-1} + (p-1)a(t)\exp\left(\int_{t_0}^{t}\frac{r(\tau)}{a(\tau)}d\tau\right)\left(x^{(n-1)}\right)^{p-2} x^{(n)}(t) < 0.$$

From the last inequality, we see that

$$x^{(n)}(t) < 0 \quad \text{for } t \ge t_1. \tag{2.7}$$

Thus, from Lemma 2.1, there exist an integer $l \in \{1,3,\ldots,n-1\}$ such that (2.1) and (2.2) hold for $t \ge t_1$. Now, from (2.7), and Lemma 2.1 with the fact that $l \in \{1,3,\ \ldots,\ n-1\}$, we see that

$$x'(t) > 0, \quad \text{for } t \ge t_1, \tag{2.8}$$

Hence, there exist a constant $c > 0$ such that

$$x(t) \ge c \quad \text{for } t \ge t_1. \tag{2.9}$$

Since $\lim_{t\to\infty} g(t) = \infty$, we can choose $t_2 \ge t_1$ such that $g(t) \ge t_1$ for all $t \ge t_2$, and so we have

$$x(g(t)) \ge c \quad \text{for } t \ge t_2. \tag{2.10}$$

We now assert that $l = n-1$. To this end, we suppose that

$$x^{(n-2)}(t) < 0 \quad \text{and} \quad x^{(n-3)}(t) > 0 \quad \text{for } t \ge t_2.$$

Using (2.6) in equation (1.1), we can write (1.1) in the form

$$\left(a(t)\left(x^{(n-1)}(t)\right)^{p-1}\right)' + r(t)\left(x^{(n-1)}(t)\right)^{p-1} + q(t)\left(x(g(t))\right)^{p-1} = 0, \quad \text{for } t \ge t_2. \tag{2.11}$$

Integrating (2.11) from $t \ge t_2$ to $u \ge t$, letting $u \to \infty$, and using (2.6) and (2.10), we get

$$a(t)\left(x^{(n-1)}(t)\right)^{p-1} \ge \int_t^\infty r(s)\left(x^{(n-1)}(s)\right)^{p-1} ds + \int_t^\infty q(s)\left(x(g(s))\right)^{p-1} ds \ \ge c^{p-1}\int_t^\infty q(s)\,ds.$$

That is,

$$x^{(n-1)}(t) \ge c\left(\frac{1}{a(t)}\int_t^\infty q(s)\,ds\right)^{1/(p-1)}. \tag{2.12}$$

Integrating (2.12) from t to v, and letting $v \to \infty$, we obtain

$$-x^{(n-2)}(t) \ge c\int_t^\infty \left(\frac{1}{a(u)}\int_u^\infty q(s)\,ds\right)^{1/(p-1)} du.$$

Integrating the last inequality from t_2 to t gives

$$\int_{t_2}^t \left[\int_v^\infty \left(\frac{1}{a(u)}\int_u^\infty q(s)\,ds\right)^{1/(p-1)} du\right] dv \le \frac{1}{c}\left(x^{(n-3)}(t_2) - x^{(n-3)}(t)\right) \le \frac{1}{c}x^{(n-3)}(t_2).$$

Letting $t \to \infty$, we have

$$\int_{t_2}^\infty \left[\int_v^\infty \left(\frac{1}{a(u)}\int_u^\infty q(s)\,ds\right)^{1/(p-1)} du\right] dv \le \frac{1}{c}x^{(n-3)}(t_2) < \infty,$$

which contradicts (2.3), and so we have $l = n-1$. In view of Lemma 2.1, (2.7), and the fact that $l = n-1$, we conclude that

$$x^{(n)}(t) < 0 \quad \text{and} \quad x^{(i)}(t) > 0, \quad i = 0,1,\ldots,n-1 \qquad \text{for } t \ge t_2. \tag{2.13}$$

Thus,

$$x^{(n-2)}(t) = x^{(n-2)}(t_2) + \int_{t_2}^{t} x^{(n-1)}(s)\,ds \ge (t-t_2)\,x^{(n-1)}(t) \qquad \text{for } t \ge t_2.$$

