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Evaluation of Schwarz Child's Exterior and Interior Solutions in Bimetric Theory of Gravitation

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Abstract

In this note, we have investigated Schwarz child's exterior and interior solutions in bimetric theory of gravitation by solving Rosen's field equations and concluded that the physical singularities at $r = 0$ and at $r = 2M$ do not appear in our solution. We have ruled out the discrepancies between physical singularity and mathematical singularity at $r = 2M$ and proved that the Schwarzschild exterior solution is always regular and does not break-down at $r = 2M$. Further, we observe that the red shift of light in our model is much stronger than that of Schwarz child's solution in general relativity and other geometrical and physical aspects are studied.

Keywords: Relativity and gravitational theory; Space-time singularities

MSC (2010) No.: 83C15, 83C75

1. Introduction

Shortly after Einstein (1915) published his field equations of general theory of relativity, Schwarzschild (1916) solved the first and still the most important exact solution of the Einstein vacuum field equations which represents the field outside a spherical symmetric mass in otherwise empty space. Later it was also recognized as the solution representing both the outside and inside of a non-rotating black hole [Rindler (2001)]. Schwarzschild used spherical symmetric space-time in order to get the solution of gravitational mass inside and outside the star, since spherical symmetry has its own importance in general relativity. In order to get the solution, for outside the stable non rotating star which is static, Schwarzschild used spherically symmetric space-time mass distribution metric and by solving Einstein field equation in empty space (no matter but gravitation is there), arrived at the famous Schwarzschild exterior solution given by

$$ds^2 = -\frac{1}{\left(1-\frac{2M}{r}\right)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) + \left(1-\frac{2M}{r}\right)dt^2, \tag{1}$$

where M is the mass of the gravitating body. To get the solution inside the non-rotating stable star, he again used spherical symmetric space-time mass distribution and solved the Einstein law of gravitation in the presence of matter which is a consequence of the Schwarzschild interior solution and is given by:

$$ds^2 = -\frac{1}{\left(1-\frac{r^2}{R^2}\right)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) + \left(A - B\sqrt{1-\frac{r^2}{R^2}}\right)^2 dt^2. \tag{2}$$

The most noteworthy thing of the Schwarzschild exterior solution is that it enables us to study the gravitational field surrounding the Sun and heavenly bodies and it is used in discussing the three crucial tests namely: i) advance of perihelion of Mercury, ii) bending of light and iii) gravitational red shift which bring out the distinction between the predictions of Newtonian theory of gravitation and the General theory of relativity. The Schwarzschild interior solution gives the gravitational field inside the Sun and other details are given in Borkar et al. (2003) and Karade (2004).

Schwarzschild (1916) found two types of singularities in his solution, one is real (or physical) singularity at $r = 0$ the other is mathematical (can be removed) singularity at $r = 2M$. This is also called Schwarzschild singularity. The issue of the Schwarzschild singularity at $r = 2M$ has created interest but of different opinions: some believe that it is a mathematical singularity while the others opine that it is a real or physical singularity [see Finkelstein (1958), Kruskal (1960), Synge (1964), Penrose (1965), Hilton (1965), Geroch (1966, 1968), Bel (1969), Rosen (1970), Finley III (1971), Karade et al. (1975, 1976), Karade (1975, 1976), Borkar et al. (2003)].

Several alternative theories to the general theory of gravitation have been formulated and the spherical symmetric models have been investigated to understand the secret of the nature of the universe. Rosen's bimetric theory of gravitation is one of the alternatives to general relativity and it is free from the singularities appearing in the big bang theory of cosmological models; it obeys the principle of covariance and equivalence of the general relativity. In Rosen's bimetric theory of gravitation (1973, 1974, 1977), at every point of space-time, there are two metrics:

$$ds^2 = g_{ij}\, dx^i\, dx^j \tag{3}$$

and

$$d\eta^2 = \gamma_{ij}\, dx^i\, dx^j. \tag{4}$$

The first metric is Riemannian metric in which g_{ij} are the fundamental metric tensors representing gravitational potentials described in the geometry of curved space-time and the second is a flat metric in which γ_{ij} represents as background metric describing the inertial forces associated with the acceleration of the frame of reference. The interpretation of these two metric tensors in bimetric theory of relativity is not unique.