Integrating this inequality $(n-3)$ – times from t_2 to t and using the fact that $x^{(n-1)}(t)$ is decreasing on $[t_2,\infty)$, we have

$$x'(t) \ge \frac{(t-t_2)^{n-2}}{(n-2)!}\,x^{(n-1)}(t) \qquad \text{for } t \ge t_2. \tag{2.14}$$

Let $k \in (0,1)$. Then for $t \ge t_2/(1-k) =: t_k \ge t_2$, we have $t - t_2 \ge kt$. Now, (2.14) implies

$$x'(t) \ge \frac{k^{n-2}t^{n-2}}{(n-2)!}\,x^{(n-1)}(t) \qquad \text{for } t \ge t_k. \tag{2.15}$$

Integrating (2.15) for $t \ge t_k \ge t_2$ yields

$$x(t) \ge \frac{k^{n-2}t^{n-1}}{(n-1)!}\,x^{(n-1)}(t) \qquad \text{for } t \ge t_k. \tag{2.16}$$

Now consider the Riccati substitution,

$$w(t) = \delta(t)\frac{a(t)\left(x^{(n-1)}(t)\right)^{p-1}}{\left(x(t)\right)^{p-1}} \qquad \text{for } t \ge t_k. \tag{2.17}$$

Clearly, $w(t) > 0$, and we have by (1.1), (2.6), (2.15), and (2.17) that

$$\begin{aligned} w'(t) = \frac{\delta'(t)}{\delta(t)}w(t) - \delta(t)\frac{r(t)\left(x^{(n-1)}(t)\right)^{p-1}}{\left(x(t)\right)^{p-1}} - \delta(t)\frac{q(t)\left(x(g(t))\right)^{p-1}}{\left(x(t)\right)^{p-1}} \\ -\delta(t)\frac{(p-1)a(t)\left(x^{(n-1)}(t)\right)^{p-1}\left(x(t)\right)^{p-2}x'(t)}{\left(x(t)\right)^{2p-2}} \\ = -\delta(t)\frac{q(t)\left(x(g(t))\right)^{p-1}}{\left(x(t)\right)^{p-1}} + \left(\frac{\delta'(t)}{\delta(t)} - \frac{r(t)}{a(t)}\right)w(t) \\ -\delta(t)\frac{(p-1)a(t)\left(x^{(n-1)}(t)\right)^{p-1}x'(t)}{\left(x(t)\right)^{p}} \end{aligned}$$

$$\le -\delta(t)\frac{q(t)\left(x(g(t))\right)^{p-1}}{\left(x(t)\right)^{p-1}}+\left(\frac{\delta'(t)}{\delta(t)}-\frac{r(t)}{a(t)}\right)w(t)-\delta(t)\frac{(p-1)k^{n-2}t^{n-2}a(t)\left(x^{(n-1)}(t)\right)^{p}}{(n-2)!\left(x(t)\right)^{p}}$$

$$=-\delta(t)\frac{q(t)\left(x(g(t))\right)^{p-1}}{\left(x(t)\right)^{p-1}}+\left(\frac{\delta'(t)}{\delta(t)}-\frac{r(t)}{a(t)}\right)w(t)-\frac{(p-1)k^{n-2}t^{n-2}w^{p/(p-1)}(t)}{(n-2)!\left(\delta(t)a(t)\right)^{1/(p-1)}}. \qquad (2.18)$$

By Lemma 2.2, we can easily deduce from (2.13) that

$$\frac{x(t)}{x'(t)}\ge\frac{t}{n-1}\quad\text{for}\quad t\ge t_2.$$

From this, we see that x/t^{n-1} is nonincreasing, and hence

$$\frac{x(g(t))}{x(t)}\ge\frac{g^{n-1}(t)}{t^{n-1}}\quad\text{for}\quad t\ge t_2.$$

Using the latter inequality in (2.18), we obtain, for all $t\ge t_k$,

$$w'(t)\le-\delta(t)q(t)\left(\frac{g^{n-1}(t)}{t^{n-1}}\right)^{p-1}+\left(\frac{\delta'(t)}{\delta(t)}-\frac{r(t)}{a(t)}\right)w(t)-\frac{(p-1)k^{n-2}t^{n-2}w^{p/(p-1)}(t)}{(n-2)!\left(\delta(t)a(t)\right)^{1/(p-1)}}. \qquad (2.19)$$

Replacing in (2.19) t with s, multiplying both sides by $H(t,s)K(s)$ and integrating with respect to s from t_k to t, we get,

$$\int_{t_k}^{t}H(t,s)K(s)\delta(s)q(s)\left(\frac{g^{n-1}(s)}{s^{n-1}}\right)^{p-1}ds$$

$$\le-\int_{t_k}^{t}H(t,s)K(s)w'(s)\,ds+\int_{t_k}^{t}H(t,s)K(s)\left(\frac{\delta'(s)}{\delta(s)}-\frac{r(s)}{a(s)}\right)w(s)\,ds$$

$$-\int_{t_k}^{t}H(t,s)K(s)\frac{(p-1)k^{n-2}s^{n-2}}{(n-2)!\left(\delta(s)a(s)\right)^{1/(p-1)}}w^{p/(p-1)}(s)\,ds$$

$$=H(t,t_k)K(t_k)w(t_k)-\int_{t_k}^{t}\left[-\frac{\partial}{\partial s}\left(H(t,s)K(s)\right)-H(t,s)K(s)\left(\frac{\delta'(s)}{\delta(s)}-\frac{r(s)}{a(s)}\right)\right]w(s)\,ds$$

$$-\int_{t_k}^{t}H(t,s)K(s)\frac{(p-1)k^{n-2}s^{n-2}}{(n-2)!\left(\delta(s)a(s)\right)^{1/(p-1)}}w^{p/(p-1)}(s)\,ds$$

$$\leq H(t,t_k)K(t_k)w(t_k)+\int_{t_k}^{t}|h(t,s)|w(s)ds$$
$$-\int_{t_k}^{t}H(t,s)K(s)\frac{(p-1)k^{n-2}s^{n-2}}{(n-2)!\left(\delta(s)a(s)\right)^{1/(p-1)}}w^{p/(p-1)}(s)ds. \qquad (2.20)$$

Now setting

$$X=\left[H(t,s)K(s)\frac{(p-1)k^{n-2}s^{n-2}}{(n-2)!}\right]^{(p-1)/p}\frac{w(s)}{\left(\delta(s)a(s)\right)^{1/p}},$$

and

$$Y=\left(\frac{p-1}{p}\right)^{p-1}|h(t,s)|^{p-1}\left(\frac{\delta(s)a(s)}{\left[H(t,s)K(s)\frac{(p-1)k^{n-2}s^{n-2}}{(n-2)!}\right]^{p-1}}\right)^{(p-1)/p},$$

in Lemma 2.3 with $\lambda=\frac{p}{p-1}$, we conclude that

$$|h(t,s)|w(s)-H(t,s)K(s)\frac{(p-1)k^{n-2}s^{n-2}}{(n-2)!\left(\delta(s)a(s)\right)^{1/(p-1)}}w^{p/(p-1)}(s)\leq\frac{\delta(s)a(s)\left(\frac{|h(t,s)|}{p}\right)^{p}}{\left[H(t,s)K(s)\frac{k^{n-2}s^{n-2}}{(n-2)!}\right]^{p-1}},$$

So that, by (2.20) we obtain

$$\int_{t_k}^{t}\left[H(t,s)K(s)\delta(s)q(s)\left(\frac{g^{n-1}(s)}{s^{n-1}}\right)^{p-1}-\frac{\delta(s)a(s)}{\left[H(t,s)K(s)\frac{k^{n-2}s^{n-2}}{(n-2)!}\right]^{p-1}}\left(\frac{|h(t,s)|}{p}\right)^{p}\right]ds$$
$$\leq H(t,t_k)K(t_k)w(t_k)\leq H(t,t_0)K(t_k)w(t_k).$$