The Rosen's field equation in bimetric theory of gravitation is

$$N_i^j - \frac{1}{2} N\, \delta_i^j = -8\pi\, T_i^j, \tag{5}$$

where $N_i^j = \frac{1}{2}\gamma^{\alpha\beta}\left(g^{sj} g_{si\,|\,\alpha}\right)_{|\beta}$, $N = g_{ij} N^{ij}$ is the Rosen scalar. Here and hereafter the vertical bar $(\,|\,)$ stands for $\gamma-$ covariant differentiation where $g = \det(g_{ij})$ and $\gamma = \det(\gamma_{ij})$.

As the Rosen's bimetric theory of gravitation is free from singularity, in this paper we embrace the task of investigating the Schwarzschild exterior and interior solution in bimetric theory of gravitation by solving Rosen's field equation and try to clarify the nature of Schwarzschild singularities at $r = 0$ and at $r = 2M$. This is the motive of work to observe whether the space-time is break-down or not at $r = 0$ and at $r = 2M$ and also to see the geometrical and physical behavior of the Schwarzschild model in bimetric theory of gravitation. It is concluded that the Schwarzschild exterior solution in bimetric theory of gravitation is free from singularities at $r = 0$ and at $r = 2M$ and the space-time is regular everywhere and the discrepancies of physical singularity and mathematical singularity at $r = 2M$ is ruled out.

2. Schwarzschild Exterior Solution

We consider static spherically symmetric, non-rotating line element in the form

$$ds^2 = -e^{\lambda} dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) + e^{\upsilon}\, dt^2, \tag{6}$$

where the metric potentials λ and υ are functions of r only.

The flat metric corresponding to metric (6) is

$$d\eta^2 = -dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) + dt^2. \tag{7}$$

Therefore, Rosen's field equation (5) by using the equations (6) and (7) in empty space-time are takes the form as follows

$$-\frac{\lambda''}{2} = 0, \tag{8}$$

$$\frac{1}{r^2} = 0, \tag{9}$$

$$\frac{1}{r^2}\left(1 + \cos ec^2\theta\right) = 0, \tag{10}$$

$$-\frac{\upsilon''}{2} = 0. \tag{11}$$

Adding equation (8) and equation (11), we have

$$\left(\lambda'' + \upsilon''\right) = 0. \tag{12}$$

On integrating, we get

$$\left(\lambda' + \upsilon'\right) = k, \tag{13}$$

where k is the constant of integration.

As we know the particle diminishes as we go infinite distance outside the star indicated $\lambda, \upsilon \to 0$ as $r \to \infty$ so that $\lambda', \upsilon' \to 0$ as $r \to \infty$. This yield $k = 0$. Thus,

$$\left(\lambda' + \upsilon'\right) = 0,$$

which on integrating gives

$$\left(\lambda + \upsilon\right) = k_1, \tag{14}$$

where k_1 is the integrating constant. Choosing $k_1 = 0$, we write

$$\lambda = -\upsilon. \tag{15}$$

Thus,

$$e^{\lambda} = e^{-\upsilon}. \tag{16}$$

On integrating equation (8) twice, we obtain

$$\lambda = k_2 r + k_3, \tag{17}$$

where k_2 and k_3 are constant of integrations. In particular, $k_2 = 1$ and $k_3 = -2M$, we write

$$\lambda = (r - 2M) \quad \text{or} \quad \lambda = \left[r\left(1 - \frac{2M}{r}\right)\right] \tag{18}$$

so that

$$e^{\lambda} = e^{\left[r\left(1-\frac{2M}{r}\right)\right]} \quad \text{and} \quad e^{\upsilon} = e^{-\left[r\left(1-\frac{2M}{r}\right)\right]}. \tag{19}$$

Therefore line element (6), reduced to

$$ds^2 = -e^{\left[r\left(1-\frac{2M}{r}\right)\right]} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + e^{-\left[r\left(1-\frac{2M}{r}\right)\right]} dt^2. \tag{20}$$

This is the Schwarzschild exterior solution in bimetric theory of gravitation.

It is observed that our Schwarzschild exterior solution (20) is regular and the space-time is not break-down at $r = 0$ and at $r = 2M$. Though, $r = 0$ is a physical singularity and $r = 2M$ is mathematical (or physical) singularity in Schwarzschild exterior solution in general relativity, these singularities are not appeared in the Schwarzschild solution in bimetric theory of gravitation. Therefore, our model of Schwarzschild metric in bimetric theory of gravitation has a physical significance that it is regular everywhere and it is singularity free solution.