Now,

$$\int_{t_0}^{t}\left[H(t,s)K(s)\delta(s)q(s)\left(\frac{g^{n-1}(s)}{s^{n-1}}\right)^{p-1}-\frac{\delta(s)a(s)}{\left[H(t,s)K(s)\frac{k^{n-2}s^{n-2}}{(n-2)!}\right]^{p-1}}\left(\frac{|h(t,s)|}{p}\right)^{p}\right]ds$$

$$=\int_{t_0}^{t_k}\left[H(t,s)K(s)\delta(s)q(s)\left(\frac{g^{n-1}(s)}{s^{n-1}}\right)^{p-1}-\frac{\delta(s)a(s)}{\left[H(t,s)K(s)\frac{k^{n-2}s^{n-2}}{(n-2)!}\right]^{p-1}}\left(\frac{|h(t,s)|}{p}\right)^{p}\right]ds$$

$$+\int_{t_k}^{t}\left[H(t,s)K(s)\delta(s)q(s)\left(\frac{g^{n-1}(s)}{s^{n-1}}\right)^{p-1}-\frac{\delta(s)a(s)}{\left[H(t,s)K(s)\frac{k^{n-2}s^{n-2}}{(n-2)!}\right]^{p-1}}\left(\frac{|h(t,s)|}{p}\right)^{p}\right]ds$$

$$\leq\int_{t_0}^{t_k}H(t,s)K(s)\delta(s)q(s)\left(\frac{g^{n-1}(s)}{s^{n-1}}\right)^{p-1}ds+H(t,t_0)K(t_k)w(t_k)$$

$$\leq H(t,t_0)\left[\int_{t_0}^{t_k}K(s)\delta(s)q(s)\left(\frac{g^{n-1}(s)}{s^{n-1}}\right)^{p-1}ds+K(t_k)w(t_k)\right],$$

for all $t\geq t_k$. This gives

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)K(s)\delta(s)q(s)\left(\frac{g^{n-1}(s)}{s^{n-1}}\right)^{p-1}\right.$$

$$\left.-\frac{\delta(s)a(s)}{\left[H(t,s)K(s)\frac{k^{n-2}s^{n-2}}{(n-2)!}\right]^{p-1}}\left(\frac{|h(t,s)|}{p}\right)^{p}\right]ds$$

$$\leq\int_{t_0}^{t_k}K(s)\delta(s)q(s)\left(\frac{g^{n-1}(s)}{s^{n-1}}\right)^{p-1}ds+K(t_k)w(t_k)<\infty,$$

which contradicts (2.4) and completes the proofs of the theorem.

For the case when $n=2$ in equation (1.1), we do not need condition (2.3), and any constant $k\in(0,1)$. Herewith, by using the Riccati substitution

$$w(t)=\delta(t)\frac{a(t)\left(x'(t)\right)^{p-1}}{\left(x(t)\right)^{p-1}} \quad \text{for } t\ge t_1,\ t_1\ge t_0,$$

we have the following result.

Theorem 2.2.

Assume that conditions (C1) – (C3), and (1.2) hold, and let D_0, D, H be as in Theorem 2.1. Suppose further that there exist functions $h\in C(D_0, R)$, $K,\delta\in C^1([t_0,\infty),(0,\infty))$ such that (1.3) is satisfied, and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)K(s)\delta(s)q(s)\left(\frac{g(s)}{s}\right)^{p-1}-\frac{\delta(s)a(s)\left(\frac{|h(t,s)|}{p}\right)^{p}}{\left(H(t,s)K(s)\right)^{p-1}}\right]ds=\infty. \quad (2.21)$$

Then, equation (1.1) is oscillatory.

Proof:

The proof is standard and so the details are omitted.

Theorem 2.3.

Assume that $p\ge 2$, $n\ge 4$ is even, and conditions (C1) – (C3), (1.2) and (2.3) are satisfied. Let D_0, D, H be as in Theorem 2.1. Suppose also that there exist function $K,\delta\in C^1([t_0,\infty),(0,\infty))$, $h\in C(D_0, R)$ such that (1.3) holds for some constant $k\in(0,1)$,

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)K(s)\delta(s)q(s)\left(\frac{g^{n-1}(s)}{s^{n-1}}\right)^{p-1}\right.$$
$$\left.-\frac{(n-2)!\delta(s)a(s)}{4H(t,s)K(s)(p-1)k^{n-2}s^{n-2}}\left(\frac{(n-1)!}{k^{n-2}s^{n-1}}\right)^{p-2}h^2(t,s)\right]ds=\infty. \quad (2.22)$$

Then equation (1.1) is oscillatory.

Proof:

Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(t)$ is an eventually positive solution of equation (1.1). Then there exist a $t_1 \geq t_0$ such that $x(t) > 0$, and $x(g(t)) > 0$ for all $t \geq t_1$. Proceeding as in the proof of Theorem 2.1, we arrive at inequality (2.19) which can be written as, for all $t \geq t_k$,

$$w'(t) \leq -\delta(t)q(t)\left(\frac{g^{n-1}(t)}{t^{n-1}}\right)^{p-1} + \left(\frac{\delta'(t)}{\delta(t)} - \frac{r(t)}{a(t)}\right)w(t) - \frac{(p-1)k^{n-2}t^{n-2}}{(n-2)!(\delta(t)a(t))^{1/(p-1)}} w^{\frac{1}{p-1}-1}(t)w^2(t). \tag{2.23}$$

From (2.16), we have

$$\frac{x(t)}{x^{(n-1)}(t)} \geq \frac{k^{n-2}t^{n-1}}{(n-1)!}, \qquad \text{for } t \geq t_k.$$

Now,

$$\begin{aligned} w^{\frac{1}{p-1}-1}(t) &= \delta^{\frac{1}{p-1}-1}(t)a^{\frac{1}{p-1}-1}(t)\left[\left(\frac{x^{(n-1)}(t)}{x(t)}\right)^{p-1}\right]^{\frac{1}{p-1}-1} \\ &= (\delta(t)a(t))^{\frac{1}{p-1}-1}\left(\frac{x^{(n-1)}(t)}{x(t)}\right)^{2-p} \\ &= (\delta(t)a(t))^{\frac{1}{p-1}-1}\left(\frac{x(t)}{x^{(n-1)}(t)}\right)^{p-2} \\ &\geq (\delta(t)a(t))^{\frac{1}{p-1}-1}\left(\frac{k^{n-2}t^{n-1}}{(n-1)!}\right)^{p-2}. \end{aligned} \tag{2.24}$$

Using (2.24) in (2.23), we have, for $t \geq t_k$,

$$w'(t) \leq -\delta(t)q(t)\left(\frac{g^{n-1}(t)}{t^{n-1}}\right)^{p-1} + \left(\frac{\delta'(t)}{\delta(t)} - \frac{r(t)}{a(t)}\right)w(t) - \frac{(p-1)k^{n-2}t^{n-2}}{(n-2)!\delta(t)a(t)}\left(\frac{k^{n-2}t^{n-1}}{(n-1)!}\right)^{p-2} w^2(t). \tag{2.25}$$

Replacing in (2.25) t with s, multiplying both sides $H(t,s)K(s)$ and integrating with respect to s from t_k to t, we obtain

$$\int_{t_k}^{t} H(t,s)K(s)\delta(s)q(s)\left(\frac{g^{n-1}(s)}{s^{n-1}}\right)^{p-1} ds$$

$$\leq -\int_{t_k}^{t} H(t,s)K(s)w'(s)\,ds + \int_{t_k}^{t} H(t,s)K(s)\left(\frac{\delta'(s)}{\delta(s)} - \frac{r(s)}{a(s)}\right)w(s)\,ds$$

$$-\int_{t_k}^{t} H(t,s)K(s)\frac{(p-1)k^{n-2}s^{n-2}}{(n-2)!\delta(s)a(s)}\left(\frac{k^{n-2}s^{n-1}}{(n-1)!}\right)^{p-2} w^2(t)\,ds$$

$$= H(t,t_k)K(t_k)w(t_k) - \int_{t_k}^{t}\left[-\frac{\partial}{\partial s}\left(H(t,s)K(s)\right) - H(t,s)K(s)\left(\frac{\delta'(s)}{\delta(s)} - \frac{r(s)}{a(s)}\right)\right]w(s)\,ds$$

$$-\int_{t_k}^{t} H(t,s)K(s)\frac{(p-1)k^{n-2}s^{n-2}}{(n-2)!\delta(s)a(s)}\left(\frac{k^{n-2}s^{n-1}}{(n-1)!}\right)^{p-2} w^2(s)\,ds$$

$$\leq H(t,t_k)K(t_k)w(t_k) + \int_{t_k}^{t}\left|h(t,s)\right|w(s)\,ds$$

$$-\int_{t_k}^{t} H(t,s)K(s)\frac{(p-1)k^{n-2}s^{n-2}}{(n-2)!\delta(s)a(s)}\left(\frac{k^{n-2}s^{n-1}}{(n-1)!}\right)^{p-2} w^2(s)\,ds$$