Gravitational Red Shift

Suppose light waves emitted from the source in the gravitational field from which number of waves of frequency v be emitted in proper time ds is given by

$$\eta = v\, ds. \tag{21}$$

An observer on the earth receiving the same number of waves of different frequency i.e., v' in proper time ds' is given by

$$\eta = v'\, ds'. \tag{22}$$

From equations (21) and (22), we get

$$v\,ds = v^{'}\,ds^{'}. \tag{23}$$

From our Schwarzschild exterior metric (20), we obtain the relation between proper time ds and coordinate time dt by assuming r, θ and ϕ constant as

$$\frac{ds}{dt} = e^{-\left[\frac{r}{2}\left(1-\frac{2M}{r}\right)\right]}. \tag{24}$$

For the radius of the Sun $r = R,$ we write

$$\frac{ds}{dt} = e^{-\left[\frac{R}{2}\left(1-\frac{2M}{R}\right)\right]}. \tag{25}$$

Hence successive light waves emitted at the coordinate interval dt are separated by the proper time $ds^{'}$ period on reaching to the earth. So that

$$ds^{'} = dt. \tag{26}$$

From equation (25) and (26), we obtain

$$\frac{ds}{ds^{'}} = e^{-\left[\frac{R}{2}\left(1-\frac{2M}{R}\right)\right]}. \tag{27}$$

From equation (23) and (27), we write

$$\frac{ds}{ds^{'}} = \frac{v^{'}}{v} = e^{-\left[\frac{R}{2}\left(1-\frac{2M}{R}\right)\right]}. \tag{28}$$

Let $\delta v = (v^{'} - v)$ be the change in frequency, then the red shift (shift towards red spectrum) is given by $\frac{\delta v}{v}$ and from equation (28) it is

$$\frac{\delta v}{v} = \frac{(v^{'} - v)}{v} = \left(\frac{v^{'}}{v} - 1\right) = -\left(\frac{R}{2} - M\right). \tag{29}$$

Here,

Mass of the Sun, $M = 2\times10^{33}$ gm and Radius of the Sun, $R = 6.96\times10^{10}$ cm

Gravitational constant, $\gamma = 6.67\times10^{-8}$ cm^3/gm, Velocity of light, $c = 3\times10^{10}$ cm/sec

Therefore,

mass of the Sun in centimeters, $m = \dfrac{\gamma M}{c^2} = 1.4822\times10^5$ cm

For the sun, equation (29) gives the red shift as

$$\frac{\delta \nu}{\nu} = 3.4799\times10^{10}. \tag{30}$$

Equation (29) is the required formula for the red shift. From equation (29), it is observed that negative sign shows that the frequency of light waves decreases as it leaves the sun. When it reaches to the earth, we get the shift towards red end of spectrum. It seems that the atoms on the sun were vibrating in slow motion when we observe them from the earth. Comparing the value of our red shift given by equation (30) in bimetric theory of gravitation to that of red shift 2.12×10^{-6} in Schwarzschild exterior solution in general relativity, we concluded that our value of red shift is much stronger than that of general relativity by 3.4799×10^{10}.

3. Schwarzschild Interior Solution

The energy momentum tensor $T_i^{\,j}$ of the incompressible perfect fluid is given by

$$T_i^{\,j} = (\rho + p)\,\upsilon_i\,\upsilon^j - p\,g_i^{\,j}, \tag{31}$$

where ρ and p are the proper energy density and pressure respectively. The quantity θ is the scalar of expansion which is given by,

$$\theta = \upsilon^i{}_{|i} \tag{32}$$

and υ^i is the flow vector satisfying the relation

$$g_{ij}\,\upsilon^i\,\upsilon^j = 1. \tag{33}$$

We assume that coordinates to be co-moving, so that

$$\upsilon^1 = \upsilon^2 = \upsilon^3 = 0, \quad \upsilon^4 = 1. \tag{34}$$

Equation (31) of energy momentum tensor yield

$$T_1^1 = T_2^2 = T_3^3 = -p, \quad T_4^4 = \rho. \tag{35}$$

The Rosen's field equations (5) for the metric (6) and (7) with the help of (35) gives

$$\left(\frac{\upsilon'' - \lambda''}{4} - \frac{1}{r^2}\left(1 + \frac{\cos ec^2\theta}{2}\right)\right) = 8\pi\ p, \tag{36}$$

$$\left(\frac{\upsilon'' + \lambda''}{4} - \frac{\cos ec^2\theta}{2r^2}\right) = 8\pi\ p, \tag{37}$$

$$\left(\frac{\upsilon'' + \lambda''}{4} + \frac{\cos ec^2\theta}{2r^2}\right) = 8\pi\ p, \tag{38}$$

$$\left(\frac{\upsilon'' - \lambda''}{4} + \frac{1}{r^2}\left(1 + \frac{\cos ec^2\theta}{2}\right)\right) = 8\pi\ \rho, \tag{39}$$

where $\lambda'' = \dfrac{d^2\lambda}{dr^2}$ etc.