$$= H(t,t_k)K(t_k)w(t_k) + \frac{1}{4}\int_{t_k}^{t}\frac{(n-2)!\delta(s)a(s)}{H(t,s)K(s)(p-1)k^{n-2}s^{n-2}}\left(\frac{(n-1)!}{k^{n-2}s^{n-1}}\right)^{p-2}\left|h(t,s)\right|^2 ds$$

$$-\int_{t_k}^{t}\left[\sqrt{H(t,s)K(s)\frac{(p-1)k^{n-2}s^{n-2}}{(n-2)!\delta(s)a(s)}\left(\frac{k^{n-2}s^{n-1}}{(n-1)!}\right)^{p-2}}\,w(s)\right.$$

$$\left.-\frac{1}{2}\sqrt{\frac{(n-2)!\delta(s)a(s)}{H(t,s)K(s)(p-1)k^{n-2}s^{n-2}}\left(\frac{(n-1)!}{k^{n-2}s^{n-1}}\right)^{p-2}}\left|h(t,s)\right|\right]^2 ds$$

$$\leq H(t,t_k)K(t_k)w(t_k)$$

$$+\frac{1}{4}\int_{t_k}^{t}\frac{(n-2)!\delta(s)a(s)}{H(t,s)K(s)(p-1)k^{n-2}s^{n-2}}\left(\frac{(n-1)!}{k^{n-2}s^{n-1}}\right)^{p-2}\left|h(t,s)\right|^2 ds. \tag{2.26}$$

From (2.26), we see that

$$\int_{t_k}^{t}\left[\begin{array}{l} H(t,s)K(s)\delta(s)q(s)\left(\dfrac{g^{n-1}(s)}{s^{n-1}}\right)^{p-1} \\ \qquad -\dfrac{(n-2)!\delta(s)a(s)}{4H(t,s)K(s)(p-1)k^{n-2}s^{n-2}}\left(\dfrac{(n-1)!}{k^{n-2}s^{n-1}}\right)^{p-2}h^2(t,s)\end{array}\right]ds.$$
$$\le H(t,t_k)K(t_k)w(t_k)\le H(t,t_0)K(t_k)w(t_k).$$

Proceeding as in the proof of Theorem 2.1, we obtain

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\Bigg[&H(t,s)K(s)\delta(s)q(s)\left(\frac{g^{n-1}(s)}{s^{n-1}}\right)^{p-1}\\ &-\frac{1}{4}\frac{(n-2)!\delta(s)a(s)}{H(t,s)K(s)(p-1)k^{n-2}s^{n-2}}\left(\frac{(n-1)!}{k^{n-2}s^{n-1}}\right)^{p-2}h^2(t,s)\Bigg]ds\\ \le\int_{t_0}^{t_k}\Bigg[&K(s)\delta(s)q(s)\left(\frac{g^{n-1}(s)}{s^{n-1}}\right)^{p-1}ds+K(t_k)w(t_k)<\infty\ ,\end{aligned}$$

which contradicts (2.22) and completes the proof of the theorem.

For the case when $n=2$ in equation (1.1) we do not condition (2.3), any constant $k\in(0,1)$. Hence, we have the following oscillation criterion.

Theorem 2.4.

Assume that $p\ge 2$, and conditions (C1) – (C3), and (1.2) are satisfied. Let D_0, D, H be as in Theorem 2.1. Suppose also that there exist functions $h\in C(D_0,\ R)$, $K,\delta\in C^1([t_0,\infty),(0,\infty))$ such that (1.3) holds, and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)K(s)\delta(s)q(s)\left(\frac{g(s)}{s}\right)^{p-1}-\frac{\delta(s)a(s)s^{2-p}h^2(t,s)}{4(p-1)H(t,s)K(s)}\right]ds=\infty.$$

Then equation (1.1) is oscillatory.

Example 2.5.

Consider the fourth-order delay differential equation with damping

$$\left(tx'''(t)\right)' + \frac{1}{t^3}x'''(t) + \frac{1}{t^2}x\left(\frac{t}{5}\right) = 0, \tag{2.27}$$

for $t \in [1,\infty)$, where $n = 4,\ p = 2,\ a(t) = t,\ r(t) = 1/t^3,\ q(t) = \frac{1}{t^2}$ and $g(t) = t/5$. Then,

$$\int_{t_0}^{\infty}\left[\frac{1}{a(s)}\exp\left(-\int_{t_0}^{s}\frac{r(\tau)}{a(\tau)}d\tau\right)\right]^{\frac{1}{p-1}}ds = \int_{1}^{\infty}\frac{1}{s}\exp\left(-\int_{1}^{s}\frac{1}{\tau^4}d\tau\right)ds$$
$$= \int_{1}^{\infty}\frac{1}{s}\exp\left(\frac{1}{3s^3} - \frac{1}{3}\right)ds \ge \int_{1}^{\infty}\frac{1}{s}\exp\left(-\frac{1}{3}\right)ds = \infty,$$

and

$$\int_{t_0}^{\infty}\left[\int_{v}^{\infty}\left(\frac{1}{a(u)}\int_{u}^{\infty}q(s)ds\right)^{1/(p-1)}du\right]dv = \int_{1}^{\infty}\left[\int_{v}^{\infty}\left(\frac{1}{u}\int_{u}^{\infty}\frac{1}{s^2}ds\right)du\right]dv$$
$$= \int_{1}^{\infty}\left[\int_{v}^{\infty}\frac{1}{u^2}du\right]dv = \int_{1}^{\infty}\frac{1}{v}dv = \infty,$$

so (1.2) and (2.3) are satisfied, respectively. To apply Theorem 2.1, it remains to show that condition (2.4) holds. To see this, note that if $H(t,s) = (t-s)^2$, $K(t) = 1$, and $\delta(t) = t$, then $h(t,s) = (t-s)\left[2-(t-s)\left(s^{-1}-s^{-4}\right)\right]$. Now, for $k = 1/\sqrt{2}$, we have

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)K(s)\delta(s)q(s)\left(\frac{g^{n-1}(s)}{s^{n-1}}\right)^{p-1} - \frac{\delta(s)a(s)\left(\frac{|h(t,s)|}{p}\right)^{p}}{\left(H(t,s)K(s)\frac{k^{n-2}s^{n-2}}{(n-2)!}\right)^{p-1}}\right]ds$$

$$= \limsup_{t\to\infty}\frac{1}{(t-1)^2}\int_{1}^{t}\left[\frac{1}{125s}(t-s)^2 - \frac{1}{2k^2}\left[2-(t-s)\left(s^{-1}-s^{-4}\right)\right]^2\right]ds$$
$$= \limsup_{t\to\infty}\frac{1}{(t-1)^2}\left(t - \frac{35}{42t^2} + \frac{1}{105t^5} - \frac{573}{875}t^2 - \frac{4244}{375}t + 6t\ln t + \frac{1}{125}t^2\ln t + \frac{2949}{250}\right) = \infty,$$

which implies that (2.4) holds. Therefore, every solution of (2.27) is oscillatory by Theorem 2.1.

Example 2.6.

Consider the sixth-order delay differential equation with damping

$$\left(tx^{(5)}(t)\right)' + \frac{1}{t^5}x^{(5)}(t) + t^{-3/2}x\left(\frac{t}{3}\right) = 0, \tag{2.28}$$

for $t \in [1,\infty)$, where $n=6$, $p=2$, $a(t)=t$, $r(t)=1/t^5$, $q(t)=t^{-3/2}$ and $g(t)=t/3$. Then,

$$\int_{t_0}^{\infty}\left[\frac{1}{a(s)}\exp\left(-\int_{t_0}^{s}\frac{r(\tau)}{a(\tau)}d\tau\right)\right]^{\frac{1}{p-1}}ds = \int_{1}^{\infty}\left[\frac{1}{s}\exp\left(-\int_{1}^{s}\frac{1}{\tau^6}d\tau\right)\right]ds$$
$$= \int_{1}^{\infty}\frac{1}{s}\exp\left(\frac{5}{s^5}-5\right)ds \geq \int_{1}^{\infty}\frac{1}{s}\exp(-5)\,ds = \infty$$

and

$$\int_{t_0}^{\infty}\left[\int_{v}^{\infty}\left(\frac{1}{a(u)}\int_{u}^{\infty}q(s)\,ds\right)^{1/(p-1)}du\right]dv = \int_{t_0}^{\infty}\left[\int_{v}^{\infty}\left(\frac{1}{u}\int_{u}^{\infty}s^{-3/2}ds\right)du\right]dv$$
$$= \int_{t_0}^{\infty}\int_{v}^{\infty}\frac{2}{u^{3/2}}\,du\,dv = \infty.$$