Adding equation (36) and (39), we have

$$\left(\frac{\upsilon'' - \lambda''}{2}\right) = 8\pi\,(\rho + p) \tag{40}$$

and by adding equation (37) and (38), we get

$$\left(\frac{\upsilon'' + \lambda''}{4}\right) = 8\pi\ p\ . \tag{41}$$

Adding equation (36) and (38), we have

$$\frac{\upsilon''}{2} - \frac{1}{r^2} = 16\pi\ p \tag{42}$$

and by adding equation (37) and (39), we obtain

$$\frac{\upsilon''}{2} + \frac{1}{r^2} = 16\pi\,(\rho + p)\ . \tag{43}$$

Subtracting equation (41) from equation (40), we have

$$\lambda'' = -16\pi\rho. \tag{44}$$

On integrating twice and selecting suitable value of constant of integration, we write

$$\lambda = \left[3\left(1 - \frac{r^2}{R^2}\right)\right], \text{ where } \frac{1}{R^2} = \frac{8\pi\rho}{3}. \tag{45}$$

Thus,

$$e^{\lambda} = e^{\left[3\left(1-\frac{r^2}{R^2}\right)\right]}. \tag{46}$$

Adding equations (40) and (41), we get

$$\frac{\upsilon''}{2} = 16\pi p + 8\pi\rho, \tag{47}$$

which on integrating, yield

$$\upsilon = \left[3\left(2Ar^2 - B\left(1-\frac{r^2}{R^2}\right)\right)\right], \text{ where } A = \frac{8\pi p}{3},\ B = 1. \tag{48}$$

We can also deduced the value of υ by adding equations (42) and (43) as under

$$\upsilon = \left[\frac{3}{2}\left(3Ar^2 - B\left(1-\frac{r^2}{R^2}\right)\right)\right], \tag{49}$$

in which $A = \frac{8\pi p}{3}$ and $B = 1$.

Thus, we have

$$e^{\upsilon} = e^{\left[3\left(2Ar^2 - B\left(1-\frac{r^2}{R^2}\right)\right)\right]} \quad \text{or} \quad e^{\upsilon} = e^{\left[\frac{3}{2}\left(3Ar^2 - B\left(1-\frac{r^2}{R^2}\right)\right)\right]}. \tag{50}$$

We have two values of e^{υ} given by equation (50), from which you can take any value. Here, we take

$$e^{\upsilon} = e^{\left[3\left(2A\ r^2 - B\left(1-\frac{r^2}{R^2}\right)\right)\right]}.$$

Hence, the Schwarzschild interior solution will be

$$ds^2 = -e^{\left[3\left(1-\frac{r^2}{R^2}\right)\right]}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + e^{\left[3\left(2Ar^2 - B\left(1-\frac{r^2}{R^2}\right)\right)\right]}\, dt^2. \tag{51}$$

This is the Schwarzschild interior solution in bimetric theory of gravitation.

At the boundary of the star i.e., $r = r_1$, for Schwarzschild exterior solution (20) and Schwarzschild interior solution (51), we write

$$\left[r_1\left(1-\frac{2M}{r_1}\right)\right] = \left[3\left(1-\frac{r_1^{\,2}}{R^2}\right)\right] \tag{52}$$

and

$$-\left[r_1\left(1-\frac{2M}{r_1}\right)\right] = \left[3\left(2\,A\,r_1^{\,2} - B\left(1-\frac{r_1^{\,2}}{R^2}\right)\right)\right]. \tag{53}$$

From equation (52) and (53), we have

$$-\left[r_1\left(1-\frac{2M}{r_1}\right)\right] = -\left[3\left(1-\frac{r_1^{\,2}}{R^2}\right)\right] = \left[3\left(2A\,r_1^{\,2} - B\left(1-\frac{r_1^{\,2}}{R^2}\right)\right)\right],$$

from which, we write

$$M = \left(\frac{3\,r_1^{\,2}}{2R^2} + \frac{r_1}{2} - \frac{3}{2}\right) \text{ and } A = 0,\ B = 1. \tag{54}$$

Thus at the boundary of star $r = r_1$, we get $A = 0$ and $B = 1$. Now at the boundary $r = r_1$, the pressure $p = 0$ and hence $A = \frac{8\pi\, p}{3} = 0$. This value of A is matched to the value $A = 0$ and $B = 1$ obtained at the boundary.

4. Conclusion

We have investigated Schwarzschild exterior and interior solutions in bimetric theory of gravitation by solving Rosen's field equations. It is realized that our Schwarzschild exterior solution is free from singularities at $r = 0$ and at $r = 2M$ and our solution always regular and do not break-down at $r = 0$ and at $r = 2M$. The singularity at $r = 2M$ has created interest of

the people. Some people say that it is a physical singularity and some opined that it is a mathematical one. We ruled out these two opinions and proved that these singularities not appeared in Schwarzschild solution in bimetric theory of gravitation. Further, we observed that the red shift of light and concluded that our red shift is much more stronger than that of red shift in general relativity. At boundary of the star, we concluded that the values of A and B as $A = 0$ and $B = 1$.

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REFERENCES

Einstein, A. (1916). Relativity: The Special and General Theory, New York: Holt, H. and Company.

Schwarzschild, K. (1916). Uber das Gravitationalfeld eines Massenpunktes nach der Einstein'schen Theorie. Sitzber Dtsch. Akad. Wiss. Physik.-math KI, Vol. 189.

Rindler, W. (2001). *Relativity: Special, General and Cosmological*, Oxford University Press Publishers, Oxford.

Karade, T.M., Khadekar, G.S. and Bendre, M. (2004). *Lectures on General Relativity*, Sonu Nilu Publishers, India.

Finkelstein, D. (1958). Past-future asymmetric of the gravitational field of a point particle. Physical Review, Vol. 110, p. 965.

Kruskal, M.D. (1960). Maximal extension of Schwarzschild metric. Physical Review, Vol. 119, p. 1743-1745.

Synge, J.L. (1964). In relativity, groups and topology (ed. Dewitt C. M. and Dewitt B, S.). Gordon and Breach, New York p. 64.

Penrose, R. (1965). Gravitational collapse and space-time singularities. Physical Review Letters, Vol. 14, No. 3, p. 57-59.

Hilton, E. (1965). The singularity in the Schwarzschild space-time. Proceedings of the Royal Society A, London, Vol. 283, p. 491-497.

Geroch, R. (1966). Singularities in closed universes. Physics Review Letters, Vol. 17, No. 8, p. 445-447.

Geroch, R. (1968). What is singularity in general relativity? Annals of Physics, Vol. 48, No. 3, p. 526-540.

Bel, L. (1969). Schwarzschild singularity. Journal of Mathematical Physics, Vol. 10, p. 1501-1503.

Rosen, N. (1970). The nature of the Schwarzschild singularity 'Relativity' (ed. Carmeli M., Fickler S. I. and Witten L.). Plenum Press, New York p. 229.

Finley, III J.D. (1971). Tidal gravitational accelerations near an arbitrary time like geodesic in Schwarzschild space. Journal of Mathematical Physics, Vol. 12, No. 1, p. 32-36.

Karade, T.M. and Rao, J.R. (1975). On the exterior field of a charged particle, Acta Physica Academiae Scintiarum Hungaricae, Vol. 38, No. 3, p. 221-223.

Karade, T.M. and Rao, J.R. (1976). Effect of rotation on a test particle in the NUT field, Australian Journal of Physics, Vol. 29, No. 2, p. 107-112.

Karade, T.M. (1975). The nature of the singularity in the gravitational field of a charged particle, Acta Physica Academiae Scientiarum Hungaricae, Vol. 39, No. 4, p. 227-231.

Karade, T. M. (1976). (Ph. D. thesis), Nature of some physical space-times in general relativity revealed by the geodesic motion of a test particle.

Borkar, M.S. and Karade, T.M. (2003). On singularity of spherically symmetric space-times. Indian Journal of Pure and Applied Mathematics, Vol. 34, No. 8, p. 1219-1224.

Rosen, N. (1973). A bimetric theory of gravitation. General Relativity and Gravitation, Vol. 4, No. 6, p. 435- 447.

Rosen, N. (1974). A theory of gravitation. Annals of Physics, Vol. 84, No. (1-2), p. 455-473.

Rosen, N. (1977). Bimetric gravitation and cosmology. Astrophysical Journal, Vol. 211, No. 2, p. 357-360.