For $H(t,s)=(t-s)^2$, $K(t)=1$, $k=\sqrt[4]{\frac{1}{2}}$, and $\delta(t)=t^2$, we see that condition (2.22) holds as

$$\limsup_{t\to\infty}\frac{1}{(t-1)^2}\int_{1}^{t}\left(\frac{1}{3^5}(t-s)^2 s^{1/2} - \frac{12}{s}\left[2-(t-s)\left(2s^{-1}-s^{-6}\right)\right]^2\right)ds$$
$$= \limsup_{t\to\infty}\frac{1}{(t-1)^2}\left[\frac{2274524}{13365}t - 192\ln t - \frac{92597}{5103}t^2 + \frac{16}{25515}t^{\frac{7}{2}}\right.$$
$$\left. + \frac{1}{385t^{10}}\left(7-64680t^{10}-792t^5\right) + \frac{30616}{1701}\right] = \infty.$$

Therefore, every solution of (2.28) is oscillatory by Theorem 2.3.

Example 2.7.

Consider the second-order delay differential equation with damping

$$x''(t)+\exp(-t)x'(t)+\exp(2t)x\left(\frac{t}{2}\right)=0, \tag{2.29}$$

for $t\in[1,\infty)$, where $n=2,\ p=2,\ a(t)=1,\ r(t)=\exp(-t),\ q(t)=\exp(2t)$ and $g(t)=t/2$. Then,

$$\begin{aligned}\int_{t_0}^{\infty}\left[\frac{1}{a(s)}\exp\left(-\int_{t_0}^{s}\frac{r(\tau)}{a(\tau)}d\tau\right)\right]^{\frac{1}{p-1}}ds &= \int_{1}^{\infty}\exp\left(-\int_{t_0}^{s}\exp(-\tau)d\tau\right)ds\\ &=\int_{1}^{\infty}\exp\big(\exp(-s)-\exp(-1)\big)ds\\ &\geq \int_{1}^{\infty}\exp\big(\exp(-1)\big)ds=\infty .\end{aligned}$$

Since $\int_{u}^{\infty}\exp(2s)ds=\infty$, for $u\geq 1$, we get

$$\int_{t_0}^{\infty}\left[\int_{v}^{\infty}\left(\frac{1}{a(u)}\int_{u}^{\infty}q(s)ds\right)^{1/(p-1)}du\right]dv=\int_{1}^{\infty}\left[\int_{v}^{\infty}\left(\int_{u}^{\infty}\exp(2s)ds\right)du\right]dv=\infty .$$

Let $H(t,s)=(t-s)^2$, $K(t)=1$, and $\delta(t)=1$. Then, $h(t,s)=(t-s)\left[2+(t-s)\exp(-s)\right]$. Now, condition (2.21) yields

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{(t-1)^2}\int_{1}^{t}\left[\frac{1}{2}(t-s)^2\exp(2s)-\frac{1}{4}\left[2+(t-s)\exp(-s)\right]^2\right]ds\\ &=\limsup_{t\to\infty}\frac{1}{(t-1)^2}\left[\frac{1}{16}\left(e^{-2t}-5e^{-2}\right)-e^{-t}+\frac{1}{8}\left(3te^{-2}+e^{2t}-e^{2}-t^2e^{-2}\right)\right.\\ &\qquad\left.+e^{-1}(2-t)+1-t+\frac{te^{2}}{4}(1-t)\right]=\infty ,\end{aligned}$$

where $\exp(u):=e^{u}$. Therefore, every solution of (2.29) is oscillatory by Theorem 2.2.

3. Conclusion

An even order trinomial functional differential equation with a damping term is considered. The oscillatory behavior of solutions of this equation is discussed. In proving our results, we employ

Riccati type transformations and integral averaging technique. Some examples are also constructed to illustrate our theoretical findings.

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